PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCLV.

VOL. 145.

LONDON:

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MDCCCLV.

ADVERTISEMENT.

The Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions,* take this opportunity to acquaint the Public, that it fully appears, as well from the Council-books and Journals of the Society, as from repeated declarations which have been made in several former *Transactions,* that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgement of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

The Meteorological Journal hitherto kept by the Assistant Secretary at the Apartments of the Royal Society, by order of the President and Council, and published in the Philosophical Transactions, has been discontinued. The Government, on the recommendation of the President and Council, has established at the Royal Observatory at Greenwich, under the superintendence of the Astronomer Royal, a Magnetical and Meteorological Observatory, where observations are made on an extended scale, which are regularly published. These, which correspond with the grand scheme of observations now carrying out in different parts of the globe, supersede the necessity of a continuance of the observations made at the Apartments of the Royal Society, which could not be rendered so perfect as was desirable, on account of the imperfections of the locality and the multiplied duties of the observer.

A List of Public Institutions and Individuals, entitled to receive a copy of the Philosophical Transactions of each year, on making application for the same directly or through their respective agents, within five years of the date of publication.

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Senftenberg, Baron von	Prague.
Wartmann, Professor Elie	Geneva.
Younghusband, Capt., R.A.	Woolwich.

Adjudication of the Medals of the Royal Society for the year 1855 by the President and Council.

The Copley Medal to M. Léon Foucault, for his various researches in Experimental Physics.

A Royal Medal to John Russell Hind, Esq., Superintendent of the Nautical Almanack, for his discovery of ten Planetoids, the computation of their orbits, and various other astronomical discoveries.

A Royal Medal to John Obadiah Westwood, Esq., for his various Monographs and Papers on Entomology.

The Bakerian Lecture was delivered by John Tyndall, PhD., F.R.S., Professor of Natural Philosophy in the Royal Institution, and entitled "On the Nature of the Force by which Bodies are repelled from the Poles of a Magnet; to which is prefixed, an Account of some Experiments on Molecular Influences."

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OF VOL. 145.

Appendix.

PHILOSOPHICAL TRANSACTIONS.

I. THE BAKERIAN LECTURE.—*On the nature of the Force by which Bodies are repelled from the Poles of a Magnet; to which is prefixed, an Account of some Experiments on Molecular Influences. By* JOHN TYNDALL, *Ph.D., F.R.S., Membre de la Société Hollandaise des Sciences; Foreign Member of the Physical Society of Berlin, and Professor of Natural Philosophy in the Royal Institution.*

Received October 31, 1854,—Read January 25, 1855.

CONTENTS:—

INTRODUCTION.

FROM the published account of his researches it is to be inferred, that the same heavy glass, by means of which he produced the rotation of the plane of polarization of a luminous ray, also led Mr. FARADAY to the discovery of the diamagnetic force. A square prism of the glass, 2 inches long and 0·5 of an inch thick, was suspended with its length horizontal between the two poles of a powerful electro-magnet: on developing the magnetism the prism moved round its axis of suspension, and finally set its length at right angles to a straight line drawn from the centre of one pole to that of the other. A prism of ordinary magnetic matter, similarly suspended, would, as is well known, set its longest dimension from pole to pole. To distinguish the two positions here referred to, Mr. FARADAY introduced two new terms, which have since come into general use: he called the direction parallel to the line joining the poles, the *axial* direction, and that perpendicular to the said line, the *equatorial* direction.

The difference between this new action, and the ordinary magnetic action, was further manifested, when a fragment of the heavy glass was suspended before a single magnetic pole: the fragment was repelled when the magnetism was excited; and to the force which produced this repulsion Mr. FARADAY gave the name of *diamagnetism.*

Numerous other substances were soon added to the heavy glass, and, among the metals, it was found that bismuth possessed the new property in a comparatively exalted degree. A fragment of this substance was forcibly repelled by either of the poles of a magnet; while a thin bar of the substance, or a glass tube containing the bismuth in fragments, or in powder, suspended between the two poles of a horseshoe magnet, behaved exactly like the heavy glass, and set its longest dimension equatorial.

These exhaustive researches, which rendered manifest to the scientific world the existence of a pervading natural force, glimpses of which merely had been previously obtained by BRUGMANN and others, were made public in 1846; and in the following year M. PLÜCKER made known his beautiful discovery of the action of a magnet upon crystallized bodies. His first result was, that when any crystal whatever was suspended between the poles of a magnet, with its optic axis horizontal, a repulsive force was exerted on the said axis, in consequence of which it receded from the poles and finally set itself at right angles to the line joining them. Subsequent experiments, however, led to the conclusion, that the axes of optically negative crystals only experienced this repulsion, while the axes of positive crystals were attracted; or, in other words, set themselves from pole to pole.

The attraction and repulsion, here referred to, were ascribed by M. PLÜCKER to the action of a new force, entirely independent of the magnetism or diamagnetism of the mass of the crystal *. Shortly after the publication of M. PLÜCKER's first memoir, Mr. FARADAY observed the remarkable magnetic properties of crystallized bismuth; and his researches upon this, and one or two kindred points, formed the subject of the Bakerian Lecture before the Royal Society for the year 1849.

* "*The force which produces this repulsion is independent of the magnetic or diamagnetic condition of the mass of the crystal; it diminishes less, as the distance from the poles of the magnet increases, than the magnetic and diamagnetic forces emanating from these poles and acting upon the crystal.*"—Prof. PLÜCKER in POGGENDORFF's Annalen, vol. lvii. No. 10; TAYLOR's Scientific Memoirs, vol. v. p. 353.

The forces emanating from the poles of a magnet are thus summed up by M. PLÜCKER:—

1st. The magnetic force in a strict sense.

2nd. The diamagnetic action discovered by FARADAY.

3rd. The action exerted on the optic axis of crystals (and that producing the rotation of the plane of polarization which probably corresponds to it). *The second diminishes more with the distance than the first, and the first more than the third.*—TAYLOR's Scientific Memoirs, vol. v. p. 380.

The crystal (cyanite) does not point according to the magnetism of its substance, *but only in obedience to the magnetic action upon its optic axes.*—Letter to Mr. FARADAY, Phil. Mag. vol. xxxiv. p. 451. The italics in all cases are M. PLÜCKER's own.

M. DE LA RIVE states the view of M. PLÜCKER to be:—"that the axis in its quality as axis, and independently of the very nature of the substance of the crystal, enjoys peculiar properties, more frequently in opposition to those possessed by the substance itself, or which at least are altogether independent of it."—Treatise on Electricity, vol. i. p. 359.

Through the admirable lectures of Professor Bunsen on Electro-chemistry in 1848, I was first made acquainted with the existence of the diamagnetic force; and in the month of November 1849 my friend Professor Knoblauch, then of Marburg, now of the University of Halle, suggested to me the idea of repeating the experiments of M. Plücker and Mr. Faraday. He had procured the necessary apparatus with the view of prosecuting the subject himself, but the pressure of other duties prevented him from carrying out his intention. I adopted the suggestion and entered upon the inquiry in M. Knoblauch's cabinet. Our frequent conversations upon the subject led to the idea of our making a joint publication of the results: this we accordingly did, in two papers, the first of which, containing a brief account of some of the earliest experiments, appeared in the Philosophical Magazine for March 1850, and some time afterwards in Poggendorff's Annalen; while the second and principal memoir appeared in the Philosophical Magazine for July 1850, and in Poggendorff's Annalen about January 1851*. I afterwards continued my researches in the private laboratory of Professor Magnus of Berlin, who with prompt kindness and a lively interest in the furtherance of the inquiry, placed all necessary apparatus at my disposal. The results of this investigation are described in a paper published in the Philosophical Magazine for September 1851, and in Poggendorff's Annalen, vol. lxxxiii.

In these memoirs it was shown that the law according to which the axes of positive crystals are attracted and those of negative crystals repelled, was contradicted by the deportment of numerous crystals both positive and negative. It was also proved that the force which determined the position of the optic axes in the magnetic field was not independent of the magnetism or diamagnetism of the mass of the crystal; inasmuch as two crystals, of the same form and structure, exhibited altogether different effects, when one of them was magnetic and the other diamagnetic. It was shown, for example, that pure carbonate of lime was diamagnetic, and always set its optic axis equatorial; but that when a portion of the calcium was replaced by an isomorphous magnetic constituent, which neither altered the structure nor affected the perfect transparency of the crystal, the optic axis set itself from pole to pole. The various complex phenomena exhibited by crystals in the magnetic field were finally referred to the modification of the magnetic and diamagnetic forces by the peculiarities of molecular arrangement.

This result is in perfect conformity with all that we know experimentally regarding the connexion of matter and force. Indeed it may be safely asserted that every force which makes matter its vehicle of transmission must be influenced by the manner in which the material particles are grouped together. The phenomena of double refraction and polarization illustrate the influence of molecular aggregation upon light. Wertheim has shown that the velocity of sound through wood, *along* the fibre, is

* The memoirs in the Philosophical Magazine were written by me, and the second one has, I believe, been translated into German by Dr. Krönig: the papers in Poggendorff's Annalen were edited by my colleague. —J. T.

about five times its velocity *across* the fibre: De la Rive, DeCandolle and myself have shown the influence of the same molecular grouping upon the propagation of heat. In the first section of the present paper, the influence of the molecular structure of wood upon its magnetic deportment is described: De Senarmont has shown that the structure of crystals endows them with different powers of calorific conduction in different directions: Knoblauch has proved the same to be true, with regard to the transmission of radiant heat: Wiedeman finds the passage of frictional electricity along crystals to be affected by structure; and some experiments, which I have not yet had time to follow out, seem to prove, that bismuth may, by the approximation of its particles, be caused to exhibit, in a greatly increased degree, those singular effects of induction which are so strikingly exhibited by copper, and other metals of high conducting power.

Indeed the mere *à priori* consideration of the subject must render all the effects here referred to extremely probable. Supposing the propagation of the forces to depend upon a subtle agent, distinct from matter, it is evident that the progress of such an agent from particle to particle must be influenced by the manner in which these particles are arranged. If the particles be twice as near each other in one direction as in another, it is certain that the agent of which we speak will not pass with the same facility in both directions. Or supposing the effects to which we have alluded to be produced by motion of some kind, it is just as certain that the propagation of this motion must be affected by the manner in which the particles which transmit it are grouped together. Whether, therefore, we take the old hypothesis of imponderables, or the new, and more philosophic one, of modes of motion, the result is still the same.

If this reasoning be correct, it would follow, that, if the molecular arrangement of a body be changed, such a change will manifest itself by an alteration of deportment towards any force operating upon the body: the action of compressed glass upon light, which Wertheim in his recent researches* has so beautifully turned to account in the estimation of pressures, is an illustration in point; and the inference also receives the fullest corroboration from experiments, some of which are recorded in the papers alluded to, and which show that all the phenomena of magnecrystallic action may be produced by simple mechanical agency. What the crystalline forces do in one case, mechanical force, under the control of the human will, accomplishes in the other. A crystal of carbonate of iron, for example, suspended in the magnetic field, exhibits a certain deportment: the crystal may be removed, pounded into the finest dust, and the particles so put together that the mass shall exhibit the same deportment as before. A bismuth crystal suspended in the magnetic field, with its planes of principal cleavage vertical, will set those planes equatorial; but if the crystalline planes be squeezed sufficiently together by a suitable mechanical force, this deportment is quite changed, and the line which formerly set equatorial now sets axial†.

* Phil. Mag. October and November 1854.

† Phil. Mag. vol. ii. Ser. 4. p. 183.

Thus we find that the influence of crystallization may be perfectly imitated, and even overcome, by simple mechanical agencies. It would of course be perfectly unintelligible were we to speak of any direct action of the magnetic force upon the force by which the powdered carbonate of iron, or the solid cube of bismuth, is compressed; such an idea, however, appears scarcely less tenable than another which seems to be entertained by some who feel an interest in this subject; namely, that there is a direct action of the magnet upon the molecular forces which built the crystal. The function of such forces, as regards the production of the effects, is, I believe, *mediate*; the molecular forces are exerted in placing the particles in position, and the subsequent phenomena, whether exhibited in magnecrystallic action, in the bifurcation and polarization of a luminous ray, or in the modification of any other force transmitted through the crystal, are not due to the action of force upon force, except through the intermediation of the particles referred to*.

The foregoing introductory statement will, perhaps, sufficiently indicate the present aspect of this question. The object I proposed to myself in commencing the inquiry now laid before the Royal Society, is to obtain, if possible, clearer notions of the nature of the diamagnetic force than those now prevalent; for though, in the preceding paragraphs, we have touched upon some of the most complex phenomena of magnetism and diamagnetism, and are able to produce these phenomena at will, the greatest diversity of opinion still prevails as to the real relationship of the two forces. The magnetic force, we know, embraces both attraction and repulsion, thus exhibiting that wonderful dual action which we are accustomed to denote by the term polarity. Mr. Faraday was the first who proposed the hypothesis that diamagnetic bodies, operated on by magnetic forces, possess a polarity "the same in kind as, but the reverse in direction, of that acquired by iron, nickel, and ordinary magnetic bodies under the same circumstances†." M. W. Weber sought to confirm this hypothesis by a series of experiments, wherein the excitement of the supposed diamagnetic polarity was applied to the generation of induced currents—apparently with perfect success. Mr. Faraday afterwards showed, and his results were confirmed by M. Verdet, that effects similar to those described by the distinguished German, were to be attributed, not to the excitement of diamagnetic polarity, but to the generation of ordinary induced currents in the metallic mass. On the question of polarity Mr. Faraday's results were negative, and he therefore, with philosophic caution, holds himself unpledged to his early opinion. M. Weber, however, still retains his belief in the reverse polarity of diamagnetic bodies, whereas Weber's countryman M. von Feilitsch, in a series of memoirs recently published in Poggendorff's Annalen, contends that the polarity of

* The influence of molecular aggregation probably manifests itself on a grand scale in nature. The Snowdon range of mountains, for example, is principally composed of slate rock, whose line of strike is nearly north and south. The magnetic properties of this rock I find, by some preliminary experiments, to be very different along the cleavage from what they are across it. I cannot help thinking that these vast masses, in their present position, must exert a different action on the magnetic needle from that which would be exerted if the line of strike were east and west.

† Experimental Researches, 2429, 2430.

diamagnetic bodies is precisely the same as that of magnetic ones. In this unsettled state of the question nothing remained for me but a complete examination of the nature of the diamagnetic force, and a thorough comparison of its phenomena with those of ordinary magnetism. This has been attempted in the following pages, with what success it must be left to the reader to decide.

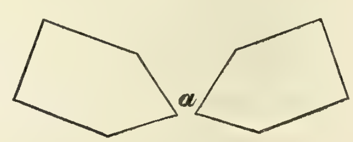
Before entering upon the principal inquiry, I will introduce one or two points which arose incidentally from the investigation, and which appear to be worth recording.

I. ON THE MAGNETIC PROPERTIES OF WOOD.

No experiments have yet been made, to determine the influence of structure upon the magnetic deportment of this substance; and even on the question whether it is magnetic, like iron, or diamagnetic, like bismuth, differences of opinion appear to prevail. Such differences are to be referred to the extreme feebleness of the force proper to the wood itself, and its consequent liability to be masked by extraneous impurity. In handling the substance intended for experiment the fingers must be kept perfectly clean, and frequent washing is absolutely necessary. After reducing the substance to a regular shape, so as to annul the influence of exterior form, its outer surface must be carefully removed by glass, and the body afterwards suspended by a very fine fibre between the poles of a strong electro-magnet.

The first step in the present inquiry was to ascertain whether the substance examined was paramagnetic* or diamagnetic. It is well known, that, in experiments of this kind, moveable masses of soft iron are placed upon the ends of the electro-magnet, the distance between the masses being varied to suit the experiment. In front of a pointed mass of iron of this kind, a cube of wood was suspended, and if, on exciting the magnet, the cube was repelled by the point, it was regarded as diamagnetic; if attracted, it was considered to be paramagnetic. The force was considerably intensified by placing the two moveable poles as in fig. 1, and suspending the cube at *a* on the same level with the points; a diamagnetic body placed there is, on the development of the magnetic force, forcibly driven *from* the line which unites the points, while a magnetic body is forcibly drawn in between them.

Fig. 1.



Having thus observed the deportment of the mass, the cube was next suspended between the *flat* ends of the poles sketched in fig. 1. The parallel faces were about three-quarters of an inch apart, and in each case the fibre of the suspended wood was horizontal. The specimen first examined was Beef-wood: suspended in the position *a*, fig. 1, the mass was repelled: suspended between the flat poles, on exciting the mag-

* The effects exhibited by iron and by bismuth come properly under the general designation of *magnetic* phenomena: to render their subdivision more distinct Mr. Faraday has recently introduced the word *paramagnetic* to denote the old magnetic effects, of which the action of iron is an example. Wherever the word *magnetic* occurs, without the prefix, it is always the old action that is referred to.

net, the cube, if in an oblique position, turned and set its fibre equatorial. By suitably breaking and closing the circuit, the cube could be turned 180° round and held in this new position. The axial position of the ligneous fibre was one of unstable equilibrium, from which, if it diverged in the slightest degree right or left, the cube turned and finally set its fibre equatorial. The following is a statement of the results obtained with thirty-five different kinds of wood:—

Table I.

Name of wood.	Deportment of mass.	Deportment of structure.	Remarks.
1. Beef-wood	Diamagnetic.	Fibre equatorial.	
2. Black Ebony	"	"	
3. Box-wood	"	"	
4. Second specimen	"	"	
5. Brazil-wood	"	"	
6. Braziletto	"	"	Action decided.
7. Bullet-wood	"	"	Action decided.
8. Cam-wood	"	"	
9. Cocoa-wood	"	"	
10. Coromandel-wood	"	"	Action strong.
11. Green Ebony	"	"	Action strong.
12. Green-heart	"	"	Action strong.
13. Iron-wood	"	"	
14. King-wood	"	"	Action strong.
15. Locust-wood	"	"	
16. Maple	"	"	Action decided.
17. Lance-wood	"	"	Action decided.
18. Olive-tree	"	"	
19. Peruvian-wood	"	"	Action strong.
20. Prince's-wood	"	"	
21. Camphor-wood	"	"	
22. Sandal-wood	"	"	
23. Satin-wood	"	"	
24. Tulip-wood	"	"	
25. Zebra-wood	"	"	
26. Botany Bay Oak	"	"	Action strong.
27. Mazatlan-wood	"	"	Action decided.
28. Tamarind-wood	"	"	
29. Sycamore	"	"	Action decided.
30. Beech	"	"	Action decided.
31. Ruby-wood	"	"	
32. Jacca	"	"	
33. Oak	"	"	Action strong.
34. Yew	"	"	Action feeble.
35. Black Oak	Paramagnetic.	"	Action decided.

The term "decided" is here used to express an action more energetic than ordinary, but in no case does the result lack the decision necessary to place it beyond doubt. It must also be remarked that the term "strong" is used in relation to the general deportment of wood; for, compared with the action of many other diamagnetic bodies, the strongest action of wood is but feeble. Simple as the problem may appear, it required considerable time and care to obtain the results here recorded. During a first examination of the cubes eight anomalies presented themselves—in eight cases the fibre set either oblique or axial. The whole thirty-five specimens were carefully rescraped with glass and tested once more; still two remained, which, though repelled as masses, persistently set with the fibre axial, and oscillated round this position so steadily as to lead to the supposition that the real deportment of the substance was thus exhibited. I scraped these cubes ten times successively, and washed them with all care, but the deportment remained unchanged. The cubes, for the sake of reference, had been stamped with diminutive numbers by the maker of them; and I noticed at length, that in these two cases a trace of the figures remained: on removing the whole surface which bore the stamp from each, the cubes forsook the axial position, and set, like the others, with the fibre equatorial.

The influence of the mere *form* of an impurity was here very prettily exhibited. The cubes in question had been stamped (probably by an iron tool) with the numbers 33 and 37, which lay in the line of the fibre; the figures, being dumpy little ones, caused an elongation of the magnetic impurity along the said line, and the natural consequence of this elongation was the deportment above described.

Of the thirty-five specimens examined one proved to be paramagnetic. Now, it may be asked, if the views of molecular action stated in the foregoing pages be correct, how is it that this paramagnetic cube sets its fibre equatorial? The case is instructive. The substance (bog oak) had been evidently steeped in a liquid containing a small quantity of iron in solution, whence it derived its magnetism; but here we have no substitution of paramagnetic molecules for diamagnetic ones, as in the cases referred to. The extraneous magnetic constituent is practically indifferent as to the direction of magnetization, and it therefore accommodates itself to the directive action of the wood to which it is attached.

II. ON THE ROTATION OF BODIES BETWEEN POINTED MAGNETIC POLES.

In his experiments on charcoal, wood-bark and other substances, M. Plücker discovered some very curious phenomena of rotation, which occurred on removing the substance experimented on from one portion of the magnetic field to another. To account for these phenomena, he assumed, that in the substances which exhibited the rotation two antagonist forces were perpetually active—a repulsive force, which caused the substance to assume one position; and an attractive force, which caused it to assume a different position: that, of these two forces, the repulsive diminished more quickly than the attractive, when the distance of the body from the poles was augmented. Thus, the former, which was predominant close to the poles, succumbed to the latter when a suitable distance was attained, and hence arose the observed rotation.

Towards the conclusion of the memoir published in the September number of the Philosophical Magazine for 1851, I stated that it was my intention further to examine this highly ingenious theory. I shall now endeavour to fulfil the promise then made, and to state, as briefly as I can, the real law which regulates these complex phenomena.

The masses of soft iron sketched in fig. 1 were placed upon the ends of the electromagnet, with their points facing each other; between the points the body to be examined was suspended by a fine fibre, which, passing round a groove, the substance could be raised or lowered by turning a milled head. The body was first suspended on the level of the points and its deportment noted, it was then slowly elevated, and the change of position, if any, was observed. It was finally permitted to sink below the points and its deportment there noted also.

The following is a statement of the results; the words 'equatorial' (E.) and 'axial' (A.) imply that the longest horizontal dimension of the substance examined took up the position denoted by each of these words respectively. The manner in which the bars were prepared is explained further on.

Table II.

Name of substance.	Horizontal dimensions.	Deportment of mass.	Position. Between poles.	Above.	Below.
1. Tartaric acid	0·5 × 0·1	Diamagnetic.	E.	A.	A.
2. A second specimen	0·4 × 0·1	"	E.	A.	A.
3. Red ferrocyanide of potassium	0·6 × 0·1	Paramagnetic.	A.	E.	E.
4. A second prism	0·9 × 0·12	"	A.	E.	E.
5. Citric acid	0·55 × 0·25	Diamagnetic.	E.	A.	A.
6. A second specimen	0·48 × 0·2	"	E.	A.	A.
7. Beryl	0·45 × 0·1	Paramagnetic.	A.	E.	E.
8. Saltpetre	0·6 × 0·3	Diamagnetic.	E.	A.	A.
9. Nitrate of soda	0·6 × 0·12	"	E.	A.	A.
10. Sulphate of iron	0·7 × 0·15	Paramagnetic.	A.	E.	E.
11. A second specimen	0·6 × 0·03	"	A.	E.	E.
12. A third specimen	1·0 × 0·13	"	A.	E.	E.
13. Calcareous spar	0·5 × 0·1	Diamagnetic.	E.	A.	A.
14. A full crystal	....	"	E.	A.	A.
15. Carbonate of iron	0·5 × 0·1	Paramagnetic.	A.	E.	E.
16. Carbonate of iron powdered and compressed	0·9 × 0·18	"	A.	E.	E.
17. Compressed disk	0·8 × 0·08	"	A.	E.	E.
18. Bismuth	0·95 × 0·15	Diamagnetic.	E.	A.	A.
19. The same compressed	0·7 × 0·05	"	E.	A.	A.
20. The same powdered and compressed	0·6 × 0·07	"	E.	A.	A.
21. Cylinder of the same	1·0 × 0·15	"	E.	A.	A.
22. Tourmaline	2·1 × 0·1	Paramagnetic.	A.	E.	E.
23. A second specimen	1·1 × 0·1	"	A.	E.	E.
24. A third	0·9 × 0·1	"	A.	E.	E.
25. Sulphate of nickel	0·9 × 0·3	"	A.	E.	E.
26. A second specimen	0·6 × 0·2	"	A.	E.	E.
27. Heavy spar	0·38 × 0·18	Diamagnetic.	E.	A.	A.
28. A second specimen	0·4 × 0·18	"	E.	A.	A.
29. Carbonate of tin powdered and compressed	0·34 × 0·04	"	E.	A.	A.
30. A second specimen	length 6 times width	"	E.	A.	A.
31. Ammonio-phosphate of magnesia powdered and compressed	0·3 × 0·06	"	E.	A.	A.
32. A second specimen	0·5 × 0·07	"	E.	A.	A.
33. Carbonate of magnesia powdered and compressed	0·45 × 0·04	"	E.	A.	A.
34. Sulphate of magnesia	0·32 × 0·2	"	E.	A.	A.
35. A second specimen	0·25 × 0·15	"	E.	A.	A.
36. Flour compressed	0·24 × 0·04	"	E.	A.	A.
37. Oxalate of cobalt	0·6 × 0·08	Paramagnetic.	A.	E.	E.

These experiments might be extended indefinitely, but sufficient are here to enable us to deduce the law of action. In the first place we notice, that all those substances which set equatorial between the points, and axial above and below them, are *diamagnetic*; while all those which set axial between the points, and equatorial above and below them, are *paramagnetic*. When any one of the substances here named is reduced to the spherical form, this rotation is not observed. I possess, for example, four spheres of calcareous spar, and when any one of them is suspended between the points, it takes up a position which is not changed when the sphere is raised or lowered; the crystallographic axis sets equatorial in all positions. A sphere of compressed carbonate of iron, suspended between the points, also sets that diameter along which the pressure is exerted from pole to pole, and continues to do so when raised or lowered. A sphere of compressed bismuth, on the other hand, sets its line of compression always equatorial. The position taken up by the spheres depends upon the *molecular structure* of the substances which compose them; but, when the mass is *elongated*, another action comes into play. Such a mass being suspended with its length horizontal, the *repulsion of its ends* constitutes a mechanical couple which increases in power with the length of the mass; and when the body is long enough, and the local repulsion of the ends strong enough, the couple, when it acts in opposition to the directive tendency due to structure, is able to overcome the latter and to determine the position of the mass. In all the cases cited, it was so arranged that the length of the body and its structure should act in opposition to each other. Tartaric acid and citric acid cleave with facility in one direction, and, in the specimens used, the planes of cleavage were perpendicular to the length of the body. In virtue of the structure these planes tended to set equatorial, but the repulsion of the elongated mass by the points prevented this, and caused the planes to set axial. When however the body was raised or lowered out of the sphere of this local repulsion, and into a position where the distribution of the force was more uniform, the advantage due to length became so far diminished that it was overcome, in turn, by the influence of structure, and the planes of cleavage turned into the equatorial position. In the specimen of saltpetre the shortest horizontal dimension was parallel to the axis of the crystal, which axis, when the influence of form is destroyed, always sets equatorial. A full crystal of calcareous spar will, when the magnetic distribution is tolerably uniform, always set its axis at right angles to the line joining the poles; but the axis is the shortest dimension of the crystal, and, between the points, this mechanical disadvantage compels the influence of structure to succumb to the influence of shape. A cube of calcareous spar, in my possession, may be caused to set the optic axis from pole to pole between the points, but this is evidently due to the elongation of the mass along the diagonals; for, when the corner of the cube succeeds in passing the point of the pole, the mass turns its axis with surprising energy into the equatorial position, round which it oscillates with great vivacity. Counting the oscillations, I found that eighty-two were performed by the cube, when its axis was equatorial, in the time required to perform fifty-nine, when the axis stood from pole to pole. Heavy spar and cœlestine are beautiful examples of directive action. These crystals, as is well known, can be cloven into prisms with rhombic bases: the principal cleavage is parallel to the base of the prism, while the two subordinate cleavages constitute the sides. If a short prism be suspended in a tolerably uniform field of force, so that its rhombic ends shall be horizontal, on exciting the magnet the short diagonal will set equatorial, as shown in fig. 2. If the prism be suspended with its axis and the short diagonal horizontal, the long diagonal being therefore vertical, the short diagonal will retain the equatorial position, while the axis of the prism sets axial as in fig. 3. If the prism be suspended with its long diagonal and axis horizontal, the short diagonal being vertical, and its directive power therefore annulled, the axis will take up the equatorial position, as in fig. 4. Now as the line which sets equatorial in diamagnetic bodies is that in which the magnetic force acts most strongly*, the crystal before us furnishes a perfect example of a substance possessing three rectangular magnetic axes, no two of which are equal. In the experiment cited in Table II., the mass was so cut that the short diagonal of the rhombic base was perpendicular to the length of the specimen. Carbonate of tin, and the other powders, were compressed by placing the powder between two clean plates of copper, and squeezing them together in a strong vice. The line of compression in diamagnetic bodies always sets equatorial, when the field of force is uniform, or approximately so; but between points the repulsion of the ends furnishes a couple strong enough to overcome this directive action, causing the longest dimension of the mass to set equatorial, and consequently its line of compression axial.

Fig. 2.

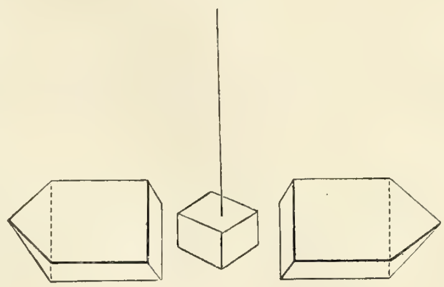


Fig. 3.

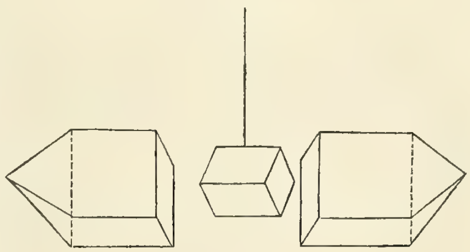
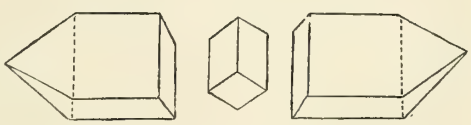


Fig. 4.



The antithesis between the deportment of diamagnetic bodies and of paramagnetic ones is perfect. Between the points the former class set equatorial, the latter axial. Raised or lowered, the former set axial, the latter equatorial. The simple substitution of an attractive for a repulsive force produces this effect. A sphere of ferrocyanide of potassium, for example, always sets the line perpendicular to the crystallographic axis from pole to pole; but when we take a full crystal, whose dimension along its axis, as in one of the cases before us, is six times the dimension at right angles to the axis, the attraction of the ends of such a mass is sufficient to overcome the directive

* Phil. Mag. Ser. 4. vol. ii. p. 177.

action due to structure, and to pull the crystal into the axial position between the points. In a field of uniform force, or between flat poles, the length sets equatorial, and it is the partial attainment of such a field, at a distance from the points, that causes the crystal to turn from axial to equatorial when it is raised or lowered. Beryl is a paramagnetic crystal, and when the influence of form is annulled, it always sets a line perpendicular to the axis of the crystal from pole to pole; a cube of this crystal, at present in my possession, shows this deportment whether the poles are pointed or flat; but in the specimen examined the dimension of the crystal along its axis was greatest, and hence the deportment described. It is needless to dwell upon each particular paramagnetic body: the same principle was observed in the preparation and choice of all of them; namely, that the line which, in virtue of the internal structure of the substance, would set axial, was transverse to the length of the body. The directive action due to structure was thus brought into opposition with the tendency of magnetic bodies to set their longest dimension from pole to pole: between the points the latter tendency was triumphant; at a distance, on the contrary, the influence of structure prevailed. The substance which possesses this directive action in the highest degree is carbonate of iron: when a lozenge, cloven from the crystalline mass, is suspended from the angle at which the crystallographic axis issues, there is great difficulty in causing the plate to set axial. If the points are near, on exciting the magnetism the whole mass springs to one or the other of the points; and when the points are distant, the plate, although its length may be twenty times its thickness, will set strongly equatorial. An excitation by one cell was sufficient to produce this result. In the experiment cited the residual magnetism was found to answer best, as it permitted the ends of the plate to be brought so near to the points that the mass was pulled into the axial position. When the magnet was more strongly excited, and the plate raised so far above the points as to prevent its springing to either of them, it was most interesting to watch the struggle of the two opposing tendencies. Neither the axial nor the equatorial position could be retained; the plate would wrench itself spasmodically from one position into the other, and, like the human spirit operated on by conflicting passions, find rest nowhere.

The conditions which determine the curious effects described in the present chapter may be briefly expressed as follows:—

An elongated diamagnetic body being suspended in the magnetic field, if the shortest horizontal dimension tend, in virtue of the internal structure of the substance, to set equatorial, it is opposed by the tendency of the longest dimension to take up the same position. Between the pointed poles the influence of length usually predominates; above the points and below them the directive action due to structure prevails.

Hence, the rotation of such a diamagnetic body, on being raised or lowered, is always from the equatorial to the axial position.

If the elongated mass be magnetic, and the shortest dimension of the mass tend, in virtue of its structure, to set from pole to pole, it is opposed by the tendency of the longest dimension to take up the same position. Between the points the influence of length is paramount, above and below the points the influence of structure prevails.

Hence, the rotation of magnetic bodies, on being raised or lowered, is always from the axial to the equatorial position.

The error of the explanation which referred many of the above actions to the presence of two conflicting forces, one of which diminished with the distance in a quicker ratio than the other, lies in the supposition, that the assuming of the axial position proved a body to be magnetic, while the assuming of the equatorial position proved a body to be diamagnetic. This assumption was perfectly natural in the early stages of diamagnetic research, when the modification of magnetic force by structure was unknown. Experience however proves that the total mass of a magnetic body continues to be attracted after it has assumed the equatorial position, while the total mass of a diamagnetic body continues to be repelled after it has taken up the axial one.

III. ON THE DISTRIBUTION OF THE MAGNETIC FORCE BETWEEN TWO FLAT POLES.

In experiments where a uniform distribution of the magnetic force is desirable, flat poles, or magnetized surfaces, have been recommended. It has long been known that the force proceeds with great energy from the edges of such poles: the increase of force from the centre to the edge has been made the subject of a special investigation by M. VON KOLKE*. The central portion of the magnetic field, or space between two such magnetized surfaces, has hitherto been regarded as almost perfectly uniform, and indeed for all ordinary experiments the uniformity is sufficient. But, when we examine the field carefully, we find that the uniformity is not perfect. Substituting, for the sake of convenience, the edge of a pole for a point, I studied the phenomena of rotation described in the last section, in a great number of instances, by comparing the deportment of an elongated body, suspended in the centre of the space between two flat poles, with its deportment when suspended between the top or the bottom edges. Having found that the fibre of wood, in masses where form had no influence, always set equatorial, I proposed to set this tendency to contend with an elongation of the mass in a direction at right angles to the fibre. For this purpose thirty-one little wooden bars were carefully prepared and examined, the length of each bar being about twice its width, and the fibre coinciding with the latter dimension. The bars were suspended from an extremely fine fibre of cocoon silk, and in the centre of the magnetic field each one of them set its length axial and consequently its fibre equatorial. Between the top and bottom edges, on the contrary, each piece set its longest dimension equatorial, and, consequently, the fibre axial.

For some time I referred the axial setting of the mass, in the centre of the field, to

* POGGENDORFF's Annalen, vol. lxxxi. p. 321.

the directive action of the fibre, though, knowing the extreme feebleness of this directive action, I was surprised to find it able to accomplish what the experiments exhibited. The thought suggested itself, however, of suspending the bars with the fibre vertical, in which position the latter could have no directive influence. Here also, to my surprise, the directive action, though slightly weakened, was the same as before; in the centre of the field the bars took up the axial position. Bars of sulphur, wax, salt of hartshorn, and other diamagnetic substances were next examined: they all acted in the same manner as the wood, and thus showed that the cause of the rotation lay, not in the structure of the substances, but in the distribution of the magnetic force around them. This distribution in fact was such, that the straight line which connected the centre of one pole with that of the opposite one was the line of weakest force. Ohm represents the distribution of electricity upon the surfaces of conductors by regarding the tensions as ordinates, and erecting them from the points to which they correspond, the steepness of the curve formed by uniting the ends of the ordinates being the measure of the increase or diminution of tension. Taking the centre of the magnetic field as the origin, and drawing lines axial and equatorial; if we erect the magnetic tensions along these lines, we shall find a steeper curve in the equatorial than in the axial direction. This may be proved by suspending a bit of carbonate of iron in the centre of the magnetic field; on exciting the magnet, the suspended body will not move to the nearest portion of the flat pole, though it may be not more than a quarter of an inch distant, but will move equatorially towards the edges, though they may be two inches distant. The little diamagnetic bars referred to were therefore pushed into the axial position by the force acting with superior power in an equatorial direction.

The results just described are simply due to the recession of the ends of an elongated body from places of stronger to those of weaker force; but it is extremely instructive to observe how this result is modified by structure. If, for example, a plate of bismuth be suspended between the poles with the plane of principal cleavage vertical, the plate will assert the equatorial position from top to bottom; and in the centre with almost the same force as between the edges. The cause of this lies in the structure of the bismuth. Its position in the field depends not so much upon the distribution of the magnetic force around it, as upon the direction of the force *through* it. I will not, however, anticipate matters by entering further upon this subject at present.

IV. COMPARATIVE VIEW OF PARAMAGNETIC AND DIAMAGNETIC PHENOMENA.

1. *State of Diamagnetic Bodies under magnetic influence.*

When a piece of soft iron is brought near to a magnet, it is attracted by the latter: this attraction is not the act of the magnet alone, but results from the mutual action of the magnet and the body upon which it operates. The soft iron in this case is said to be magnetized by influence; it becomes itself a magnet, and the intensity of its magnetization varies with the strength of the influencing magnet. Poisson figured the act of magnetization as consisting of the decomposition of a neutral magnetic fluid into north and south magnetism, the amount of the decomposition being proportional to the strength of the magnet which produces it. Ampère, discarding the notion of magnetic fluids, figured the molecules of soft iron as surrounded by currents of electricity, and conceived the act of magnetization to consist in setting the planes of these molecular currents parallel to each other: the degree of parallelism, or in other words, the intensity of the magnetization, depending, as in Poisson's hypothesis, upon the strength of the influencing magnet.

The state into which the soft iron is here supposed to be thrown is a state of constraint, and when the magnet is removed, the substance returns to its normal condition. Poisson's separated fluids rush together once more, and Ampère's molecular currents return to their former irregular positions. As our knowledge increases, we shall probably find both hypotheses inadequate to represent the phenomena; the only thing certain is, that the soft iron, when acted upon by the magnet, is thrown into an unusual condition, in virtue of which it is attracted; and that the intensity of this condition is a function of the force which produces it.

There are, however, certain bodies which, unlike soft iron, offer a great resistance to the imposition of the magnetic state, but when once they are magnetized they do not, on the removal of the magnet, return to their neutral condition, but on the contrary retain the magnetism impressed on them. It is in virtue of this quality that steel can be formed into compass needles and permanent magnets. This power of resistance and retention is named by Poisson coercive force.

Let us conceive a body already magnetized, and in which coercive force exists in a very high degree—a piece of very hard steel for example—to be brought near a magnet, the strength of which is not sufficient to magnetize the steel further. To simplify the matter let us fix our attention upon the south pole of the magnet, and conceive it to act upon the north pole of the piece of steel. Let the magnetism of the said south pole, referred to any unit, be M, and of the north pole of the steel, M′; then their mutual attraction, at the unit of distance, is expressed by the product MM′. Conceive now the magnet to increase in power from M to nM, the steel being still supposed hard enough to resist magnetization by influence; the mutual attraction now will be

$$n\mathrm{MM}',$$

or n times the former attraction; hence when a variable magnetic pole acts on an opposite one of constant power, the attraction is proportional to the strength of the former.

Let us now take a body whose magnetization varies with that of the magnet: a south pole of the strength M induces in such a body a north pole of the strength M′, and the attraction which results from their mutual action is

$$\mathrm{MM}'.$$

Let the strength of the influencing south pole increase from M to nM; then, assuming the magnetism of the body under influence to increase in the same ratio, the strength of the above-mentioned north pole will become nM′, and the attraction, expressed by the product of both, will be

$$n^2 MM';$$

that is to say, the attraction of a body magnetized by influence, and whose magnetism varies as the strength of the influencing magnet, is proportional to the *square of the strength* of the latter.

Here then is a mark of distinction between those bodies which have their power of exhibiting magnetic phenomena conferred upon them by the magnet, and those whose actions are dependent upon some constant property of the mass: in the latter case the resultant action will be simply proportional to the strength of the magnet, while in the former case a different law of action will be observed*.

The examination of this point lies at the very foundation of our inquiries into the nature of the diamagnetic force. Is the repulsion of diamagnetic bodies dependent merely on the mass considered as ordinary matter, or is it due to some condition impressed upon the mass by the influencing magnet? This question admits of the most complete answer either by comparing the increase of repulsion with the increase of power in the magnet which produces the repulsion, or by comparing the attraction of a paramagnetic body, which we know to be thrown into an unusual condition, with the repulsion of a diamagnetic body, whose condition we would ascertain.

Bars of iron and bismuth, of the same dimensions, were submitted to the action of an electromagnet, which was caused gradually to increase in power; commencing with an excitation by one cell, and proceeding up to an excitation by ten or fifteen. The strength of the current was in each case accurately measured by a tangent galvanometer. The bismuth bar was suspended between the two flat poles, and, when the magnet was excited, took up the equatorial position. The iron bar, if placed directly between the poles, would, on the excitation of the magnetism, infallibly spring to one of them; hence it was removed to a distance of 2 feet 7 inches from the centre of the space between the poles, and in a direction at right angles to the line which united them. The magnet being excited, the bar was drawn a little aside from its position of equilibrium and then liberated, a series of oscillations of very small amplitude followed, and the number of oscillations accomplished in a minute was carefully ascertained. Tables III. and IV. contain the results of experiments made in the manner described with bars of iron and bismuth of the same dimensions.

* This test was first pointed out in a paper on the Polarity of Bismuth, Phil. Mag. Nov. 1851, p. 333. I have reason, however, to know that the same thought occurred to M. Poggendorff previous to the publication of my paper.—J. T.

TABLE III.

Bar of soft iron, No. 1.
length 0·8 of an inch.
width 0·13 of an inch.
depth 0·15 of an inch.

Strength of current.	Attraction.
168	168^2
214	204^2
248	253^2
274	275^2
323	313^2
362	347^2
385	374^2
411	385^2

TABLE IV.

Bar of bismuth, No. 1.
length 0·8 of an inch.
width 0·13 of an inch.
depth 0·15 of an inch.

Strength of current.	Repulsion.
78	78^2
136	135^2
184	191^2
226	226^2
259	259^2
287	291^2
341	322^2
377	359^2
411	386^2

These experiments prove, that, up to a strength of about 280, the attractive force operating upon the iron, and the repulsive force acting upon the bismuth, are each accurately proportional to the square of the strength of the magnetising current. For higher powers, both attraction and repulsion increase in a smaller ratio; but it is here sufficient to show that the diamagnetic repulsion follows precisely the same law as the magnetic attraction. So accurately indeed is this parallelism observed, that while the forces at the top of the tables produce attractions and repulsions exactly equal to the square of the strength of the current, the same strength of 411, at the bottom of both tables, produces in iron an attraction of 385^2, and in bismuth a repulsion of 386^2. The numbers which indicate the strength of current in the first column are the tangents of the deflections observed in each case: neglecting the indices, the figures in the second column express the number of oscillations accomplished in a minute, multiplied by a constant factor to facilitate comparison: the forces operating upon the bars being proportional to the squares of the number of oscillations, the simple addition of the index figure completes the expression of these forces.

In these experiments the bismuth bar set *across* the lines of magnetic force, while the bar of iron set *along* them; the former was so cut from the crystalline mass, that the plane of principal cleavage was parallel to the length of the bar, and in the experiments hung vertical. I thought it interesting to examine the deportment of a bar of bismuth which should occupy the same position, with regard to the lines of force, as the bar of iron; that is to say, which should set its length axial. Such a bar is obtained when the planes of principal cleavage are transverse to the length.

TABLE V.

Bar of bismuth, No. 2.
length 0·8 of an inch.
width 0·13 of an inch.
depth 0·15 of an inch.

Set axial between the excited poles.

Strength of current.	Repulsion.
68	67^2
182	187^2
218	218^2
248	249^2
274	273^2
315	309^2
364	350^2
401	366^2

A deportment exactly similar to that exhibited in the foregoing cases is observed here also: up to about 280 the repulsions are accurately proportional to the squares of the current strengths, and from this point forward they increase in a less ratio.

A paramagnetic substance was next examined which set its length at right angles to the lines of magnetic force: the substance was carbonate of iron. The native crystallized mineral was reduced to powder in a mortar, and the powder was compressed. It was suspended, like the bismuth, between the flat poles, with its line of compression horizontal. When these poles were excited the compressed bar set the line of pressure from pole to pole, and consequently its length equatorial.

Table VI.

Bar of compressed carbonate of iron.
length 0·95 of an inch.
width 0·17 of an inch.
depth 0·23 of an inch.
Set equatorial between the excited poles.

Strength of current.	Attraction.
74	74^2
135	133^2
179	180^2
214	218^2
249	248^2
277	280^2
341	330^2
381	353^2

It is needless to remark upon the perfect similarity of deportment here exhibited to the cases previously recorded.

In the following instances the same law of increase is observable.

Table VII.

Sulphate of iron, No. 1.
length 0·75 of an inch.
width 0·22 of an inch.
depth 0·27 of an inch.
Set axial between the excited poles.

Strength of current.	Attraction.
71	70^2
132	133^2
217	220^2
280	275^2
328	333^2
359	348^2

Table VIII.

Sulphate of iron, No. 2.
length 0·75 of an inch.
width 0·22 of an inch.
depth 0·27 of an inch.
Set equatorial between the excited poles.

Strength of current.	Attraction.
70	68^2
121	123^2
203	207^2
271	268^2
331	308^2
370	334^2

In sulphate of iron there is one direction which, in virtue of the molecular structure of the substance, sets strongly from pole to pole. The bar No. 1. was so cut that this direction was parallel to its length, which therefore set axial; while No. 2. had the same direction *across* it, thus causing the length of the bar to set equatorial.

Two comparative series were finally made with two prisms of iron and of bismuth, more massive than those previously examined.

Table IX.

Bar of iron, No. 2.
length 1·0 inch.
width 0·3 inch.
depth 0·3 inch.

Strength of current.	Attraction.
70	71^2
122	122^2
167	168^2
206	204^2
268	260^2
322	311^2
356	339^2

Table X.

Bar of bismuth, No. 3.
length 1·0 inch.
width 0·3 inch.
depth 0·3 inch.

Strength of current.	Repulsion.
70	72^2
126	121^2
164	166^2
206	205^2
246	248^2
276	279^2
364	344^2

These experiments can leave little doubt upon the mind, that if a magnetic body be attracted in virtue of its being converted into a magnet, a diamagnetic body is repelled *in virtue of its being converted into a diamagnet.* On no other assumption can it be explained, why the repulsion of the diamagnetic body, like the attraction of the magnetic one, increases in a so much quicker ratio than the force of the magnet which produces the repulsion. But, as this is a point of great importance, I will here introduce corroborative evidence, derived from modes of experiment totally different from the method already described. By a series of measurements with the torsion balance, in which the attractive and repulsive forces were determined directly, with the utmost care, the relation of the strength of the magnet to the force acting upon the substances named in Tables XI., XII. and XIII. was found to be as follows:—

Table XI.

Spheres of native sulphur.

Strength of magnet.	Ratio of repulsions.
96	95^2
153	158^2
222	224^2
265	264^2
316	316^2

Table XII.

Spheres of carbonate of lime.

Strength of magnet.	Ratio of repulsions.
134	134^2
172	173^2
213	212^2
259	264^2
310	311^2
370	374^2

Table XIII.

Spheres of carbonate of iron.

Strength of magnet.	Ratio of attractions.
66	66^2
89	89^2
114	114^2
141	141^2

In confirmation of these results I will cite a series obtained by M. E. Becquerel*,

* Ann. de Chim. et de Phys. 3rd Series. vol. xxviii. p. 302.

whose experiments first showed that the repulsion of diamagnetic bodies follows the same law as the attraction of magnetic ones.

Bar of sulphur.
length 25 millims.
weight 840 milligrms.

Squares of the magnetic intensities.	Quotients of the repulsions by the magnetic intensities.
36·58	0·902
27·60	0·929
26·84	0·906
16·33	0·920

The constancy of the quotient in the second column proves that the ratio of the repulsions to the squares of the magnetic intensities is a ratio of equality.

I will also cite a series of experiments by Mr. Joule *, which he adduces in confirmation of the results obtained by M. E. Becquerel and myself.

Bar of bismuth.

Strength of magnet.	Repulsions.
1	1^2
2	2^2
4	4^2

Let us contrast these with the results obtained by the same gentleman, by permitting the magnet to act upon a hard magnetic needle.

Magnetic needle.
length 1·5 inch.

Strength of magnet.	Attraction.
1	1
2	2
4	4

Here we find experiment in strict accordance with the theoretical deduction stated at the commencement of the present chapter. The intensity of the magnetism of the steel needle is constant, for the steel resists magnetization by influence; the consequence is that the attraction is simply proportional to the strength of the magnet.

A consideration of the evidence thus adduced from independent sources, and obtained by different methods, must, I imagine, render the conclusion certain that diamagnetic bodies, like magnetic ones, exhibit their phenomena in virtue of a state

* Phil. Mag. 4th Series. vol. iii. p. 32.

of magnetization induced in them by the influencing magnet. This conclusion is in no way invalidated by the recent researches of M. Plücker, on the law of induction in paramagnetic and diamagnetic bodies, but on the contrary derives support from his experiments. With current strengths which stand in the ratio of 1 : 2, M. Plücker finds the repulsion of bismuth to be as 1 : 3·62, which, though it falls short of the ratio of 1 : 4, as the law of increase according to the square of the current would have it, is sufficient to show that the bismuth was not passive, but acted the part of an induced diamagnet in the experiments. In the case of the soft iron itself M. Plücker finds a far greater divergence; for here currents which stand in the ratio of 1 : 2 produce attractions only in the ratio of 1 : 2·76.

2. *Duality of Diamagnetic Excitement.*

Having thus safely established the fact, that diamagnetic bodies are repelled, in virtue of a certain state into which they are cast by the influencing magnet, the next step of our inquiry is;—Will the state evoked by one magnetic pole facilitate, or prevent, the repulsion of the diamagnetic body by a second pole of an opposite quality? If the force of repulsion were an action on the mass, considered as ordinary matter, this mass, being repelled by both the north and the south pole of a magnet, when they operate upon it separately, ought to be repelled by the sum of the forces of the two poles where they act upon it together. But if the excitation of diamagnetic bodies be of a *dual* nature, as is the case with magnetic bodies, then it may be expected, that the state excited by one pole will not facilitate, but on the contrary prevent, the repulsion of the mass by a second opposite pole.

To solve this question the apparatus sketched in fig. 5*a*. Plate II. was made use of. AB and CD are two helices of copper wire 12 inches long, of 2 inches internal, and of 5½ inches external diameter. Into them fit soft iron cores 2 inches thick: the cores are bent as in the figure, and reduced to flat surfaces along the line *ef*, so that when the two semicylindrical ends are placed together, they constitute a cylinder of the same diameter as the cores within the helices*. In front of these poles a bar of pure bismuth *gh* was suspended by cocoon silk; by imparting a little torsion to the fibre, the end of the bar was caused to press gently against a plate of glass *ik*, which stood between it and the magnets. By means of a current reverser the polarity of one of the cores could be changed at pleasure; thus it was in the experimenter's power to excite the cores, so that the poles PP′ should be of the same quality, or of opposite qualities.

The bar, being held in contact with the glass by a very feeble torsion, a current was sent round the cores, so that they presented two poles of the same name to the suspended bismuth; the latter was promptly repelled, and receded to the position dotted in the figure. On interrupting the current it returned to the glass as before.

* The ends of the semicylinders were turned so as to present the blunted apex of a cone to the mass of bismuth.

The cores were next excited, so that two poles of opposite qualities acted upon the bismuth; the latter remained perfectly unmoved*.

This experiment shows that the state, whatever it may be, into which bismuth is cast by one pole, so far from being favourable to the action of the opposite pole, completely neutralizes the effect of the latter. A perfect analogy is thus established between the deportment of the bismuth and that of soft iron under the same circumstances; for it is well known that a similar neutralization occurs in the latter case. If the repulsion depended upon the *abstract strength* of the poles, without reference to their *quality*, the repulsion, when the poles are of opposite names, ought to be *greater* than when they are alike; for in the former case the poles are greatly strengthened by their mutual inductive action, while, in the latter case, they are enfeebled by the same cause. But the fact of the repulsion being dependent on the quality of the pole, demonstrates that the substance is capable of assuming a condition peculiar to each pole, or in other words, is capable of a *dual* excitation†. The experiments from which these conclusions are drawn are a manifest corroboration of those made by M. Reich with steel magnets.

If we suppose the flat surfaces of the two semicylinders which constitute the ends of the cores to be in contact, and the cores so excited that the poles P and P′ are of different qualities, the arrangement, it is evident, forms a true electro-magnet of the horseshoe form; and here the pertinency of a remark made by M. Poggendorff, with his usual clearness of perception, becomes manifest; namely, that if the repulsion of diamagnetic bodies be an indifferent one of the mass merely, there is no reason why they should not be repelled by the centre of a magnet, as well as by its ends.

3. *Separate and joint action of a Magnet and a Voltaic Current on Paramagnetic and Diamagnetic Bodies.*

In operating upon bars of bismuth with the magnet, or the current, or both combined, it was soon found that the gravest mistakes might be committed, if the question of structure was not attended to; that it is not more indefinite to speak of the volume of a gas without giving its temperature, than to speak of the deportment of bismuth without stating the relation of the form of the mass to the planes of crystallization. Cut in one direction, a bar of bismuth will set its length parallel to an electric current passing near it; cut in another direction, it will set its length perpendicular

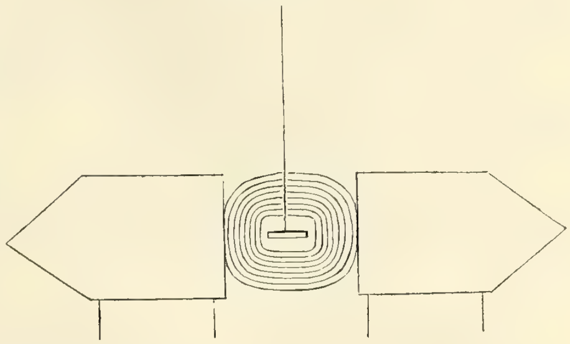
* A shorter bar of bismuth than that here sketched, with a light index attached to it, makes the repulsion more evident. It may be thus rendered visible throughout a large lecture-room.

† Since the above was written, the opinion has been expressed to me, that the action of the *unlike* poles, in the experiment before us, is "diverted" from the bismuth upon each other, the absence of repulsion being due to this diversion, and not to the neutralization of inductions in the mass of the bismuth itself. Many, however, will be influenced by the argument as stated in the text, who would not accept the interpretation referred to in this note; I therefore let the argument stand, and hope at no distant day to return to the subject.—J. T., 5th May, 1855.

to the same current. It was necessary to study the deportment of both of these bars separately.

A helix was formed of covered copper wire one-twentieth of an inch thick: the space within the helix was rectangular, and was 1 inch long, 0·7 inch high, and 1 inch wide: the external diameter of the helix was 3 inches. Within the rectangular space the body to be examined was suspended by a fibre which descended through a slit in the helix. The latter was placed between the two flat poles of an electro-magnet, and could thus be caused to act upon the bar within it, either alone or in combination with the magnet. The disposition will be at once understood from fig. 6, which gives a front view of the arrangement.

Fig. 6.



Action of Magnet alone: division of Bars into Normal and Abnormal.—A bar of soft iron suspended in the magnetic field will set its longest dimension from pole to pole: this is the normal deportment of paramagnetic bodies. A bar of bismuth, whose planes of principal cleavage are throughout parallel to its length, suspended in the magnetic field with the said planes vertical, will set its longest dimension at right angles to the line joining the poles: this is the normal deportment of diamagnetic bodies. We will therefore, for the sake of distinction, call the former a *normal paramagnetic bar*, and the latter a *normal diamagnetic bar*.

A bar of compressed carbonate of iron dust, whose shortest dimension coincides with the line of pressure, will, when suspended in the magnetic field with the said line horizontal, set its length equatorial. A bar of compressed bismuth dust, similarly suspended, or a bar of bismuth whose principal planes of crystallization are transverse to its length, will set its length axial in the magnetic field. We will call the former of these an *abnormal paramagnetic bar*, and the latter an *abnormal diamagnetic bar*.

Action of Current alone on normal and abnormal bars.—A normal paramagnetic bar was suspended in the helix above described; when a current was sent through the latter, the bar set its longest horizontal dimension parallel to the axis of the helix, and consequently perpendicular to the coils.

An *abnormal paramagnetic bar* was suspended in the same manner; when a current was sent through the helix, the bar set its longest dimension perpendicular to the axis of the helix, and consequently parallel to the coils.

A *normal diamagnetic bar* was delicately suspended in the same helix; on the passage of the current it acted precisely as the abnormal magnetic bar; setting its longest dimension perpendicular to the axis of the helix and parallel to the coils.

When a fine fibre and sufficient power are made use of, this deportment is obtained without difficulty.

An *abnormal diamagnetic bar* was suspended as above; on the passage of the current it acted precisely as the normal magnetic bar; it set its length parallel to the axis of the helix and perpendicular to the coils. Here also, by fine manipulation, the result is obtained with ease and certainty.

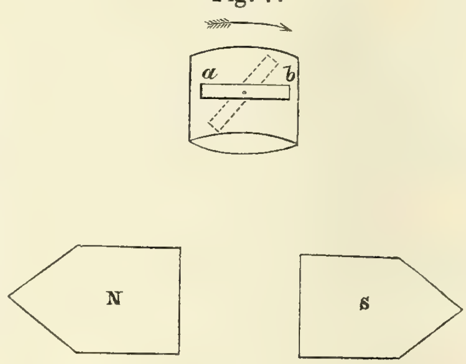
Action of Magnet and Current combined.—In examining this subject eight experiments were made with each particular bar; it will be remembered that fig. 6 gives a general view of the arrangement.

1. Four experiments were made in which the *magnet* was excited first, and after the suspended bar had taken up its position of equilibrium, the deflection produced by the passage of a current through the surrounding helix was observed.

2. Four experiments were made in which the *helix* was excited first, and when the bar within it had taken up its position of equilibrium, the magnetism was developed and the consequent deflection observed.

Normal Paramagnetic Bar.—In experimenting with the soft iron it was necessary to place it at some distance from the magnet; otherwise the attraction of the entire mass by one or the other pole would completely mask the action sought. Fig. 7 represents the disposition of things in these experiments: N and S indicate the north and south poles of the magnet; *ab* is the bar of iron; the helix within which the bar was suspended is shown in outline around it; the arrow shows the direction of the current in the *upper half* of the helix; its direction in the under portion would, of course, be the reverse.

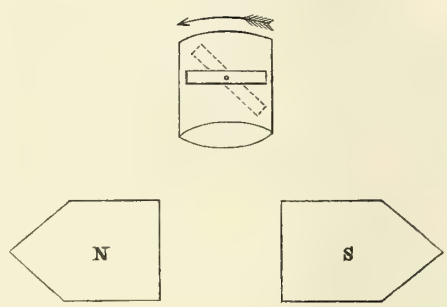
Fig. 7.



On exciting the magnet, the bar of soft iron set itself parallel to the line joining the poles, as shown by the unbroken line in fig. 7.

When the direction of the current in the helix was that indicated by the arrow, the bar was deflected towards the position dotted in the figure.

Fig. 8.



Interrupting the current in the helix, and permitting the magnet to remain excited, the bar returned to its former position: the current was now sent through the helix in the direction of the arrow, fig. 8; the consequent deflection was towards the dotted position.

Both the current which excited the magnet and that which passed through the helix were now interrupted, and the polarity of the magnet was reversed. On sending a current through the helix in the direction of the arrow, the deflection of the bar was from the position of the defined line to that of the dotted one, fig. 9.

Fig. 9.

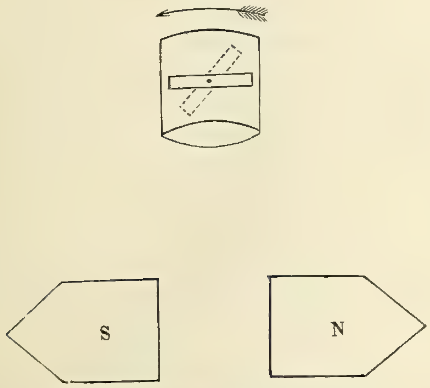
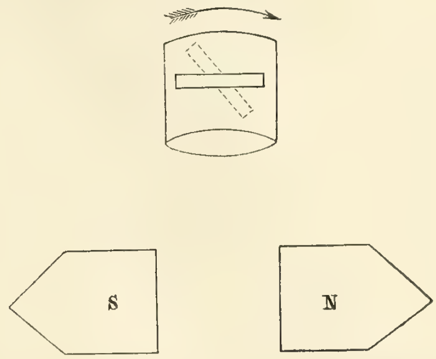


Fig. 10.



Interrupting the current through the helix, and permitting the bar to come to rest under the influence of the magnet alone, a current was sent through the helix in a direction opposed to its former one: the deflection produced was that shown in fig. 10.

The position of equilibrium finally assumed by the bar depends, of course, upon the ratio of the forces acting upon it: in these experiments, the bar, in its final position, enclosed an angle of about 50 degrees with the axial line.

A series of experiments were next made, in which the bar was first acted on by the current passing through the helix, the magnet being brought to bear upon it afterwards. On the passage of the current through the helix, in the direction shown in fig. 11, the bar set its length parallel to the axis of the latter. On exciting the magnet so that its polarity was that indicated by the letters N and S in the figure, the deflection was towards the dotted position.

Fig. 11.

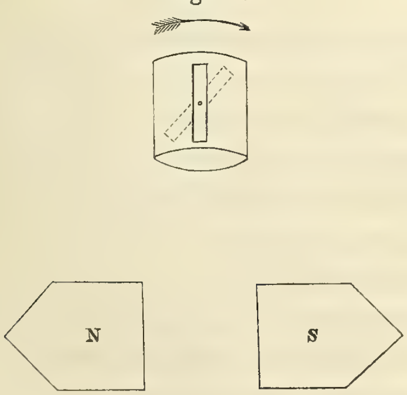
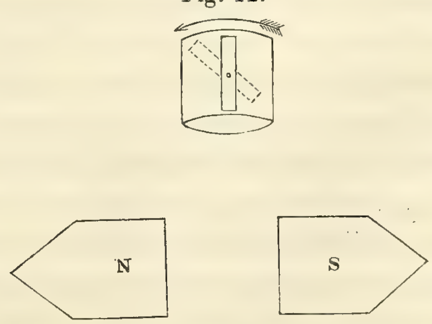


Fig. 12.



Interrupting the current through both magnet and helix, and reversing the current through the latter, the bar came to rest, as before, parallel to the axis: on exciting the magnet, as in the last case, the deflection was that shown in fig. 12.

Preserving the same current in the helix, and reversing the polarity of the magnet, the deflection was that shown in fig. 13.

Preserving the magnet poles as in the last experiment, and reversing the current in the helix, the deflection was that shown in fig. 14.

Fig. 13.

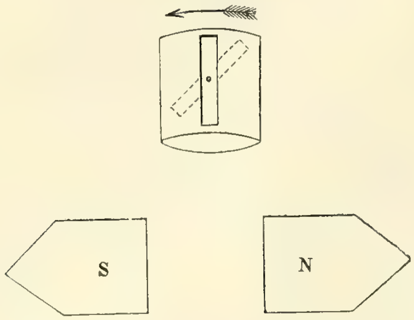
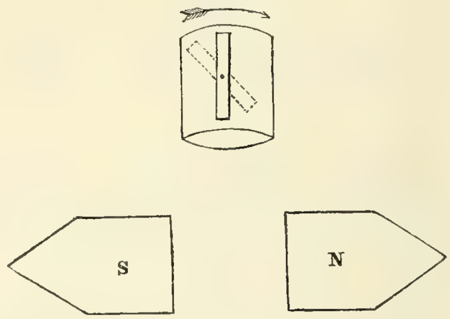


Fig. 14.



In these cases, the bar, in its final position of equilibrium, enclosed an angle of about 40 degrees with the axial line.

Normal Diamagnetic Bar.—The above experiments exhibit to us the deportment of the normal paramagnetic body under a great variety of conditions, and our next step is to compare with it the deportment of the normal diamagnetic body under the same circumstances.

For the sake of increasing the force, the helix was removed from its lateral position and placed between the two poles, as in fig. 6, p. 25. The normal diamagnetic bar was suspended within the helix and submitted to the self-same mode of examination as that applied in the case of the paramagnetic body.

The polarity first excited was that shown in fig. 9, Plate I., and the position of rest, when the magnet alone acted, was at right angles to the line joining the poles; on sending a current through the helix in the direction of the arrow, the deflection was towards the dotted line.

Preserving the magnetic polarity as in the last experiment, the direction of the current through the helix was reversed, and the deflection was that shown in fig. 10.

Reversing the polarity of the magnet, and sending the current through the helix in the direction of the last experiment, the deflection was that shown in fig. 11.

Preserving the last magnetic poles, and sending the current through the helix in the opposite direction, the deflection was that shown in fig. 12.

In the following four experiments the helix was excited first.

Operated upon by the helix alone, the suspended bar set its length parallel to the convolutions, and perpendicular to the axis of the coil: the direction of the current was first that shown in fig. 13: when the magnet was excited, the bar was deflected towards the dotted position.

Interrupting both currents, and reversing the current in the helix; when the magnet was excited, as in the last experiment, the deflection was that shown in fig. 14.

Preserving the helix current as in the last experiment; when the polarity of the magnet was reversed, the deflection was that shown in fig. 15.

Interrupting both, and reversing the current in the helix; when the magnet was excited as in the last experiment, the deflection was that shown in fig. 16.

In a paper on the Polarity of Bismuth* published in the Philosophical Magazine, Ser. 4. vol. ii., and in Poggendorff's Annalen, vol. lxxxvii., an experiment is recorded in which the deportment exhibited by fig. 11 of the present series was obtained. In a recent memoir on the same subject, M. v. Feilitsch† states that he has sought this result in vain. Sometimes he observed the deflection at the moment of closing the circuit, but conceived that it must be ascribed to the action of induced currents; for immediately afterwards a deflection in the opposite direction was observed, which deflection proved to be the permanent one.

I have repeated the experiment here referred to with all possible care; and the result is that described in the remarks which refer to fig. 11. This result agrees in all respects with that described in my former paper. To enable myself, however, to appeal to quantitative measurement, a small graduated circle was constructed and placed underneath the bar of bismuth suspended within the helix. The effect, as will be seen, is not one regarding which a mistake could be made on account of its minuteness: operating delicately, and choosing a suitable relation between the strength of the magnet and that of the spiral‡, on sending a current through the latter as in fig. 11, the bar was deflected so forcibly that the limit of its first impulsion reached 120° on the graduated circle underneath. The permanent deflection of the bar amounted to 60° in the same direction, and hence the deportment could in no wise be ascribed to the action of induced currents, which vanish immediately. Before sending the current through the helix, the bar was acted on by the magnet alone, and pointed to zero.

Though it was not likely that the shape of the poles could have any influence here, I repeated the experiment, using the hemispherical ends of two soft iron cores as poles: the result was the same.

A pair of poles with the right and left hand-edges rounded off showed the same deportment.

A pair of poles presenting chisel edges to the helix showed the same deportment.

Various other poles were made use of, some of which appeared to correspond exactly with those figured by M. v. Feilitsch; but no deviation from the described deportment was observed. To test the polarity of the magnet, a magnetic needle was always at hand: once or twice the polarity of the needle became reversed, which, had it not been noticed in time, would have introduced confusion into the experiments.

* From the notices of this paper which have appeared in the continental journals, I am obliged to infer that it is in some respects obscurely written. The conclusion I intended to express is that bismuth possesses a polarity opposed to that of iron.—J. T.

† Poggendorff's Annalen, vol. xcii. p. 395.

‡ In most of these experiments the spiral was excited by ten cells, the magnet by two.

Here is a source of error, against which, however, M. v. FEILITSCH has probably guarded himself. Some irregularity of crystalline structure may, perhaps, have influenced the result. With "chemically pure zinc" M. v. FEILITSCH obtained the same deflection that I obtained with bismuth: now chemically pure zinc is *diamagnetic**, and hence its deportment is corroborative of that which I have observed. M. v. FEILITSCH, however, appears to regard the zinc used by him as magnetic; but if this be the case it cannot have been chemically pure. It is necessary to remark that I have called the north pole of the electro-magnet that which attracts the south, or unmarked end, of a magnetic needle; and I believe this is the custom throughout Germany.

Abnormal Paramagnetic Bar.—This bar consisted of compressed carbonate of iron dust, and was suspended within the helix with the line of compression, which was its shortest dimension, horizontal. As in the cases already described, it was first acted upon by the magnet alone; having attained its position of equilibrium, a current was sent through the helix, and the subsequent deflection was observed.

The magnet being excited as in fig. 17, Plate I., the bar set its length equatorial; on sending a current through the helix in the direction of the arrow, the bar was deflected to the dotted position.

Reversing the current in the helix, but permitting the magnet to remain as before, the deflection was that shown in fig. 18.

Interrupting all, and reversing the polarity of the magnet; on sending the current through as in the last case, the deflection was that shown in fig. 19.

Reversing the current, but preserving the last condition of the magnet, the deflection was that shown in fig. 20.

In the subsequent four experiments the helix was excited first. Whatever might be the direction of the current through the helix, the bar always set its length perpendicular to the axis of the latter, and parallel to the coils.

When the direction of the helix current, and the polarity of the magnet, were those shown in fig. 21, the deflection was to the dotted position.

Interrupting all, and reversing the current in the helix; on exciting the magnet the deflection was that shown in fig. 22.

Changing the polarity of the magnet, and preserving the helix current in its former direction, the deflection was that shown in fig. 23.

Interrupting all, and reversing the current through the helix; when the magnetism was developed the deflection was that shown in fig. 24.

Abnormal Diamagnetic Bar.—This bar consisted of a prism of bismuth whose principal planes of crystallization were perpendicular to its length: the mode of experiment was the same as that applied in the other cases.

Acted upon by the magnet alone, the bar set its length from pole to pole: the magnetic excitation being that denoted by fig. 29, a current was sent through the helix in the direction of the arrow; the bar was deflected to the dotted position.

* Phil. Mag. vol. xxviii. p. 456.

Reversing the current through the helix, the deflection was that shown in fig. 30.

Interrupting both currents and reversing the magnetic poles; on sending a current through the helix as in the last experiment, the deflection was that shown in fig. 31.

Reversing the current through the helix, the deflection was that shown in fig. 32.

In the subsequent four experiments the helix was excited first.

Sending a current through the helix in the direction denoted by the arrow, the bar set its length at right angles to the convolutions, and parallel to the axis of the helix; when the magnetism was excited as in fig. 25, the deflection was to the dotted position.

When the current was sent through the helix in an opposite direction, the deflection was that shown in fig. 26.

Interrupting both currents, and reversing the poles of the magnet; on sending a current through the helix as in the last experiment, the deflection was that shown in fig. 27.

Reversing the current in the helix, the deflection was that shown in fig. 28.

In all these cases the position of equilibrium due to the first force was attained, before the second force was permitted to act.

It will be observed, on comparing the deportment of the normal paramagnetic bar with that of the normal diamagnetic one, that the position of equilibrium taken up by the latter, when operated on by the helix alone, is the same as that taken up by the former when acted on by the magnet alone: in both cases the position is from pole to pole of the magnet. A similar remark applies to the abnormal para- and diamagnetic bars. It will render the distinction between the deportment of both classes of bodies more evident, if the position of the two bars, before the application of the second force, be one and the same. When both the bars, acted on by one of the forces, are axial, or both equatorial, the contrast or coincidence, as the case may be, of the deflections from this common position by the second force will be more strikingly evident.

To effect the comparison in the manner here indicated, the figures have been collected together and arranged upon Plate I. The first column represents the deportment of the normal paramagnetic bar under all the conditions described; the second column, that of the normal diamagnetic bar; the third shows the deportment of the abnormal paramagnetic bar, and the fourth that of the abnormal diamagnetic bar.

A comparison of the first two columns shows us that the deportment of the normal magnetic bar is perfectly antithetical to that of the normal diamagnetic one. When, on the application of the second force, an end of the former is deflected to the right, the same end of the latter is deflected to the left. When the position of equilibrium of the magnetic bar, under the joint action of the two forces, is from N.E. to S.W., then the position of equilibrium for the diamagnetic bar is invariably from N.W. to S.E. There is no exception to this antithesis, and I have been thus careful to vary the conditions of experiment in all possible ways, on account of the divergent results obtained by other inquirers. In his recent memoirs upon this subject, M. v. FEILITSCH states that he has found the deflection of diamagnetic bodies, under the circumstances here described, to be precisely the same as that of paramagnetic bodies: this result is of course opposed to mine; but when it is remembered that the learned German worked confessedly with the "roughest apparatus," and possessed no means of eliminating the effects of structure, there seems little difficulty in referring the discrepancy between us to its proper cause.

The same perfect antithesis will be observed in the case of the abnormal bars, on a comparison of the third and fourth columns. In all cases then, whether we apply the magnet singly, or the current singly, or the magnet and current combined, the deportment of the normal diamagnetic bar is opposed to that of the normal paramagnetic one, and the deportment of the abnormal paramagnetic bar is opposed to that of the abnormal diamagnetic one. But if we compare the normal paramagnetic with the abnormal diamagnetic bar, we see that the deportment of one is identical with that of the other*. The same identity of action is observed when the normal diamagnetic bar is compared with the abnormal paramagnetic one. The necessity of taking molecular structure into account in experiments of this nature could not, I think, be more strikingly exhibited.

For each of the bars, under the operation of the two forces, there is an oblique position of equilibrium: on the application of the second force, the bar swings like a pendulum beyond this position, oscillates round it, and finally comes to rest there. Hence, if before the application of the second force the bar occupy the axial position, the deflection, when the second force is applied, appears to be from the axis to the equator; but if it first occupy the equatorial position, the deflection appears to be from the equator to the axis.

We have already shown that the repulsion of diamagnetic bodies is to be referred to a state of excitement induced by the magnet which acts upon them: it has been long known that the attraction of paramagnetic bodies is due to the same cause. The experiments just described exhibit to us bars of both classes of bodies moving in the magnetic field: such motions occur in virtue of the induced state of the body, and the relation of that state to the forces which act upon the mass. We have seen that in all cases the antithesis between both classes of bodies is maintained. Whatever therefore the state of the paramagnetic bar, under magnetic excitement, may be, a precisely antithetical state would produce all the phenomena of the diamagnetic bar. If the bar of iron be polar, a reverse polarity on the part of bismuth would

* Identical to the eye, but not to the mind. The notion appears to be entertained by some, that, by changing molecular structure, I had actually converted paramagnetic substances into diamagnetic ones, and *vice versâ*. No such change, however, can cause *the mass* of a diamagnetic body suspended by its centre of gravity to be *attracted*, or the mass of a paramagnetic body to be *repelled*. But by a change of molecular structure one of the forces may be so caused to apply itself that it shall present to the eye all the *directive* phenomena exhibited by the other.—J. T., May 5, 1855.

produce the effects observed. From this point of view all the movements of diamagnetic bodies become perfectly intelligible, and the experiments to be recorded in the next chapter are not calculated to diminish the probability of the conclusion that diamagnetic bodies possess a polarity opposed to that of magnetic ones.

The phenomena to which we have thus far referred consist in the rotations of elongated bars about their axes of suspension. The same antithesis, however, presents itself when we compare the *motion of translation* of a paramagnetic body, within the coil, with that of a diamagnetic one. A paramagnetic sphere was attached to the end of a horizontal beam and introduced into the coil: the magnet being excited the sphere could be made to traverse the space within the coil in various directions, by properly varying the current through the coil. A diamagnetic sphere was submitted to the same examination, and it was found that the motions of both spheres, when operated on by the same forces, were always in opposite directions.

V. FURTHER COMPARISON OF PARAMAGNETIC AND DIAMAGNETIC PHENOMENA:— DIAMAGNETIC POLARITY.

When an iron bar is placed within a helix, it is well known that on sending a current through the latter the bar is converted into a magnet, one end of the bar thus excited being attracted, and the other end repelled by the same magnetic pole. In this *twoness* of action consists what is called the *polarity* of the bar: we will now consider whether a bar of bismuth exhibits similar effects.

Fig. 39 Plate II. represents the disposition of the apparatus used in the examination of this question. AB is a helix of covered copper wire one-fifteenth of an inch in thickness: the length of the helix is 5 inches, external diameter 5 inches, and internal diameter 1·5 inch. Within this helix a bar of bismuth $6\frac{1}{2}$ inches long and 0·4 of an inch thick was suspended. The suspension was effected by means of a light beam, from two points of which, sufficiently distant from each other, depended two silver wires each ending in a loop: into these loops, *ll'*, the bar of bismuth was introduced, and the whole was suspended by a number of fibres of unspun silk from a suitable point of support. Fig. 39 *a* is a side view of the arrangement used for the suspension of the bar. Before introducing the latter within the helix, it was first suspended in a receiver, which protected it from air currents, and in which it remained until the torsion of the fibre had exhausted itself: the bar was then removed, and the beam, without permitting the fibre to twist again, was placed over the helix so as to receive the bar introduced through the latter. From the ends of this helix two wires passed to a current reverser R, from which they proceeded further to the poles of a voltaic battery. CD and EF are two electro-magnetic spirals, each 12 inches long, $5\frac{1}{2}$ inches external and 2 inches internal diameter. The wire composing them is one-tenth of an inch thick, and so coiled that the current could be sent through four wires simultaneously. Within these spirals were introduced two cores of soft iron 2 inches thick and 14 inches long: the ends of the cores appear at P and P'. The spirals were so connected together that the same current excited both, thus developing the same magnetic strength in the poles P P′. From the ends of the spirals proceeded wires to the current reverser R′, and thence to a second battery of considerably less power than the former. By means of the reverser R′ the polarity of the cores could be changed; P′ could be converted from a south pole to a north pole, at the same time that P was converted from a north pole to a south pole. Lastly, by a change of the connexions between the two spirals, the cores could be so excited as to make the poles of the same quality, both north or both south.

The diameter of the cylindrical space, within which the bismuth bar was suspended, was such as to permit of a free play of the ends of the bar through the space of an inch and a half. Having seen that the bar swung without impediment, and that its axis coincided as nearly as possible with the axis of the helix, a current from the battery was sent through the latter. The magnetism of the cores P and P′ was then excited, and the action upon the bismuth bar observed. M. v. FEILITSCH has attempted a similar experiment to that here described, but without success: when, however, sufficient power is combined with sufficient delicacy, the success is complete, and the most perfect mastery is obtained over the motions of the bar.

The helix above described is the one which I have found most convenient for the experiments; various other helices, however, were tried with a result equally certain, if less energetic. The one first made use of was 4 inches long, 3 inches exterior diameter and three-quarters of an inch interior diameter, with wire one-fifteenth of an inch in thickness, the bar being suspended by a fibre which passed through a slit in the helix: sending through this helix a current from a battery of 10 cells, and exciting the cores by a current from 1 cell, the phenomena of repulsion and *attraction* were exhibited with all desirable precision.

I shall now proceed to describe the results obtained by operating in the manner described. The bismuth bar being suitably suspended, a current was sent through the helix, so that the direction of the current *in the upper half* was that indicated by the arrow in fig. 40. On exciting the magnet, so that the pole N was a north pole and the pole S a south pole, the ends of the bar of bismuth were *repelled.* The final position of the bar was against the side of the helix most remote from the magnets: it is shown by dots in the figure.

By means of the reverser R the current was now sent through the helix in the direction shown in fig. 41: the bar promptly left its position, crossed the space in which it could freely move, and came to rest as near the magnets as the side of the helix would permit it. *It was manifestly attracted by the magnets.*

Permitting the current in the helix to flow in the last direction, the polarity of the cores of soft iron was reversed: we had then the state of things sketched in fig. 42; the bismuth bar instantly loosed from the position it formerly occupied, receded from the magnet and took up finally the position marked by the dots.

After this new position had been attained, the current through the helix was reversed: the bar promptly sailed across the field towards the magnets, and finally came to rest in the dotted position, fig. 43. In all these cases, when the bar was freely moving in any direction, under the operation of the forces acting upon it, the reversion either of the current in the helix or of the polarity of the cores arrested the motion; approach was converted into recession and recession into approach.

The ends of the helix in these experiments were not far from the ends of the soft iron cores; and it might therefore be supposed that the action was due to some modification of the cores by the helix, or of the helix by the cores. It is manifest that the magnets can have no *permanent* effect upon the helix; the current through the latter, measured by a tangent galvanometer, is just as strong when the cores are excited as when they are unexcited. The helix may certainly have an effect upon the cores, and this effect is either to enfeeble the magnetism of the cores or to strengthen it; but if the former, and the bar were the simple bismuth which it is when no current operates on it, the action, though weakened, *would still be repulsive*, and if the latter, the increase would simply augment the repulsion. The fact, however, of the ends of the bar being *attracted*, proves that the bar has been thrown into a peculiar condition by the current circulating in the surrounding coil. Changing the direction of the current in the coil, we find that the self-same magnetic forces which were formerly attractive are now repulsive; to produce this effect the condition of the bar must have changed with the change of the current; or, in other words, the bar is capable of accepting *two different states* of excitement, which depend upon the direction of the current.

In order, however, to reduce as far as possible the action of the helix upon the cores, I repeated the experiments with the small helix referred to in fig. 6, page 25. It will be remembered that this helix is but an inch in length, and that the bismuth bar is $6\frac{1}{2}$ inches long. I removed the magnets further apart, so that the centres of the cores were half an inch beyond the ends of the bismuth bar, while the helix encircled only an inch of its central portion: in this position, when the helix was excited, there was no appreciable magnetism excited by it in the dormant cores; at least, if such were excited, it was unable to attract the smallest soft iron nail. Here then we had cores and helix sensibly independent of each other, but the phenomena appeared as before. The bar could be held by the cores against the side of the helix, with its ends only a quarter of an inch distant from the ends of the cores; on reversing either current the ends instantly receded, but the recession could be stopped by again changing the direction of the current. With a tranquil atmosphere, and an arrangement for reversing the current without shock or motion, the bar obeyed in an admirable manner the will of the experimenter, and, under the operation of the same forces, exhibited all the deflections sketched in figs. 40, 41, 42 and 43.

The motion of the bar cannot be referred to the action of induced currents. The bar was brought into the centre of the hollow cylinder in which it swung and held there; the forces were all in action, and therefore all phenomena of induction passed; the arrangement of the forces being that shown in fig. 40, on releasing the bar it was driven from the cores, whereas when the arrangement was that shown in fig. 41, it was drawn towards them.

But it does not sufficiently express the facts to say that the bar is capable of two different states of excitement; it must be added, that both states exist simultaneously in the excited bar. We have already proved that the state necessary for the action of one pole is not that which enables an opposite pole to produce the same action; hence, when the two ends of the bar are attracted or repelled, at the same time, by two opposite poles, it is a proof that these two ends are in different states. But if this be correct, we can test our conclusion by reversing one of the poles; the direction of its force being thereby changed, it ought to hold the other pole in check and prevent all motion in the bar. This is the case: if, in any one of the instances cited, the polarity of either of the cores be altered; if the south be converted into a north, or the north into a south pole, thus making both poles of the same quality, the repulsion of the one is so nearly balanced by the attraction of the other, that the bar remains without motion towards either of them.

To carry the argument a step further, let us fix our attention for an instant upon fig. 40. The end of the bar nearest to the reader is repelled by a south pole; the same end ought to be *attracted* by a north pole. In like manner, the end of the bar most distant from the reader is repelled by a north pole, and hence the state of that end ought to fit it for *attraction* by a south pole. If, therefore, our reasoning be correct, when we place a north pole opposite to the lower end of the bar, and on the same side of it as the upper north pole, and a south pole opposite the upper end of the bar and on the same side of it as the lower south pole, the simultaneous action of these four poles ought to be more prompt and energetic than when only two poles are used. This arrangement is shown in Plate III.: the two poles to the right of the bismuth bar must be of the same name, and the two to the left of the bar of the opposite quality. If those to the right be both north, those to the left must be both south, and *vice versâ*. The current reverser for the magnets appears in front, that for the helix is hidden by the figure. The above conclusion is perfectly verified by experiments with this apparatus, and the twofold deflection of the bismuth bar is exhibited with remarkable energy*.

The bar used in these cases is far heavier than those commonly made use of in experiments on diamagnetism; but the dimensions stated do not mark the practical limit of the size of the bar. A solid bismuth cylinder, 14 inches long and 1 inch in diameter, was suspended in a helix 5·7 inches long, 1·8 inch internal diameter, 4 inches external diameter, and composed of copper wire 0·1 of an inch in thickness: when a current of twenty cells was sent through the helix, and the magnets (only

* These experiments, and almost all the others mentioned in this memoir, may be exhibited in the lecture room. By attaching indexes of wood to the bars of bismuth, and protecting the indexes from air currents by glass shades, the motions may be made visible to several hundreds at once. See a description of a Polymagnet, Phil. Mag. June 1855.—J. T.

two of them were used) were excited by one cell, all the phenomena exhibited by figs. 40, 41, 42 and 43, were distinctly exhibited.

A considerable difference is always necessary between the strength of the current passing through the helix and that which excites the cores, so as to prevent the induction of the cores, which, of itself, would be followed by repulsion, from neutralizing, or perhaps inverting, the induction of the helix. When *two* magnets were used and the helix was excited by ten cells, I found the magnetic excitement by one or two cells to be most advantageous; when the cores were excited by ten, or even five cells, the action was always repulsive*. When four magnets were applied and the helix was excited by a battery of ten or fifteen cells, a power of five cells for the magnets was found efficient.

The deportment of paramagnetic bodies is so well known that it might be left to the reader to discern that in all the cases described it is perfectly antithetical to that of the diamagnetic body. I have nevertheless thought it worth while to make the corresponding experiments with an iron bar; to facilitate comparison the results are placed side by side in Plate II. with those obtained with the bar of bismuth. It must be left to the reader to decide whether throughout this inquiry the path of strict inductive reasoning has been adhered to: if this be the case, then the inference appears unavoidable, *that the diamagnetic force is a polar force, the polarity of diamagnetic bodies being opposed to that of paramagnetic ones under the same conditions of excitement*†.

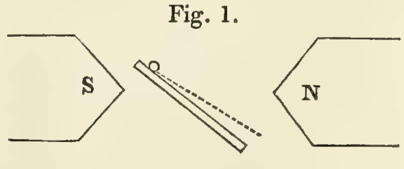
* The perfect similarity of this deportment to that of soft iron under the same circumstances is evident.

† I would gladly refer to M. Plücker's results in connexion with this subject had I been successful in obtaining them; I will here however introduce the description of his most decisive experiment in his own words. (See Scien. Mem. New Ser. p. 336.)

"From considerations of which we shall speak afterwards, it appeared to me probable that bismuth not only assumes polarity in the vicinity of a magnetic pole, but that it also retains the polarity for some time after the excitation has taken place; or, in other words, that bismuth retains a portion of its magnetism permanently, as steel, unlike soft iron, retains a portion of the magnetism excited in it by induction. My conjecture has been corroborated by experiment.

"I hung a bar of bismuth, 15 millims. long and 5 millims. thick, between the pointed poles of the large electro-magnet; it was suspended horizontally from a double cocoon-thread, fig. 1. The distance between the points was diminished until the bar could barely swing freely between them. A little rod of glass was brought near to one of the points, so that the bismuth bar, before the magnetism was excited, and in consequence of the torsion, leaned against the glass rod. On exciting the magnet by a current of three of Grove's elements, the bismuth, prevented from assuming the equatorial position, pressed more forcibly against the glass rod; when the current was interrupted, the bar remained still in contact with the rod, while its free end vibrated round its position of equilibrium. The current was closed anew and then reversed by a gyrotrope. In consequence of this reversion, the bar of bismuth, loosening from the glass rod, moved towards the axial position, but soon turned and pressed against the glass as before, or in some cases having passed quite through the axial position was driven round with the reversed ends into the equatorial....This experiment, which was made with some care, proves that the bismuth requires time to reverse its polarity."

Fig. 1.



I have repeated this experiment with great care, and have obtained in part the effect described: it is perfectly

VI. CONCLUDING OBSERVATIONS: ON M. WEBER'S THEORY OF DIAMAGNETIC POLARITY*, AND ON AMPÈRE'S THEORY OF MOLECULAR CURRENTS.

It is well known that a voltaic current exerts an attractive force upon a second current, flowing in the same direction; and that when the directions are opposed to each other the force exerted is a repulsive one. By coiling wires into spirals, AMPÈRE was enabled to make them produce all the phenomena of attraction and repulsion exhibited by magnets, and from this it was but a step to his celebrated theory of molecular currents. He supposed the molecules of a magnetic body to be surrounded by such currents, which, however, in the natural state of the body mutually neutralized each other, on account of their confused grouping. The act of magnetization he supposed to consist in setting these molecular currents parallel to each other, and starting from this principle he reduced all the phenomena of magnetism to the mutual action of electric currents.

If we reflect upon the experiments recorded in the foregoing pages from first to last; on the inversion of magnecrystallic phenomena by the substitution of a magnetic constituent for a diamagnetic; on the analogy of the effects produced in magnetic and diamagnetic bodies by compression; on the antithesis of the rotating actions described near the commencement; on the indubitable fact that diamagnetic bodies, like magnetic ones, owe their phenomena to an induced condition into which they are thrown by the influencing magnet, and the intensity of which is a function of the magnetic strength; on the circumstance that this excitation, like that of soft iron, is of a dual character; on the numerous additional experiments which have been recorded, all tending to show the perfect antithesis between the two classes of bodies;—we can hardly fail to be convinced that Mr. FARADAY's first hypothesis of diamagnetic action is the true one—that diamagnetic bodies operated on by magnetic forces possess a polarity "the same in kind as, but the reverse in direction of that acquired by magnetic bodies." But if this be the case, how are we to conceive of the *physical mechanism* of this polarity? According to COULOMB's and POISSON's theory, the act of magnetization consists in the decomposition of a neutral magnetic fluid; the north pole of a magnet, for example, possesses an attraction for the south fluid of a piece of soft iron submitted to its influence, draws the said fluid towards it, and with it the material particles with which the fluid is associated. To account

easy to produce the rotation of the bar. The cause of this rotation, however, was in my case as follows:—When the magnet was unexcited the position of equilibrium of the axis of the bar acted upon by the torsion of the fibre, was that shown by the dotted line in the figure; when the magnetism was developed, the repulsive force acting on the free end of the bar necessarily pushed it beyond the dotted line—an action which was perfectly evident when the attention was directed towards it. On reversing the current, a little time was required to change the polarity of the iron masses; during this time the free end of the bismuth *fell* towards its former position, and the velocity acquired was sufficient to carry it quite beyond the pole points. The only difference between M. PLÜCKER and myself is, that I obtained the same result by simply *intercepting* the current as by reversing it. I may remark that I have submitted ordinary bismuth to the most powerful and delicate tests, but as yet I have never been able to detect in it a trace of that retentive power ascribed to it by M. PLÜCKER.

* POGG. Ann. vol. lxxxvii. p. 145, and TAYLOR's Scien. Mem. New Ser. p. 163.

for diamagnetic phenomena this theory seems to fail altogether: according to it indeed the oft-used phrase, 'a north pole exciting a north pole, and a south pole a south pole,' involves a contradiction. For if the north fluid be supposed to be *attracted* towards the influencing north pole, it is absurd to suppose that its presence there could produce *repulsion.* The theory of AMPÈRE is equally at a loss to explain diamagnetic action; for, if we suppose the particles of bismuth surrounded by molecular currents, then according to all that is known of electro-dynamic laws, these currents would set themselves parallel to, and in the same direction as those of the magnet, and hence attraction, and not repulsion, would be the result. The fact, however, of this not being the case proves that these molecular currents are not the mechanism by which diamagnetic induction is effected. The consciousness of this, I doubt not, drove M. WEBER to the assumption that the phenomena of diamagnetism are produced by molecular currents, not *directed,* but actually *excited* in the bismuth by the magnet. Such induced currents would, according to known laws, have a direction *opposed* to those of the inducing magnet, and hence would produce the phenomena of repulsion. To carry out the assumption here made, M. WEBER is obliged to suppose that the molecules of diamagnetic bodies are surrounded by channels, in which the induced molecular currents, once excited, continue to flow without resistance.

This theory, notwithstanding its great beauty, is so extremely artificial, that I imagine the general conviction of its truth cannot be very strong; but there is one conclusion flowing from it which appears to me to be in direct opposition to experimental facts. The conclusion is, "*that the magnetism of two iron particles in the line of magnetization is increased by their reciprocal action; but that, on the contrary, the diamagnetism of two bismuth particles lying in this direction is diminished by their reciprocal action.*" The reciprocal action of the particles varies inversely as the cube of the distance between them: at a distance expressed by the number 1, for example, the enfeeblement is eight times what it would be at the distance 2.

The conclusion, as regards the iron, is undoubtedly correct; but I believe experiment proves that the mutual action of diamagnetic molecules, when caused to approach each other, *increases* their repulsive action. I have had massive iron moulds made and coated with copper by the voltaic current; into these fine bismuth powder has been introduced and submitted to powerful hydraulic pressure. No sensible fact can, I think, be more certain, than that the particles of this dust are brought into closer proximity along the line in which the pressure is exerted, and this is the line of *strongest diamagnetization.* If a portion of the compressed mass be placed upon the end of a torsion beam and the amount of repulsion measured, it will be found that the repulsion is a maximum when the line of magnetization coincides with the line of compression; or, in other words, with that line in which the particles are packed most closely together: if the bismuth were fixed, and the magnet moveable, the former would repel the latter with a maximum force with the line of compression parallel to the direction of magnetization: it is a stronger *diamagnet* in this direction than in any other. Cubes of bismuth, which, in virtue of their crystallization, possessed a line of minimum magnetization, have been placed in those moulds and pressed closely together in the direction of the said line: the approximation of the particles thus effected has converted the direction spoken of from one of minimum into one of maximum magnetization. It would be difficult for me to say how many diamagnetic bodies I have submitted to compression, some massive, some in a state of powder, but in no single instance have I discovered an exception to the law that the line of compression of purely diamagnetic bodies is the line of strongest diamagnetization. The approximation of diamagnetic particles is therefore accompanied by an augmentation of their power, instead of a diminution of it, as supposed by the theory of M. Weber.

Any hypothesis which involves the idea of the diminution of the diamagnetic action of a body by the approximation of its particles, is, I believe, opposed to facts. Such a hypothesis must, I imagine, form the basis of the following remark of Professor W. Thomson:—referring to "a thin bar or needle of a diamagnetic substance," he says, "such a needle has no tendency to arrange itself across the lines of magnetic force; but, as will be shown in a future paper, if it be very small compared with the dimensions and distance of the magnet, the direction it will assume, when allowed to turn freely round its centre of gravity, will be that of the lines of force*." I have not found in any of the subsequent numbers of the Philosophical Magazine the proof here promised†. But I doubt not the conclusion involves the assumption that the mutual action of diamagnetic particles is to weaken each other, and hence to produce a more feeble magnetization *along* a thin diamagnetic bar than *across* it—an assumption which, as already shown, is contradicted by experiment.

It is scarcely possible to reflect upon the discovery of Faraday in all its bearings, without being deeply impressed with the feeling that we know absolutely nothing of the physical causes of magnetic action. We find the magnetic force producing, by processes which are evidently similar, two great classes of effects. We have a certain number of bodies which are attracted by the magnet, and a far greater number which are repelled by the same agent. Supposing these facts to have been known to Ampère, would he have satisfied his profound mind by founding a theory which accounts for only the smaller portion of them? This theory is admirable as far as it goes, but the generalization is yet to come which shall show the true relationship of phenomena, towards whose connexion the theory of Ampère furnishes at present no apparent clue.

Royal Institution, October 1854.

* Philosophical Magazine, vol. xxxvii. p. 244.

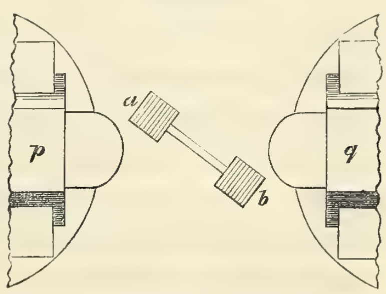
† This remark appears to have induced Mr. Thomson to publish the proof referred to in the last Number of the Philosophical Magazine. The arguments there brought forward have been long familiar to me, but I regret to say that I cannot attach much real value to them. At some future day I hope to be able to justify the scepticism which I here venture to express.—J. T., May 5, 1855.

Note *on* M. Matteucci's *objections*.

The foregoing memoir was on the point of leaving my hands for the Royal Society, when accident, backed by the kindness of Mr. Faraday, placed the '*Cours Spécial*' of M. Matteucci, recently published in Paris, in my hands. An evening's perusal of this valuable work induces me to append the following remarks to the present paper.

M. Matteucci honours the researches which bear my name, and those which I published in connexion with M. Knoblauch, with a considerable share of his attention. He corroborates all the experimental facts, but at the conclusion states three objections to the manner in which these facts have been explained. "La faveur," writes the learned Italian, "avec laquelle les idées de MM. Tyndall et Knoblauch ont été accueillies m'imposent le devoir de ne pas vous laisser ignorer les objections qui s'élèvent contre elles. La première consiste dans la différence très-grande et constant dans la force qui fait osciller entre les poles un aiguille de bismuth cristallisé, suivant que ses clivages parallèles à sa longueur sont suspendus verticalement ou dans un plan horizontal: c'est différence me parait inconciliable avec le résultat déjà rapporté de l'expérience de M. Tyndall, sur lequel se fonde l'explication des phénomènes magneto-cristallisés. Mais une objection encore plus grave est celle du mouvement d'*attraction** vers les poles qui se manifeste dans les prismes de bismuth cristallisé dont les clivages sont perpendiculaires à leur longueur. Pour rendre la conséquence de cette dernière expérience encore plus évidente, j'ai fixé deux cubes de bismuth, qui ont deux faces opposées naturelles et parallèles aux plans de clivage, aux extrémités d'un petit levier de verre, ou de sulphate de chaux, suspendu par un fil de cocon au milieu de champ magnétique entre les extrémités polaires d'un electro-aimant (fig. 27); lorsque les deux cubes ont les clivages verticaux et perpendiculaires à la longueur à l'aiguille, au moment où le circuit est fermé, l'aiguille est attiré, *quelle que soit la position* qu'elle occupe dans le champ magnétique, et se fixe en equilibre dans la ligne polaire.....Il me semble impossible d'expliquer ces mouvements du bismuth cristallisé, comme on a essayé de le faire, à la force repulsive de l'aimant, qui, suivant l'expérience de M. Tyndall†, s'exerce avec plus d'intensité parallèlement aux clivages que dans la direction perpendiculaire à ces plans.

Fig. 27*a*.



"Remarquons encore qu'on ne trouve pas constamment l'accord qui devrait exister, selon les idées de MM. Tyndall et Knoblauch, entre les phénomènes magneto-cristallisés et les effets produits par la compression dans le bismuth, si l'on considère ces plans de clivages et la ligne suivant laquelle la compression a eu lieu comme jouissant des même propriétés‡."

With regard to the first objection, I may say that it is extremely difficult to meet

* This is in reality not a 'movement of *attraction*,'—see Appendix to the present paper.—J. T. May 1855.

† This was first proved by Mr. Faraday.—J. T.

‡ Cours Spécial sur l'Induction, &c., p. 255.

one so put; it is simply an opinion, and I can scarcely say more than that mine does not coincide with it. I would gladly enter upon the subject and endeavour to give the objection a scientific form were the necessary time at my disposal, but this, I regret to say, is not the case at present. I shall moreover be better pleased to deal with the objection after it has assumed a more definite form in the hands of its proposer, for I entertain no doubt that it is capable of a sufficient answer. The second objection M. Matteucci considers to be a more grave one. The facts are as follows:—the repulsion of a mass of crystallized bismuth depends upon the direction in which the mass is magnetized. When the magnetizing force acts in a certain direction, the intensity of magnetization, and the consequent repulsion of the mass, is a maximum. This is proved by placing the mass upon the end of a torsion beam and bringing its several directions successively into the line of the magnetic force. Poisson would have called such a direction through the mass a principal axis of magnetic induction, and I have elsewhere called it a line of elective polarity. When a sphere or cube of bismuth is freely suspended in the magnetic field, with the direction referred to horizontal, in all positions except two the forces acting on the mass tend to turn it; those positions are, when the line of maximum magnetization is axial and when it is equatorial, the former being a position of unstable, and the latter a position of stable equilibrium. When the above line is oblique to the direction magnetization, the sphere or cube will turn round its axis of suspension until the direction referred to has set itself at right angles to the line joining the poles. Now if the direction of maximum magnetization be transverse to an elongated mass of bismuth, such a mass must, when the said direction recedes to the equator, set its length from pole to pole. The facts observed by M. Matteucci seem to me to be a simple corroboration of this deduction*.

The third objection is directed against an imaginary case, "si l'on considère les plans de clivage et la ligne de compression comme jouissant les même propriétés." It must be evident that a crystal like bismuth, possessing a number of cleavages of unequal values, cannot be compared in all respects with a body which has suffered pressure *in one direction* only. I have no doubt whatever that by a proper application of force, in different directions, a compressed mass might be caused to imitate to perfection every one of the actions exhibited by crystallized bismuth. Indeed I would go farther, and say, that I shall be happy to undertake to reproduce, with bismuth powder, the deportment of any diamagnetic crystal whatever that M. Matteucci may think proper to name.

In looking further over M. Matteucci's instructive book, I find another point alluded to in a manner which tempts me to make a few remarks in anticipation of a fuller examination of the subject. The point refers to the reciprocal action of the particles of magnetic and diamagnetic bodies. It is easy to see, that if the attraction of a bar of iron varies simply as the number of the particles attracted, then, inasmuch as the weight of the body varies in the same ratio, and the moment of inertia

* For a more complete examination of this subject see the "Appendix" to this paper.—J. T., May 1855.

as the weight, the times of oscillation of two masses of the same length, but possessing different numbers of attracting particles, must be the same. COULOMB indeed mixed iron filings with wax, so as to remove the particles out of the sphere of their mutual inductive action, and proved that when needles of equal lengths, but of different diameters, were formed from the same mixture, the duration of an oscillation was the same for all. From this he inferred that the attractive force is simply proportional to the number of ferruginous particles; but this could not be the case if these particles exerted any sensible reciprocal action, either tending to augment or diminish the induction due to the direct action of the magnet. On account of such a mutual action, two bars of soft iron, of the same length, and of different diameters, have not the same time of oscillation.

In examining the question whether the particles of diamagnetic bodies exert a similar reciprocal action, M. MATTEUCCI fills quills of the same length, and of different diameters, with powdered bismuth, and finds that there is no difference between the duration of an oscillation of the thick ones and the slender ones; from this he infers that there can be no reciprocal action among the particles of the bismuth.

Now it is not to be imagined that even in COULOMB's experiments with the iron filings the molecular induction was absolutely nothing, but simply that it was so enfeebled by the separation of the particles that it was insensible in the experiments. This remark applies with still greater force to M. MATTEUCCI's experiments with the bismuth powder; for the enfeeblement of a force already so weak, by the division of the diamagnetic mass into powder, must of course practically extinguish all reciprocal action of the particles, even supposing a weak action of the kind to exist when the mass is compact.

I will not here refer to my own experiments on compressed bismuth, but will take a result arrived at by M. MATTEUCCI himself while repeating and corroborating these experiments. "I made," says M. MATTEUCCI, "two cylinders of bismuth precisely of the same dimensions, the one compressed, the other in its natural state, and found that the compressed mass had a diamagnetic power *distinctly superior* to that of natural bismuth*." Now M. MATTEUCCI, in his 'Cours Spécial,' has made his own choice of a test of reciprocal molecular action; he assumes that if cylinders of the same length, but of different masses, have equal times of oscillation, it is a conclusive proof that there is no action of the kind referred to. This necessarily implies the assumption, that were the times of oscillation *different*, a reciprocal action would be demonstrated. Now, according to his experiments described in the Association Report, the times of oscillation *are* different; the diamagnetism of the compressed cylinder is "*distinctly superior*" to that of the uncompressed one: *the diamagnetic effect increases in a greater proportion than the quantity of matter*; and hence, on M. MATTEUCCI's own principles, the result negatived by his experiments on powdered bismuth is fairly established by those which he has made with the compressed substance.

* Report of British Association for 1852, Transactions of Sections, p. 7.

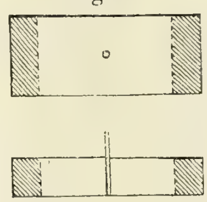
APPENDIX.

Received December 21, 1854.

Reflecting further on the subject of diamagnetic polarity, an experiment occurred to me which constitutes a kind of crucial test to which the conclusions arrived at in the foregoing memoir may be submitted.

Two square prisms of bismuth, 0·43 of an inch long and 0·2 of an inch wide, were laid across the ends of a thin plate of cedar wood, and fastened there by white wax. Another similar plate of wood was laid over the prisms, and also attached to them by wax; a kind of rectangular box was thus formed, one inch long and of the same width as the length of the prisms, the ends of the box being formed by the prisms, while its sides were open. Both plates of wood were pierced through at the centre, and in the aperture thus formed a wooden pin was fixed, which could readily be attached to a suspending fibre. Fig. 1 represents the arrangement both in plan and section.

Fig. 1.



The prisms first chosen were produced by the compression of fine bismuth powder, without the admixture of gum or any other foreign ingredient, the compressed mass being perfectly compact and presenting a surface of metallic brilliancy. If such a mass be placed on the end of a torsion balance and a magnetic pole is brought to bear upon it, I have proved the repulsion to be maximum when the direction in which the mass has been compressed is in the continuation of the axis of the magnet. A comparative view of the repulsion in this direction, and in another perpendicular to it, is given in the following Table.

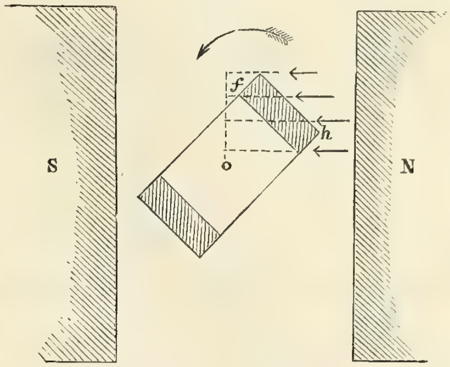
Compressed bismuth powder.

	Repulsion.	
Strength of magnet.	line of pressure axial.	line of pressure equatorial.
5·8	22	13
8·4	46	31
10·0	67	46
11·9	98	67

We see here that the repulsion, when the line of pressure is axial, exceeds what occurs when the same line is equatorial by fully one-half the amount of the latter. Now this can only be due to the more intense magnetization, or rather diamagnetization, of the bismuth along the line of pressure; and in the experiment now to be described I availed myself of this fact to render the effect more decided.

The prisms of bismuth were so constructed that the line of pressure was parallel to the length of each. The rectangular box before referred to was suspended from its centre of gravity O in the magnetic field, so that the two prisms were in the same horizontal plane. Let the position of the box thus suspended horizontally be that shown in fig. 2. For the sake of simplicity, we will confine our attention to the action of one of the poles N, which may be either flat or rounded, upon the prism *hf* adjacent to it, as indeed all the phenomena to be described can be produced before a single pole. The direction of the force emanating from N is represented by the arrows, and if this force be *purely repulsive*, the action upon every single particle of the diamagnetic mass furnishes a moment which, in the position here assumed, tends to turn the rectangular box in the direction marked by the arrow. It is perfectly impossible that such a system of forces could cause the box to turn in a direction opposed to the arrow; yet this is the precise direction in which the box turns when the magnetic force is developed.

Fig. 2.

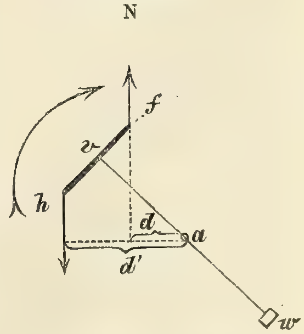


Here, then, we have a mechanical effect which is perfectly inexplicable on the supposition that the diamagnetic force is purely repulsive. But if the conclusions arrived at in the foregoing memoir be correct, if the diamagnetic force be a polar force, then we must assume that attraction and repulsion are developed simultaneously, as in the case of ordinary magnetic phenomena. Let us examine how this assumption will affect the analysis of the experiment before us.

The marked end of a magnetic needle is pulled towards the north magnetic pole of the earth; and yet, if the needle be caused to float upon a liquid, there is no motion of its mass towards the terrestrial pole referred to. The reason of this is known to be, that the south end of the needle is repelled by a force equal to that by which the north, or marked end, is attracted. These two equal and opposite forces destroy each other as regards *a motion of translation*, but they are effective in producing *a motion of rotation*. The magnetic needle, indeed, when in a position oblique to the plane of the magnetic meridian, is solicited towards that plane by a mechanical couple, and if free to move will turn and find its position of equilibrium there.

Let such a needle, *fh*, be attached, as in fig. 3, to the end of a light wooden beam, *vw*; let the beam and needle be suspended horizontally from the point *a*, round which the whole system is free to turn, the weight of the needle being balanced by a suitable counterpoise, *w*; let the north pole of the earth be towards N. Supposing the beam to occupy a position oblique to the magnetic meridian, as in the figure, the end *f*, or the marked end, of the needle is solicited towards N by a force φ, and the tendency of this force to produce rotation in the direction of the arrow is expressed by the

Fig. 3.



product of φ into the perpendicular drawn from the axis of rotation upon the direction of the force. Setting this distance $=d$, we have the moment of φ in the direction stated,

$$=\varphi d.$$

The end h of the needle is repelled by the earth's magnetic pole with a force φ': calling the distance of the direction of this latter force from the axis of rotation d', we have the moment of φ' in a direction opposed to the arrow,

$$=\varphi' d'.$$

Now as the length of the needle may be considered a vanishing quantity, as compared with its distance from the terrestrial pole, we have practically

$$\varphi=\varphi',$$

and consequently $$\varphi d<\varphi' d'.$$

The tendency to turn the lever in a direction opposed to the arrow is therefore predominant; the lever will obey this tendency and move until the needle finds itself in the magnetic meridian: when this position is attained, the predominance spoken of evidently ceases, and the system will be in equilibrium. Experiment perfectly corroborates this theoretic deduction.

In this case, the centre of gravity of the needle recedes from the north magnetic pole as if it were repelled by the latter; but it is evident that the recession is not due either to the attraction or repulsion of the needle considered as a whole, but simply to the mechanical advantage possessed by the force φ', on account of its greater distance from the axis of rotation. If the force acting upon every particle of the needle were purely *attractive*, it is evident that no such recession could take place. Supposing, then, that we were simply acquainted with the fact, that the end f of the needle is attracted by the terrestrial pole, and that we were wholly ignorant of the action of the said pole upon the end h, the experiment here described would lead us infallibly to the conclusion that the end h must be repelled. For if it were attracted, or even if it were neither attracted nor repelled, the motion of the bar must be *towards* the pole N instead of in the opposite direction.

Let us apply this reasoning to the experiment with the bismuth prisms already described. The motion of the magnetic needle in the case referred to is not more inexplicable, on the assumption of a purely attractive force, than is the motion of our rectangular box on the assumption of a purely repulsive one; and if the above experiment would lead to the conclusion that the end h of the magnetic needle is repelled, the experiment with the bismuth leads equally to the conclusion that the end f of the prism hf, fig. 2, must be *attracted* by the pole N. The assumption of such an attraction, or in other words, of diamagnetic polarity, is alone capable of explaining the effect, and the explanation which it offers is perfect.

On the hypothesis of diamagnetic polarity, the prism hf turns a hostile end h to the magnetic pole N and a friendly pole f away from it. Let the repulsive force acting upon the former be φ, and the attractive force acting upon the latter φ'. It is manifest that if φ were equal to φ', as in the case of the earth's action, or in other words, if the field of force were perfectly uniform, then, owing to the greater distance of φ' from the axis of rotation, from the moment at which the rectangular box quits the equatorial position, which is one of unstable equilibrium, to the moment when its position is axial, the box would be incessantly drawn towards the position last referred to.

But it will be retorted that the field of force is not uniform, and that the end *h*, on account of its greater proximity to the magnet, is more forcibly repelled than the end *f* is attracted: to this I would reply, that it is only in "fields" which are approximately uniform that the effects can be produced; but to produce motion towards the pole, it is not necessary that the field should be perfectly uniform: setting, as before, the distance of the direction of the force φ from the axis of rotation $=d$, and that of the force $\varphi'=d'$, a motion towards the pole N will always occur whenever

$$\frac{d'}{d}>\frac{\varphi}{\varphi'}.$$

To ascertain the diminution of the force on receding from a polar surface such as that here used, I suspended a prism of bismuth, similar to those contained in the rectangular box, at a distance of 0·9 of an inch from the surface of the pole. Here, under the action of the magnet excited by a current of ten cells, the number of oscillations accomplished in a second was 17; at 0·7 of an inch distant the number was 18; at 0·5 of an inch distant the number was 19; at 0·3 distant the number was 19·5; and at 0·2 distant the number was 20. The forces at these respective distances being so very little different from each other, it follows that a very slight deviation of the box from the equatorial position is sufficient to give the moment of φ' a preponderance over that of φ, and consequently to produce the exact effect observed in the experiment.

The consistency of this reasoning is still further shown when we operate in a field of force which diminishes speedily in intensity as we recede from the magnet. Such a field is the space immediately in front of pointed poles. Suspending our rectangular box between the points, and causing the latter to approach until the box has barely room to swing between them, it is impossible to produce the phenomena which we have just described. The intensity with which the nearest points of the bismuth bar are repelled so much exceeds the attraction of the more distant end, that the moment of attraction is not able to cope successfully with the moment of repulsion; the bars are consequently repelled *en masse*, and the length of the box takes up a position at right angles to the line which unites the poles.

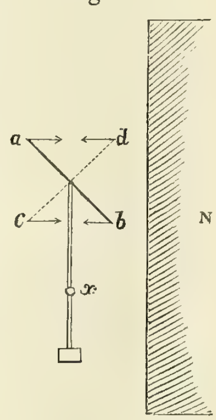
It is manifest, however, that by increasing the distance between the bismuth bar and the points acting upon it, we diminish the difference of action upon the two ends of the bar. When the distance is sufficient, we can produce, with the pointed poles, all the phenomena exhibited between flat or rounded ones.

All the effects which have been described are produced with great distinctness when, instead of compressed bismuth, two similar bars of the crystallized substance are used, in which the planes of principal cleavage are parallel to the length. Such bars are not difficult to procure, and they ought to hang in the magnetic field with the planes of cleavage vertical. It is unnecessary to describe the experiments made with such bars; they exhibit with promptness and decision all the effects observed with the compressed bismuth.

We have hitherto operated upon elongated masses of bismuth; but with the compressed substance, or with the substance crystallized uniformly in planes, as in the case last referred to, an elongation of the mass is not necessary to the production of the effects described. Previous, however, to the demonstration of this proposition, I shall introduce a kind of lemma, which will prepare the way for the complete proof.

Diamagnetic bodies, like paramagnetic ones, vary considerably in the intensity of their forces. Bismuth or antimony, for example, exhibits the diamagnetic force with greater energy than gold or silver, just as iron or nickel exhibits the magnetic force with greater energy than platinum or chromium. Let two thin bars, *ab*, *cd*, fig. 4, of two bodies of different diamagnetic powers, be placed at right angles to each other, so as to form a cross; let the cross be attached to the end of a lever and suspended horizontally from the point x, before the flat or rounded pole N of a magnet. Let the continuous line *ab* represent the needle of the powerful diamagnetic body, and the broken line *cd* that of the feeble one. On the former a mechanical couple acts in the direction denoted by the arrows at its ends; and on the latter a couple operates in the direction of the arrows at *its* ends. These two couples are evidently opposed to each other; but the former being, by hypothesis, the more powerful of the two, it will overcome the latter. The mechanical advantage possessed by the *attracted* end *a* of the more powerful bar, on account of its greater distance from the axis of suspension x, will, in an approximately uniform field of force which we here assume, cause the centre of gravity of the cross to move towards the pole N.

Fig. 4.



In the formation of such a cross, however, it is not necessary to resort to two different substances in order to find two needles of different diamagnetic powers; for in crystallized bodies, or in bodies subjected to mechanical pressure, the diamagnetic force acts with very different energies in different directions. Let a mass of a diamagnetic body which has been forcibly compressed in one direction be imagined; let two needles be taken from such a mass, the one with its length parallel, and the other with its length perpendicular to the line of pressure. Two such needles, though composed of the same chemical substance, will behave exactly as the two bars of the cross in the experiment last described; that needle whose length coincides with the line of pressure will bear the same relation to the other that the needle of the powerfully diamagnetic substance bears to that of the feeble one. An inspection of the table at page 44 will show that this must be the case.

It is also shown in the following table, that in masses of crystallized bismuth the diamagnetic repulsion acts with very different energies in different directions. Cubes were taken from a mass of bismuth with the planes of principal cleavage parallel throughout to two opposite faces of each cube. The cubes were placed upon the ends of a torsion balance, and the diamagnetic repulsion was accurately measured when the force acted parallel to the planes of cleavage. The cubes were then turned 90° round, and the repulsion was measured when the force acted perpendicular to the planes referred to.

Cubes of crystallized bismuth.

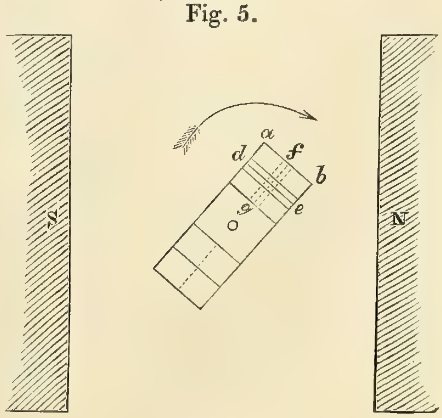
	Repulsion when the force was directed	
Strength of magnet.	along the cleavage.	across the cleavage.
3·6	11·7	8
5·7	34·8	23
8·4	78	53
10·0	111	76·5
11·9	153	110

It is manifest from this table that bismuth behaves as a body of considerably superior diamagnetic power when the force acts *along* the planes of cleavage.

Let two indefinitely thin needles be taken from such a mass, the one with its length parallel, and the other with its length perpendicular to the planes of cleavage; it is evident that if two such needles be formed into a cross and subjected to experiment in the manner above described, the former will act the part of the more powerfully diamagnetic needle, and produce similar effects in the magnetic field.

We now pass on to the demonstration of the proposition, that it is not necessary that the crystallized masses should be elongated to produce the effects exhibited by the prisms in the experiments already recorded. Let us suppose the ends of our rectangular box to be composed of cubes, instead of elongated masses, of crystallized bismuth, and let the planes of principal cleavage be supposed to be parallel to the face *a b*, fig. 5. Let the continuous line *d e* represent an indefinitely thin slice of the cube passing through its centre, and the dotted line *g f* a similar slice in a perpendicular direction. These two slices manifestly represent the case of the cross in fig. 4, and were they alone active, the rectangular box, in a uniform field of magnetic force, must turn in the direction of the arrow. Comparing similar slices *in pairs* on each side of those two central slices, it is

Fig. 5.



manifest that every pair parallel to the line *d e* represents a stronger mechanical couple than every corresponding pair parallel to *f g*. The consequence is, that a cube of crystallized bismuth suspended in the manner described, in a sufficiently uniform field of magnetic force, will move in the same direction as the cross in fig. 4: its centre of gravity will therefore *approach* the pole N, which was to be demonstrated.

This deduction is perfectly illustrated by experiment. It is manifest that the effect of the pole S upon the cube adjacent to it is to increase the moment of rotation of the rectangular box: the same reasoning applies to it as to the pole N.

Referring to fig. 27*a*, page 41, it will be seen that we have here dealt with the second and gravest objection of M. Matteucci, and converted the facts upon which the objection is based into a proof of diamagnetic polarity, so cogent that it alone would seem to be sufficient to decide this important question. Holding the opinion entertained by M. Matteucci regarding the nature of diamagnetic force*, his objection must have appeared to him to be absolutely unanswerable: I should be glad to believe that the remarks contained in this 'Appendix' furnish, in the estimation of the distinguished philosopher referred to, a satisfactory explanation of the difficulty which he has disclosed.

Let me, in conclusion, briefly direct the reader's attention to the body of evidence laid before him in the foregoing pages. It has been proved that matter is repelled by the pole of a magnet in virtue of an induced condition into which the matter is thrown by such a pole. It is shown that the condition evoked by one pole is not that which is evoked by a pole of an opposite quality—that each pole excites a condition peculiar to itself. A perfect antithesis has been shown to exist between the deportment of paramagnetic and diamagnetic bodies when acted on by a magnet alone, by an electric current alone, or by a magnet and an electric current combined. The perplexing phenomena resulting from molecular structure have been laid open, and the antithesis between paramagnetic and diamagnetic action traced throughout. It is further shown, that whatever title to polarity the deportment of a bar of soft iron, surrounded by an electric current, and acted on by other magnets, gives to this substance, a bar of bismuth possesses precisely the same title: the disposition of forces which, in the former case, produces attraction, produces in the latter case repulsion, while the repulsion of the iron finds its exact complement in the attraction of the bismuth. Finally, we have a case adduced by M. Matteucci which suggests a crucial experiment to which all our previous reasoning has been submitted, by which its accuracy has been proved, and the insufficiency of the assumption, that the diamagnetic force is not polar, is reduced to demonstration. When we remember that against all this no single experimental fact or theoretic argument† which can in any

* Il ne peut exister dans les corps diamagnétiques une polarité telle qu'on la concoit dans le fer doux."—*Cours Spécial*, p. 201.

† I ought perhaps to except an argument of Professor W. Thomson's, which professes to prove that an absolute creation of force, and the setting up of a perpetual motion, would follow, if diamagnetic polarity were

degree be considered conclusive has ever been brought forward, nor do I believe can be brought forward, the conclusion seems irresistible, that we have in the agency by which bodies are repelled from the poles of a magnet, a force of the same dual character as that by which bodies are attracted; that, in short, "diamagnetic bodies possess a polarity the same in kind but the opposite in direction to that possessed by magnetic ones."

conceded. While expressing my admiration of the ingenuity of Mr. Thomson's reasoning, it appears to me to labour under the disadvantage of proving too much, his conclusion being equally fatal to polarity of all kinds. The argument, I believe, was first publicly urged against myself at the Belfast Meeting of the British Association; but at the Liverpool Meeting last year Professor Thomson himself admitted "that he had not perfect confidence in the truth of the conclusion, as one of the assumptions on which the reasoning was founded admitted of doubt."—See Athenæum, 1854, p. 1204. Indeed, from many of his published papers, it might be inferred that Mr. Thomson actually *assumed* what I, in the present memoir, have attempted to *prove*.

I refrain from alluding to the negative results obtained by Mr. Faraday in repeating M. Weber's experiments; for though admirably suited to the exhibition of certain effects of ordinary induction, Mr. Faraday himself has shown how unsuitable the apparatus employed would be for the investigation of the question of diamagnetic polarity. See Experimental Researches (2653, 2654), vol. iii. p. 143.—J. T., May 9, 1855.

II. *On the Attraction of the Himalaya Mountains, and of the elevated regions beyond them, upon the Plumb-line in India. By the Venerable* John Henry Pratt, *M.A., Archdeacon of Calcutta. Communicated by the Rev.* J. Challis, *M.A., F.R.S. &c.*

Received October 23,—Read December 7, 1854.

1. It is now well known that the attraction of the Himalaya Mountains, and of the elevated regions lying beyond them, has a sensible influence upon the plumb-line in North India. This circumstance has been brought to light during the progress of the great trigonometrical survey of that country. It has been found by triangulation that the difference of latitude between the two extreme stations of the northern division of the arc, that is, between Kalianpur and Kaliana, is 5° 23′ 42″·294, whereas astronomical observations show a difference of 5° 23″ 37″·058, which is 5″·236* less than the former.

2. That the geodetic operations are not in fault appears from this; that two bases, about seven miles long, at the extremities of the arc having been measured with the utmost care, and also the length of the northern base having been computed from the measured length of the southern one, through a chain of triangles stretching along the whole arc, about 370 miles in extent, the difference between the measured and the computed lengths of the northern base was only 0·6 of a foot, an error which would produce, even if wholly lying in the meridian, a difference of latitude no greater than 0″·006.

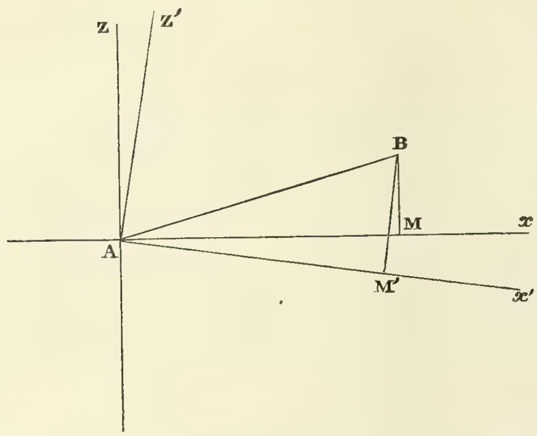
3. The difference 5″·236 must therefore be attributed to some other cause than error in the geodetic operations. A very probable cause is the attraction of the superficial matter which lies in such abundance on the north of the Indian arc. This disturbing cause acts in the right direction; for the tendency of the mountain mass must be to draw the lead of the plumb-line at the northern extremity of the arc more to the north than at the southern extremity, which is further removed from the attracting mass. Hence the effect of the attraction will be to lessen the difference of latitude, which is the effect observed. Whether this cause will account for the error in the difference of latitude in *quantity*, as well as in direction, remains to be considered, and is the question I propose to discuss in the present paper.

4. But if mountain attraction have any sensible influence at the stations on the arc, how is it that the geodetic operations are not affected by it? How is it that such a remarkable degree of exactness between the measured and computed lengths of the

* This is the difference as stated by Colonel Everest in his work on the Measurement of the Meridional Arc of India, published in 1847. See p. clxxviii.

northern base attests, it would seem, to the non-existence of any external disturbing cause? For in observing the altitude or depression of one station in the triangulation as seen from another, the error on the plumb-line must come into the calculation. The answer is, that these small errors occur in the calculation of the horizontal arc in very small terms not higher than the second order; whereas in the expression for the inclination of the two verticals at the extremities of the arc they occur in terms of the first order. This I will further illustrate.

Fig. 1.



5. Suppose the arc divided into n equal portions: and let $\nu_0\nu_1\nu_2\ldots\ldots\nu_n$ be the deflections of the plumb-line at the $n+1$ stations thus chosen. Let A be one of these stations, and B the next towards the south; Az, Ax vertical and horizontal lines through A on the supposition that there is no mountain attraction; Az', Ax' the vertical and horizontal lines as affected by attraction. Draw BM and BM$'$ perpendicular to Ax and Ax': let AM$=a$, BM$=h$, $\angle z\text{A}z'=\nu$, $\angle\text{BAM}=\alpha$. Then AM is the true horizontal distance between A and B, and AM$'$ the calculated horizontal distance. Hence the calculation makes this portion of the arc too short by

$$\text{AM}-\text{AM}'=\text{AM}\left(1-\frac{\cos(\alpha+\nu)}{\cos\alpha}\right)=a(\tan\alpha\,.\sin\nu+1-\cos\nu)=h\,.\nu+\frac{1}{2}a\,.\nu^2,$$

neglecting the cube and higher powers of ν.

Hence the whole arc is made too short by

$$h_0\nu_0+h_1\nu_1+h_2\nu_2+\ldots.+h_n\nu_n+\frac{a}{2}(\nu_0^2+\nu_1^2+\nu_2^2+\ldots.+\nu_n^2),$$

$h_0h_1h_2\ldots h_n$ being the heights of the various stations of observation above the trne horizontal line. When the Station B is below A then h is negative. These heights are all extremely small compared with a, as the arc lies through a comparatively flat country. Hence the expression for the error in the length of the arc is made up, as I said, of small terms of no higher order than the second; whereas the error in the difference of latitude ($=\nu_n-\nu_0$) has terms of the first order.

6. That this expression for the shortening of the arc is a minute quantity utterly inappreciable, may easily be shown by taking an extreme case. The quantities $h_0h_1h_2\ldots h_n$ are some of them positive and some of them negative, in such a manner that their algebraical sum equals the difference of height of Kalianpur and Kaliana above the level of the sea. From Colonel Everest's work on the Indian Arc (published in 1847) I gather, that between Kalianpur and Kaliana there are forty-seven principal stations, or, including the two terminal ones, forty-nine: and the Survey

shows that in passing from north to south there are twenty-five elevations of one station above the level of the preceding one, amounting in all to 3901·1 feet; and twenty-three depressions, amounting to 2965·2 feet (see pp. 269–273). The difference between these =935·9 feet, which is the height assigned to Kalianpur above Kaliana. I will take these, then, as the values of $h_0 h_1 h_2 \ldots h_n$ in my present example; so that the sum of the positive quantities among $h_0 h_1 h_2 \ldots = 3901{\cdot}1$ feet, and the sum of the negative =2965·2, and $n=48$. Now ν_n is the greatest and ν_0 is the least of the quantities $\nu_n \ldots \nu_0$. Hence it follows, that

$$h_0\nu_0 + h_1\nu_1 + \ldots + h_n\nu_n \text{ is less than } 3901{\cdot}1\nu_n - 2{\cdot}965\,.2\nu_0,$$

and therefore, much more, less than $3901{\cdot}1\nu_n$ feet.

Now by the Survey $\nu_n - \nu_0 = 5''{\cdot}236$, or in arcs $=0{\cdot}000025$;

$$\therefore \quad h_0\nu_0 + h_1\nu_1 + \ldots + h_n\nu_n \text{ is less than } 3901{\cdot}1 \times 0{\cdot}000025 \text{ or } 0{\cdot}097527 \text{ foot.}$$

If in this extreme case of supposing the attraction to equal its greatest value at more than half of the stations, and that at stations where its effect would be greatest, the result is so insignificant, what must it be in the actual case*? The same may be shown with respect to the other term in the expression for the shortening of the arc, viz. $\frac{1}{2}a(\nu_0^2 + \nu_1^2 + \ldots \nu_n^2)$. This quantity is less than $\frac{n+1}{2}\, a.\nu_n^2$; or, if we reckon the distance between Kaliana and Kalianpur to be 370 miles, and therefore $n.a = 370 \times 1760 \times 3$ feet and $n=48$, this quantity is less than 0·008 of a foot, which is utterly inappreciable. Hence mountain attraction may have a sensible value at the stations on the arc, and yet not affect the *geodetic* calculations in the slightest appreciable degree†.

7. I can see no ground, therefore, whatever for the process of dispersion which Colonel EVEREST describes at page clxx of the Introduction to his work, by which he distributes the error 5″·236 among the triangles. It appears to me to be unquestionable that the geodetic operations are in no way sensibly affected by mountain attraction, and therefore need no correction whatever on that account. It is the *astronomical* operation of observing the difference of latitude which requires the correction. That it is here that the correction must be applied appears again in attempting to determine the azimuths of the arc at seven stations *astronomically* (see p. xlii). It is only when the plumb-line is brought into use to determine the vertical angles of stars that the effect of attraction becomes sensible; and never in the geodetic calculations, where only horizontal angles or extremely minute vertical angles (viz. the elevations or depressions of $h_0 h_1 h_2 \ldots$) are observed.

8. The importance of accounting satisfactorily for the difference between the geodetic

* If the triangulation be carried into elevated regions some of the values of $h_0 h_1 h_2 \ldots$ will be large; and therefore the conclusion in the text will not in that case stand.

† In this paper I show that the difference caused by attraction in the latitudes of the extremities of the northern division of the arc, viz. Kaliana and Kalianpur, amounts to 15″·885, which is more than three times the angle 5″·236. But the conclusion arrived at in art. 6. is still true.

and astronomical results appears from the effect it must have upon the determination of the earth's ellipticity; an effect such, that unless this quantity be fully accounted for, it must render the great Indian Survey comparatively useless in the delicate problem of the Figure of the Earth, however valuable it may be for the purposes of mapping the vast continent of Hindostan.

9. The effect of a small error in the difference of latitude upon the determination of the ellipticity may be calculated as follows:—

Let ε be the ellipticity, a quantity known not to differ much from $\frac{1}{300}$; λ the amplitude of the arc; μ the latitude of the middle point of the arc. Then by the usual formula

$$\frac{\text{length of arc}}{\text{equatorial radius}}=\lambda-\frac{1}{2}\varepsilon(\lambda+3\sin\lambda\cos 2\mu).$$

But $\sin\lambda=\lambda-\frac{1}{6}\lambda^3+\ldots=\lambda\left(1-\frac{1}{6}\lambda^2+\ldots\right)$; $\lambda=5^\circ\ 23'\ 37''$ for the arc between Kalianpur and Kaliana $=0\cdot094$ in parts of the radius,

$$\therefore\quad \frac{1}{6}\lambda^2=0\cdot00147.$$

Hence by putting λ instead of $\sin\lambda$ in the above formula, we shall be omitting a quantity of the order $\frac{1}{2}\varepsilon\times 0\cdot00441\cos 2\mu$, which is utterly insignificant,

$$\therefore\quad \frac{\text{length of arc}}{\text{equatorial radius}}=\lambda\left(1-\frac{1}{2}\varepsilon\right)\left(1-\frac{3}{2}\varepsilon\cos 2\mu\right).$$

In the same way if L be the amplitude and M the latitude of the middle point of another arc,

$$\frac{\text{length of arc L}}{\text{equatorial radius}}=\text{L}\left(1-\frac{1}{2}\varepsilon\right)\left(1-\frac{3}{2}\varepsilon\cos 2\text{M}\right),$$

$$\therefore\quad \frac{\text{length of arc }\lambda}{\text{length of arc L}}=\frac{\lambda}{\text{L}}\left\{1-\frac{3}{2}\varepsilon(\cos 2\mu-\cos 2\text{M})\right\}.$$

Suppose the observed values of λ and μ are subject to small errors owing to mountain attraction; to find the effect on ε we must differentiate this expression, supposing the angles λ, μ and ε variable and the other quantities constant,

$$\therefore\quad 0=d\lambda\left\{1-\frac{3}{2}\varepsilon(\cos 2\mu-\cos 2\text{M})\right\}$$

$$+3\lambda\varepsilon\sin 2\mu.d\mu-\frac{3}{2}\lambda(\cos 2\mu-\cos 2\text{M})d\varepsilon,$$

$$\therefore\quad d\varepsilon=\frac{d\lambda}{\lambda}\,\frac{2}{3(\cos 2\mu-\cos 2\text{M})},$$

neglecting extremely small quantities of the higher order.

Now in the case before us,

$$\lambda=\text{ latitude of Kaliana }-\text{ latitude of Kalianpur},$$

$$=29^\circ\ 30'\ 48''-24^\circ\ 7'\ 11''=5^\circ\ 23'\ 37'',$$

$\mu=$ half the sum of these latitudes,
$=26° 49'$; $\cos 2\mu=0·59295$.

Suppose $d\lambda=1''$ only; then

$$d\varepsilon=\frac{1''}{5° 23' 38''}\frac{2}{3(0·59295-\cos 2M)}$$

$$=\frac{1}{58254}\frac{2}{0·59295-\cos 2M}.$$

This will be smallest when 2M is chosen as nearly 180° as possible. The great arc lately measured near North Cape is the one which will best meet this condition. Put therefore $M=70°$, $\cos 2M=-0·76604$, and

$$\therefore\ d\varepsilon=\frac{1}{58254}\frac{2}{1·35899}=\frac{1}{39585}$$

$$=\frac{\varepsilon}{132},\text{ if we put }\frac{1}{300}\text{ for }\varepsilon.$$

Hence for an error of $5''·236$ in defect in the amplitude, the effect on the ellipticity will be to diminish it by $\frac{5·236}{132}\varepsilon=\frac{\varepsilon}{25}$ nearly, or by nearly $\frac{1}{25}$th part of its whole value, under the most favourable circumstances. This is sufficient to show the great importance of endeavouring to account satisfactorily for the discrepancy brought to light by the Indian Survey; and that, not by merely putting it down to mountain attraction, but by calculating that attraction by some independent means, with a view to see whether its amount actually corresponds with the observed anomaly*.

10. To dissect and actually to calculate the attraction of the masses of which the Himalayas, and the regions beyond, are composed, appears, at the very thought of it, to be an herculean undertaking next to impossible. I am fully convinced, however, that no other method will succeed. It is upon this plan that the solution of the problem is conducted in this paper. It will be seen, that by selecting a peculiar law of dissection the calculation is very greatly simplified, and made to depend entirely and solely upon a knowledge of the elevations and depressions, in fact, the general contour of the surface. This information for some part of the mass is already supplied by the maps of the Trigonometrical Survey.

11. In the following pages I propose, in the first place, to develope my method of calculation, and to deduce a formula by which the attraction can be determined with a precision corresponding to the degree of accuracy to which the contour of the surface is known.

* If the effect of mountain attraction upon the northern division of the arc be what I make it, $15''·885$, then the ellipticity as determined from this and the Russian arc would be too small by $\frac{1}{8}\varepsilon$, if mountain attraction is neglected. The error in the ellipticity in comparing the whole arc between Kaliana and Damargida with the North Cape arc will, under the same circumstances, amount even to $\frac{1}{6}\varepsilon$. This will appear from the sequel, and is here mentioned only to illustrate the importance of the subject under consideration.

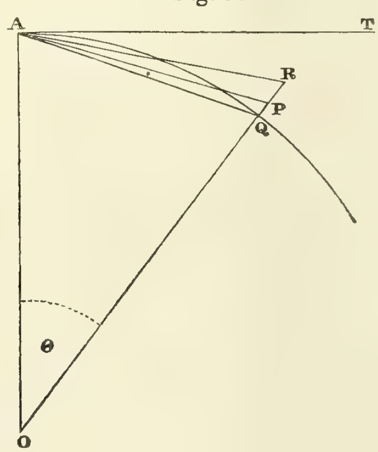
In the second place, I propose to reduce the formula to numbers, and so arrive at such an approximate value of the attraction as the data I have been able to collect will allow.

12. This approximate value is, as will be seen, larger than $5''\cdot236$, the error brought to light by the Survey. I make various suppositions with a view, if possible, to reduce my result to this, but without effect. This leads me to look in another direction for an explanation of the cause of discordance, and I arrive at a conclusion which clears up the discrepancy, confirms the calculations of this paper, and illustrates the importance of not disregarding the influence of mountain attraction.

I. *Determination of a Formula for calculating Mountain Attraction on the stations of the Indian Arc.*

13. Let O be the centre of a circle AQ, AT the tangent at A, QR a slender prism of mass M, being the prolongation of the radius through the point Q. Then if $AQ=a$, $AR=b$, $\angle QAR=\omega$, and $\angle AOQ=\theta$, the following is true:—

Fig. 2.



Lemma.—The attraction of the prism QR on the point A in the direction AT

$$=\frac{M}{ab}\cos\frac{\theta}{2}\left\{1+\tan\frac{\theta}{2}\tan\frac{\omega}{2}\right\}.$$

For let P be any point of the prism, $QP=z$, $QR=h$, $\angle PAQ=\psi$,

$\therefore$ mass of element of prism at $P=M\frac{dz}{h}$,

$$\text{attraction of this on A in direction } AP=M\frac{dz}{h}\,\frac{1}{PA^2},$$

$$\text{attraction of this on A in direction } AT=M\frac{dz}{h}\,\frac{\cos PAT}{PA^2},$$

$$\text{Cos } PAT=\cos\left(\frac{1}{2}\theta-\psi\right)$$

$$\frac{AP}{a}=\frac{\cos\frac{1}{2}\theta}{\cos\left(\frac{1}{2}\theta+\psi\right)},\quad \frac{h}{b}=\frac{\sin\omega}{\cos\frac{1}{2}\theta},$$

$$z=QP=a\frac{\sin\psi}{\cos\left(\frac{1}{2}\theta+\psi\right)}=a\left(\cos\frac{1}{2}\theta\tan\left(\frac{1}{2}\theta+\psi\right)-\sin\frac{1}{2}\theta\right)$$

$$\frac{dz}{d\psi}=a\cos\frac{1}{2}\theta\sec^2\left(\frac{1}{2}\theta+\psi\right).$$

Putting these values in the above expression, attraction of element at P on A in direction AT

$$=\frac{M}{ab\sin\omega}\cos\left(\frac{1}{2}\theta-\psi\right)d\psi;$$

$\therefore$ attraction of prism QR on A in direction AT

$$=\frac{M}{ab\sin\omega}\left\{\text{constant}-\sin\left(\frac{1}{2}\theta-\psi\right)\right\}\text{ from }\psi=0\text{ to }\psi=\omega;$$

$$=\frac{M}{ab\sin\omega}\left\{\sin\frac{1}{2}\theta-\sin\left(\frac{1}{2}\theta-\omega\right)\right\}$$

$$=\frac{M}{ab}\cos\frac{1}{2}\theta\left\{1+\frac{1-\cos\omega}{\sin\omega}\tan\frac{1}{2}\theta\right\}$$

$$=\frac{M}{ab}\cos\frac{1}{2}\theta\left\{1+\tan\frac{1}{2}\omega\tan\frac{1}{2}\theta\right\}.$$

14. *Corollary.* The above formula will reduce itself to the following in the cases to which we have to apply it.

Attraction of prism on A in direction $AT=\frac{M}{a^2}\cos\frac{1}{2}\theta.$

For in these cases AQ is the circumference, and O the centre, of the earth; QR the height of any point of the surface above the sea-level at A.

Now $$\frac{a}{b}=\frac{\cos\left(\frac{1}{2}\theta+\omega\right)}{\cos\frac{1}{2}\theta}=\cos\omega-\sin\omega.\tan\frac{\theta}{2};$$

$\therefore$ attraction of prism

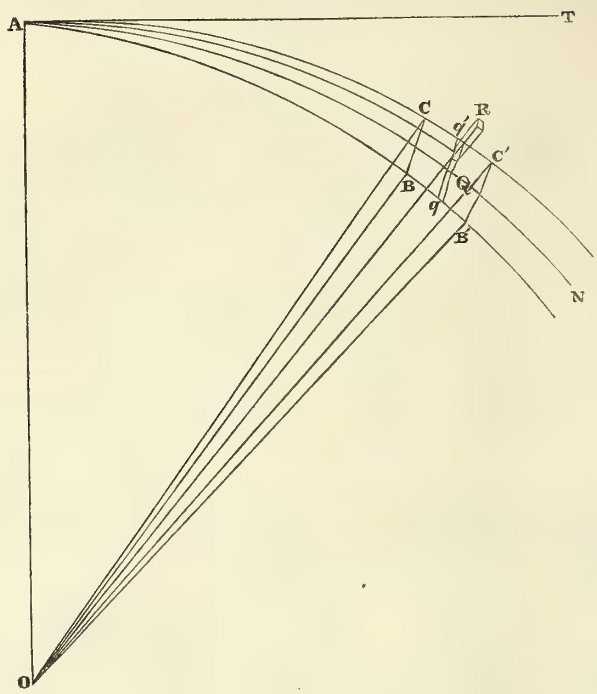
$$=\frac{M}{a^2}\cos\frac{1}{2}\theta\left\{1+\tan\frac{1}{2}\omega\tan\frac{1}{2}\theta\right\}\times\left\{\cos\omega-\sin\omega\tan\frac{1}{2}\theta\right\}$$

$$=\frac{M}{a^2}\cos\frac{1}{2}\theta\left\{1-\frac{1}{2}\omega\tan\frac{1}{2}\theta\right\},$$

neglecting the square and higher powers of ω. But ω is never greater than 2° (=0·03488 in arcs), and when it has this maximum value, θ is less than 2°; and as θ increases, ω decreases in a higher degree. Hence the second term within the brackets is of insensible importance, and the corollary as enunciated is true.

15. In order to calculate the attraction of the superficial crust of the earth upon the point A on its surface, I shall suppose a number of vertical planes to be drawn through A making any angles with each other, and thus dividing the surface through A, parallel to the sea-level, into a number of *lunes* all meeting again in a point in the antipodes of A. About A as centre suppose a number of concentric circles drawn on this surface; the law of the distances of these circles will be determined hereafter. In this way the whole surface will be divided into a number of four-sided *compartments*, two of the sides in every compartment converging to A, and the other two being parts of circles concentric in A.

Fig. 3.



Let ABB′ and ACC′ be parts of two of the great circles forming these lunes; BCC′B′ one of the four-sided compartments. Let ∠BAC$=\beta$, ∠AOB$=\alpha$, ∠BOB′$=\varphi$; Q an element of the compartment; qQq' parallel to BC; ∠QAN$=\psi$, AN being a great circle bisecting the angle β, and AT a tangent to AN; AO$=r$, ∠AOQ$=\theta$;

∴ distance of Q from AO$=r\sin\theta$;

and $r\sin\theta\, d\psi$, and $rd\theta$ are the sides of the element Q. Let k' be the height of R, the earth's surface, above Q; ϱ the mean density of the superficial matter of the earth;

∴ mass of prism QR$=\varrho k' r^2 \sin\theta\, d\theta\, d\psi$;

also chord AQ$=2r\sin\frac{1}{2}\theta$.

Hence by the corollary in Art. 14,

attraction of prism QR on A along the tangent to AQ

$$=\varrho\frac{k'r^2\sin\theta\, d\theta\, d\psi}{4r^2\sin^2\frac{1}{2}\theta}\cos\frac{1}{2}\theta;$$

∴ attraction of prism QR on A along AT

$$=\varrho\frac{k'r^2\sin\theta\, d\theta\, d\psi}{4r^2\sin^2\frac{1}{2}\theta}\cos\frac{1}{2}\theta\cos\psi.$$

Integrating with respect to ψ, from $\psi=-\frac{1}{2}\beta$ to $\frac{1}{2}\beta$, attraction on A of the mass standing on qq' in direction AT

$$=\frac{\varrho\sin\theta\cos\frac{1}{2}\theta\, d\theta}{4\sin^2\frac{1}{2}\theta}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}k'\cos\psi d\psi=\frac{\varrho\sin\theta\cos\frac{1}{2}\theta\, d\theta}{4\sin^2\frac{1}{2}\theta}.2\sin\frac{1}{2}\beta.k \text{ very nearly,}$$

k being the average value of k' on the arc qq';

$$=\varrho\sin\frac{1}{2}\beta.\frac{\cos^2\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}d\theta.k. \qquad (1)$$

16. The degree of error incurred in thus taking k for k' may be judged of by applying the formula to an extreme case. For example, let us suppose that the surface of

the earth at this part is a plane passing through the middle point of the elementary base qQq', and sloping upwards towards the side on which A lies. This is an extreme case; for the actual height will be extremely small near the middle of qq', that is, in parts where the variations of cos ψ are least; and the actual heights will be greatest towards q and q', that is, in parts where the variations of cos ψ are greatest. Hence the error incurred in replacing the actual heights by their average will be much greater in this than in a more general case. Now in this extreme instance

$$k'=(c-c\cos\psi)\tan\eta,$$

c being the distance of qQq' from A, and η the inclination of the plane to the horizon.

Hence the true value of the integral

$$\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} k'\cos\psi d\psi=c\tan\eta\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}(\cos\psi-\cos^2\psi)d\psi$$

$$=c\tan\eta\left(2\sin\frac{1}{2}\beta-\frac{1}{2}\beta-\frac{1}{2}\sin\beta\right)$$

$$=c\tan\eta\left(\frac{1}{24}\beta^3-\frac{7}{1920}\beta^5+\ldots\right).$$

And the approximate value of the integral

$$=2\sin\frac{1}{2}\beta.k=2\sin\frac{1}{2}\beta.\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}(c-c\cos\psi)\tan\eta d\psi\div\beta$$

$$=\frac{2c}{\beta}\sin\frac{1}{2}\beta\tan\eta\left(\beta-2\sin\frac{1}{2}\beta\right)$$

$$=c\tan\eta\left(\frac{1}{24}\beta^3-\frac{13}{5760}\beta^5+\ldots.\right).$$

Hence the ratio of the true value to the approximate value $=1-\frac{1}{120}\beta^2$. If $\beta=30°$, the width I intend to give to the lunes, this ratio differs from unity by the insignificant fraction 0·00228, which we may well neglect; and if this is the smallness of the error in such an extreme case, we may consider that the approximation will in the general case differ from the true value by an inappreciable quantity.

17. To find the attraction of the whole mass comprised within the compartment, the expression (1) in art. 15. should be integrated with respect to θ from $\theta=\alpha$ to $\theta=\alpha+\phi$. But to do this we ought to know what function k is of θ. There is, however, no known law of connexion between them. We must therefore resort to some other means. According to the law of dissection, which I shall in the end adopt, the value of $\frac{\cos^2\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$ does not vary much within the limits of a single compartment, that is, between the limits $\theta=\alpha$ and $\theta=\alpha+\phi$. Indeed its middle value, when $\theta=\alpha+\frac{1}{2}\phi$, is only one-fifteenth part smaller than its greatest value, when $\theta=\alpha$, and

only one-sixteenth part greater than its smallest value, when $\theta=\alpha+\phi$. I shall therefore take this middle value instead of the variable value, and the expression (1.) becomes integrable. Suppose h is the mean value of k from $\theta=\alpha$ to $\theta=\alpha+\phi$; then

Attraction of whole mass standing on the compartment BB′C′C

$$=\varrho\sin\frac{1}{2}\beta.\frac{\phi\cos^2\left(\frac{1}{2}\alpha+\frac{1}{4}\phi\right)}{\sin\left(\frac{1}{2}\alpha+\frac{1}{4}\phi\right)}h. \qquad (2.)$$

By this approximation the attraction of the nearer and narrower part of the mass is made a little too small, and that of the further and wider part a little too large. These errors tend to counterbalance each other; and the residual error, if any, will be very trifling. In some of the compartments the compensation may be exact; in others, in excess; in others, in defect—according to the variations of k. So that taking all the masses on the lune, the probabilities are that the compensation on the whole will be perfect, and that no error will be incurred.

18. That the extreme values of $\frac{\cos^2\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$ do not differ more than one-fifteenth and one-sixteenth from its middle value appears as follows:—

According to the law of dissection, which I shall soon adopt,

$\phi=\frac{1}{10}\alpha$, when α is very small;

$\phi=\frac{1}{9}\alpha$, when α is about 38°;

$\phi=\frac{1}{8}\alpha$, when α is about 52° 30′;

$\phi=\frac{1}{7}\alpha$, when α is about 65°;

$\phi=\frac{1}{6}\alpha$, when α is about 76°;

$\phi=\frac{1}{5}\alpha$, when α is about 84°;

$\phi=\frac{1}{4}\alpha$, when α is about 130°;

and ϕ has intermediate values for intermediate values of α, and of course the smaller the ratio which ϕ bears to α, the less will be the error in our approximation. Now out of 49 cases in which we have to use the formula (2.),

α is less than 38°, and ϕ is less than $\frac{1}{9}\alpha$, in 40 cases.

α is less than 52° 30′, and ϕ is less than $\frac{1}{8}\alpha$, in 3 cases.

α is less than 65°, and ϕ is less than $\frac{1}{7}\alpha$, in 2 cases.

α is less than 76°, and ϕ is less than $\frac{1}{6}\alpha$, in 1 case.

α is less than 84°, and ϕ is less than $\frac{1}{5}\alpha$, in 1 case.

α is less than 130°, and ϕ is less than $\frac{1}{4}\alpha$, in 1 case.

α is greater than 130°, and ϕ is greater than $\frac{1}{4}\alpha$, in 1 case.

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Let us take, then, the extreme case of the first 40, viz. $\alpha=38°$ and $\phi=\frac{1}{9}\alpha$. Hence $\alpha+\frac{1}{2}\phi=40°\ 6'$, and $\alpha+\phi=42°\ 13'$; and

	Difference.	Ratio to middle value.
Greatest value of $\dfrac{\cos^2\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}=2{\cdot}746$		
	0·172	$\frac{1}{15}$th.
Middle . . . . . . $=2{\cdot}574$		
	0·156	$\frac{1}{16}$th.
Least . . . . . . $=2{\cdot}418$		

Hence in the most unfavourable of the first 40 cases, viz. that in which ϕ is exactly $=\frac{1}{9}\alpha$, the extreme values will depart only $\frac{1}{15}$th and $\frac{1}{16}$th part from the middle value: and the successive pairs of values on the two sides of the middle will differ less and less as they approach the middle. Suppose that there are n such pairs, and that they form two series in arithmetic progression,

$$-\frac{1}{15}\ldots\ldots.0 \text{ and } 0\ldots\ldots.+\frac{1}{16}.$$

The sums of these two series are $-\frac{1}{15}\cdot\frac{n+1}{2}$ and $+\frac{1}{16}\cdot\frac{n+1}{2}$,

and hence the average error or the mean of these

$$=-\left(\frac{1}{15}-\frac{1}{16}\right)\frac{n+1}{2}\div(2n+1)=-\frac{1}{960}\,\frac{2n+2}{2n+1}=-\frac{1}{960},$$

when n is indefinitely increased.

Hence the probable error in this extreme 40th case is in defect, and is about $\frac{1}{1000}$th part of the whole. The probable error in the 9 cases beyond the 40th will be greater than this; but for the 39 cases before the 40th far less and less.

Hence in the whole lune we may fairly consider that no appreciable error will be incurred.

19. I now proceed to select the Law of Dissection, that is, the relation of the lengths of the respective compartments to their distances from A. Upon this depends the simplicity and the success of the method of calculation now proposed.

Let the relation between α and φ be always such, that

$$\frac{\varphi \cos^2\left(\frac{1}{2}\alpha+\frac{1}{4}\varphi\right)}{\sin\left(\frac{1}{2}\alpha+\frac{1}{4}\varphi\right)}=\text{a numerical constant}=c.$$

To fix the value of this constant I shall make $\varphi=\frac{1}{10}\alpha$ when φ and α are indefinitely small. Hence, expanding in powers of φ and α,

$$c=\frac{\varphi}{\frac{1}{2}\alpha+\frac{1}{4}\varphi}=\frac{1}{5+\frac{1}{4}}=\frac{4}{21},$$

and the Law of Dissection of the earth's crust is expressed by the equation

$$\frac{\varphi \cos^2\left(\frac{1}{2}\alpha+\frac{1}{4}\varphi\right)}{\sin\left(\frac{1}{2}\alpha+\frac{1}{4}\varphi\right)}=\frac{4}{21}. \quad \ldots \ldots \ldots \ldots \quad (3.)$$

This reduces the formula (2.) to the following:—Attraction of mass standing on any compartment

$$=\frac{4}{21}\varrho \sin\frac{1}{2}\beta.h, \quad \ldots \ldots \ldots \ldots \quad (4.)$$

which depends simply upon the average height (h) of the surface of the mass above the surface through A, and not at all upon the distance of the compartment from A.

20. It is in this that the remarkable simplicity of the method consists. We have but to calculate the angles from the Law of Dissection (3.), and lay down the circles and the lines diverging from A upon a good map on which the elevations and depressions are marked, and the attractions of the several masses standing on the compartments thus marked out will be given by the formula (4.) at once, when we determine upon their average elevations.

21. Let D be the mean density of the earth, which has been finally fixed at 5·66 of distilled water by the recent experiments of the late Mr. Baily; r the radius of the earth $=$ 4000 miles; g the measure of gravity; then

$$g=\frac{4.\pi}{3}\text{D}.r$$

$$\text{D}=\frac{3}{4\pi}\,\frac{g}{r}.$$

Hence formula (4). becomes,—Attraction of mass standing on any one compartment

$$=\frac{4}{21}\frac{\rho}{D}D\sin\frac{1}{2}\beta.h$$

$$=\frac{1}{7\pi}\frac{\rho}{D}\sin\frac{1}{2}\beta\cdot\frac{h}{r}\cdot g.$$

Let h be expressed in parts of a mile; ϱ being the density of the superficial crust of the earth, we shall take $=2\cdot75$, which is the density assigned to the mountain Schehallien; reducing to numbers, we have

Attraction of mass standing on any one compartment

$$=0\cdot000005523\times h\sin\frac{1}{2}\beta.g. \qquad (5.)$$

This gives the attraction of each mass in terms of gravity.

22. We may from this easily deduce the deflection of the plumb-line caused by the attraction of the mass; for the tangent of deflection evidently equals the expression (5.) divided by g, by the simple law of the resolution of forces. Hence

$$\text{Tangent of deflection}=0\cdot000005523\times h\sin\frac{1}{2}\beta$$

$$=\tan(1''\cdot1392)\times h\sin\frac{1}{2}\beta;$$

∴ Deflection of the plumb-line caused by the mass standing on any one compartment

$$=1''\cdot1392\,h\sin\frac{1}{2}\beta. \qquad (6.)$$

23. It remains to calculate the dimensions of the successive compartments as indicated by the Law of Dissection which I have adopted. The equation which expresses the law cannot be solved directly; we must therefore resort to approximation or trial. All pairs of values we thus find for α and φ must satisfy the equation expressing the law. That equation becomes, on our replacing the arc φ by the angle φ,

$$\varphi^0=\frac{4}{21}\frac{180}{\pi}\frac{\sin\left(\frac{1}{2}\alpha+\frac{1}{4}\varphi\right)}{\cos^2\left(\frac{1}{2}\alpha+\frac{1}{4}\varphi\right)},$$

or

$$\varphi=\log^{-1}\left\{\begin{array}{l}11\cdot0379639\\ +\log\sin\left(\frac{1}{2}\alpha+\frac{1}{4}\varphi\right)\\ -2\log\cos\left(\frac{1}{2}\alpha+\frac{1}{4}\varphi\right)\end{array}\right\} \qquad (7.)$$

This is the test which all corresponding values of α and φ must satisfy.

24. The solution of equation (3.) expressing the law may be facilitated, for values of α not exceeding 38°, by expansion and approximation. Expand in powers of α and φ, and it becomes

$$\varphi=\frac{4}{21}\left(\frac{\alpha}{2}+\frac{\varphi}{4}\right)\left\{1-\frac{1}{6}\left(\frac{\alpha}{2}+\frac{\varphi}{4}\right)^2+\left(\frac{\alpha}{2}+\frac{\varphi}{4}\right)^2\right\}$$

$$=\frac{2}{21}\left(\alpha+\frac{\varphi}{2}\right)\left\{1+\frac{5}{6}\left(\frac{\alpha}{2}+\frac{\varphi}{4}\right)^2\right\};$$

$$\therefore \frac{\alpha}{\phi}+\frac{1}{2}=\frac{21}{2}\left\{1-\frac{5}{6}\left(\frac{\alpha}{2}+\frac{\phi}{4}\right)^2\right\}$$

$$\frac{\alpha}{\phi}=10\left\{1-\frac{7}{8}\left(\frac{\alpha}{2}+\frac{\phi}{4}\right)^2\right\}$$

$$\frac{\phi}{\alpha}=\frac{1}{10}\left\{1+\frac{7}{8}\left(\frac{\alpha}{2}+\frac{\phi}{4}\right)^2\right\}$$

$$=\frac{1}{10}\left\{1+\frac{7}{8}\left(\frac{\alpha}{2}+\frac{\alpha}{40}\right)^2\right\}$$

$$=\frac{1}{10}(1+0\cdot2411\,\alpha^2);$$

or, if α be expressed in degrees, $=\frac{1}{10}(1+0\cdot000073\,\alpha^2)$ . . . . (8.)

This formula, as will appear in the issue, may be used for all values of α up to 38° without sensible error.

25. Let α_1, α_2, α_3 .... ϕ_1, ϕ_2, ϕ_3 ..... be the successive values of α and ϕ for the several compartments of a lune. Then these are connected by the equations

$$\alpha_2=\alpha_1+\phi_1,\quad \alpha_3=\alpha_2+\phi_2,\ \ldots\ldots$$

As the hills do not begin to rise north and north-east of the station Kaliana till between a distance of three-quarters and a whole degree, I shall assume* the first value of α (viz. α_1)=0°·75.

* If we wish to apply this method of calculation to any other particular case, we may make any other assumption we please regarding the first value of α, so long as we take care not to apply the method to calculate the attraction on a station *too near* to elevated ground. In short, we must see that the angle ω in art. 14. is sufficiently small to be neglected in the formula therein deduced. If it be not, we must calculate the attraction of the nearer masses by a direct method.

We may use the formula (8.) we have just deduced to calculate the values of α and ϕ nearer to A than α_1. In this way we shall obtain the following series of values, writing them backward from α_1 towards A.

α_0 =0°·6818	ϕ_0 =0°·06818
α_{-1}=0·6198	ϕ_{-1}=0·06198
α_{-2}=0·5635	ϕ_{-2}=0·05635
α_{-3}=0·5122	ϕ_{-3}=0·05122
α_{-4}=0·4657	ϕ_{-4}=0·04657
α_{-5}=0·4234	ϕ_{-5}=0·04234
α_{-6}=0·3849	ϕ_{-6}=0·03849
α_{-7}=0·3599	ϕ_{-7}=0·03599
α_{-8}=0·3181	ϕ_{-8}=0·03181
α_{-9}=0·2892	ϕ_{-9}=0·02892
α_{-10}=0·2629	ϕ_{-10}=0·02629
α_{-11}=0·2390	ϕ_{-11}=0·02390
α_{-12}=0·2172	ϕ_{-12}=0·02172
α_{-13}=0·1975	ϕ_{-13}=0·01975
α_{-14}=0·1795	ϕ_{-14}=0·01795
α_{-15}=0·1632	ϕ_{-15}=0·01632
α_{-16}=0·1484	ϕ_{-16}=0·01484
α_{-17}=0·1349	ϕ_{-17}=0·01349
&c.	&c.

The formula (8.) then gives the following successive values:—

	Distance from A of the nearer ends of the compartments.	Length of the compartments.	Distance from A of the middle points of the compartments.
1.	$\alpha_1 = 0^\circ{\cdot}75$	$\phi_1 = 0^\circ{\cdot}075$	$0^\circ{\cdot}787$
2.	$\alpha_2 = 0{\cdot}825$	$\phi_2 = 0{\cdot}0825$	0·866
3.	$\alpha_3 = 0{\cdot}907$	$\phi_3 = 0{\cdot}0907$	0·949
4.	$\alpha_4 = 0{\cdot}998$	$\phi_4 = 0{\cdot}0998$	1·048
5.	$\alpha_5 = 1{\cdot}098$	$\phi_5 = 0{\cdot}1098$	1·153
6.	$\alpha_6 = 1{\cdot}208$	$\phi_6 = 0{\cdot}1208$	1·268
7.	$\alpha_7 = 1{\cdot}329$	$\phi_7 = 0{\cdot}1329$	1·395
8.	$\alpha_8 = 1{\cdot}461$	$\phi_8 = 0{\cdot}1461$	1·534
9.	$\alpha_9 = 1{\cdot}607$	$\phi_9 = 0{\cdot}1607$	1·687
10.	$\alpha_{10} = 1{\cdot}768$	$\phi_{10} = 0{\cdot}1768$	1·856
11.	$\alpha_{11} = 1{\cdot}945$	$\phi_{11} = 0{\cdot}1945$	2·042
12.	$\alpha_{12} = 2{\cdot}140$	$\phi_{12} = 0{\cdot}2140$	2·247
13.	$\alpha_{13} = 2{\cdot}353$	$\phi_{13} = 0{\cdot}2353$	2·472
14.	$\alpha_{14} = 2{\cdot}589$	$\phi_{14} = 0{\cdot}2589$	2·719
15.	$\alpha_{15} = 2{\cdot}848$	$\phi_{15} = 0{\cdot}2848$	2·990
16.	$\alpha_{16} = 3{\cdot}133$	$\phi_{16} = 0{\cdot}3133$	3·289
17.	$\alpha_{17} = 3{\cdot}446$	$\phi_{17} = 0{\cdot}3446$	3·616
18.	$\alpha_{18} = 3{\cdot}790$	$\phi_{18} = 0{\cdot}3790$	3·980
19.	$\alpha_{19} = 4{\cdot}170$	$\phi_{19} = 0{\cdot}4170$	4·378
20.	$\alpha_{20} = 4{\cdot}586$	$\phi_{20} = 0{\cdot}4586$	4·813
21.	$\alpha_{21} = 5{\cdot}046$	$\phi_{21} = 0{\cdot}5046$	5·298
22.	$\alpha_{22} = 5{\cdot}550$	$\phi_{22} = 0{\cdot}5550$	5·828
23.	$\alpha_{23} = 6{\cdot}106$	$\phi_{23} = 0{\cdot}6106$	6·408
24.	$\alpha_{24} = 6{\cdot}716$	$\phi_{24} = 0{\cdot}6716$	7·054
25.	$\alpha_{25} = 7{\cdot}388$	$\phi_{25} = 0{\cdot}7388$	7·707
26.	$\alpha_{26} = 8{\cdot}127$	$\phi_{26} = 0{\cdot}8127$	8·533
27.	$\alpha_{27} = 8{\cdot}939$	$\phi_{27} = 0{\cdot}8939$	9·386
28.	$\alpha_{28} = 9{\cdot}833$	$\phi_{28} = 0{\cdot}9833$	10·324
29.	$\alpha_{29} = 10{\cdot}816$	$\phi_{29} = 1{\cdot}089$	11·360
30.	$\alpha_{30} = 11{\cdot}905$	$\phi_{30} = 1{\cdot}202$	12·506
31.	$\alpha_{31} = 13{\cdot}107$	$\phi_{31} = 1{\cdot}326$	13·770
32.	$\alpha_{32} = 14{\cdot}433$	$\phi_{32} = 1{\cdot}462$	15·211
33.	$\alpha_{33} = 15{\cdot}99$	$\phi_{33} = 1{\cdot}620$	16·80
34.	$\alpha_{34} = 17{\cdot}61$	$\phi_{34} = 1{\cdot}800$	18·51
35.	$\alpha_{35} = 19{\cdot}41$	$\phi_{35} = 1{\cdot}992$	20·40
36.	$\alpha_{36} = 21{\cdot}40$	$\phi_{36} = 2{\cdot}211$	22·50
37.	$\alpha_{37} = 23{\cdot}61$	$\phi_{37} = 2{\cdot}456$	24·83
38.	$\alpha_{38} = 26{\cdot}06$	$\phi_{38} = 2{\cdot}734$	27·43
39.	$\alpha_{39} = 28{\cdot}79$	$\phi_{39} = 3{\cdot}054$	30·31
40.	$\alpha_{40} = 31{\cdot}84$	$\phi_{40} = 3{\cdot}419$	33·55
41.	$\alpha_{41} = 35{\cdot}26$	$\phi_{41} = 3{\cdot}600$	37·06
42.	$\alpha_{42} = 38{\cdot}86$	$\phi_{42} = 4{\cdot}314$	41·01

When part of the attracting mass is near, and therefore ω is not so small that it may be neglected (as in art. 14.)—which will be the case in carrying the survey into the mountainous regions—a different formula must be used for calculating the deflection of the plumb-line caused by the nearer parts of the attracting mass.

Let, as before, the mass be divided by lunes, which for the parts now under consideration must be narrow; let ω and ω' be the angles which the highest and lowest points of a small vertical prism, reaching from any point of the surface down to the sea-level, makes with the horizontal line at the eye of the observer. Let θ be the horizontal distance from the observer of the middle line of the prism, in degrees; β the width of the lune, in degrees; l the length of the compartment on which the prism stands, in miles; r the radius of the earth, in miles. Then the Lemma in art. 13. leads to the following exact formula:—

26. That the formula (8.) which I have been using to determine these values does not, thus far, lead to material error may be shown by substituting the 42nd pair of values in the test given by formula (7.). In this case

$$\alpha=38^\circ{\cdot}86,\quad \phi=4^\circ{\cdot}314,\quad \frac{1}{2}\alpha+\frac{1}{4}\phi=20^\circ{\cdot}50=20^\circ\ 30',$$

and formula (7.) becomes

$$\phi=\log^{-1}\left\{\begin{array}{r}11{\cdot}0379639\\ 9{\cdot}5443253\\ \hline 20{\cdot}5822892\\ 19{\cdot}9431752\\ \hline 0{\cdot}6391140\end{array}\right\}=4^\circ{\cdot}356.$$

Hence the error $=4^\circ{\cdot}356-4^\circ{\cdot}314=0^\circ{\cdot}042$, and this equals $\frac{1}{103}$rd of the whole.

If the same test be applied to the 40th values of α and ϕ, the error is only $\frac{1}{305}$th of the whole; and as we pass further back it becomes absolutely evanescent.

27. The remaining values of α and ϕ, after the 42nd as above determined, we must find by solving equation (3.) by trial and testing the values by formula (7.).

Horizontal attraction of the prism on the station of the observer

$$=\frac{3}{4}\frac{\rho}{\mathrm{D}}\frac{\beta}{180}\frac{l}{r}\left\{\left(1+\tan\frac{1}{2}\theta\tan\frac{1}{2}\omega\right)\sin\omega+\left(1+\tan\frac{1}{2}\theta\tan\frac{1}{2}\omega'\right)\sin\omega'\right\}\cos\frac{1}{2}\theta.$$

In this $\tan\frac{1}{2}\theta\tan\frac{1}{2}\omega$ and $\tan\frac{1}{2}\theta\tan\frac{1}{2}\omega'$ may be neglected as quite insensible; for $\tan\frac{1}{2}\omega$ and $\tan\frac{1}{2}\omega'$ can neither of them be greater than 1, and in that case $\theta=0$; and when they have any other sensible value, $\tan\frac{1}{2}\theta$ is of insensible magnitude. So also as θ is never made larger than 1°, or so large, in the application of this formula, $\cos\frac{1}{2}\theta$ may be put $=1$; and the formula becomes

Horizontal attraction of the prism on the station of the observer

$$=\frac{3}{4}\frac{\rho}{\mathrm{D}}\frac{\beta}{180}\frac{l}{r}(\sin\omega+\sin\omega').$$

The values of l need not follow any law, but may be chosen in each lune according to the form of the vertical section; some values being long and some short, according as the variations of ω and ω' are slow or rapid. The angles ω and ω' must be found as follows:—The lune having been divided into compartments, the average height and depth of the top and bottom of the prism standing on each compartment, above and below the observer's horizon, must be divided by the horizontal distance of the prism from the observer. This will give the tangents of the angles ω and ω', whence the sines may be found.

The above expression must be thus calculated for all the compartments: the whole added together gives the attraction of the mass standing on the portion of the lune to which this method is to be applied. This sum, multiplied by the cosine and sine of the azimuth, will give the attraction in the meridian and in the prime vertical. The same being done all round the circle, the resultant attraction and the azimuth of the plane are easily found; whence the deflection is known, and the various angles of observation may be corrected.

In using the method of the text for the parts beyond, the heights $h_1 h_2 \ldots.$ must be measured from the sea-level, and not from the surface, parallel to the sea-level, passing through the station of the observer. This *is* done in the text in the case of Kaliana, because it appears that below the level of that place there are no variations of surface sufficient to produce any sensible alteration in the attraction of the whole mass. This, however, will not be the case with stations in the mountains.

The following applications of the test show that the values I shall now write down are correct:—

$$\alpha_{43}=43^{\circ}\cdot17,\ \phi_{43}=4^{\circ}\cdot98;\ \phi=\log^{-1}\left\{\begin{array}{r}11\cdot0379639\\ \underline{9\cdot5888296}\\ 20\cdot6267935\\ \underline{19\cdot9291416}\\ 0\cdot6976519\end{array}\right\}=4^{\circ}\cdot985.$$

$$\alpha_{44}=48^{\circ}\cdot15,\ \phi_{44}=5^{\circ}\cdot783;\ \phi=\log^{-1}\left\{\begin{array}{r}11\cdot0379639\\ \underline{9\cdot6351413}\\ 20\cdot6731052\\ \underline{19\cdot9108962}\\ 0\cdot7622090\end{array}\right\}=5^{\circ}\cdot784.$$

$$\alpha_{45}=53^{\circ}\cdot93,\ \phi_{45}=6^{\circ}\cdot80;\ \phi=\log^{-1}\left\{\begin{array}{r}11\cdot0379639\\ \underline{9\cdot6808891}\\ 20\cdot7188530\\ \underline{19\cdot8864756}\\ 0\cdot8323774\end{array}\right\}=6^{\circ}\cdot80.$$

$$\alpha_{46}=60^{\circ}\cdot73,\ \phi_{46}=8^{\circ}\cdot21;\ \phi=\log^{-1}\left\{\begin{array}{r}11\cdot0379639\\ \underline{9\cdot7292234}\\ 20\cdot7671873\\ \underline{19\cdot8528620}\\ 0\cdot9143253\end{array}\right\}=8^{\circ}\cdot21.$$

If h be the height above the observer's eye of the top of the prism, and h' the depth of the bottom of it below, then (see figure in art. 13)

$$\frac{h+r}{r}=\frac{\text{OR}}{\text{OA}}=\frac{\sin \text{OAR}}{\sin \text{ORA}}=\frac{\cos\left(\frac{1}{2}\theta-\omega\right)}{\cos\left(\frac{1}{2}\theta+\omega\right)};$$

$$\therefore\ \frac{h}{r}=\frac{2\sin\omega\sin\frac{1}{2}\theta}{\cos\left(\frac{1}{2}\theta+\omega\right)}=2\sin\omega\tan\frac{1}{2}\theta;$$

since ω is extremely small in the parts to which the method of the text is to be applied.

So also $$\frac{h'}{r}=2\sin\omega'\tan\frac{1}{2}\theta.$$

Substituting for $\sin\omega$ and $\sin\omega'$ from these in the exact formula above (which applies to all cases), and neglecting excessively small quantities,

Horizontal attraction of the prism on the station of the observer

$$=\frac{3}{4}\frac{\rho}{\text{D}}\frac{\beta}{180}\frac{l}{r}\cdot\cos\frac{1}{2}\theta\cot\frac{1}{2}\theta\cdot\frac{h+h'}{2r};$$

or the attraction of the parts to which the method of the text applies is found by taking the sum of the heights $(h+h')$, or the whole height from the sea-level up to the surface of the attracting mass. This, it will be observed, is the same whatever be the height of the station of observation. In fact, the horizontal attraction of the mass, situated so far off as 50 miles and more, upon any station, even though in the mountains, cannot differ in any appreciable degree from that upon the point where the vertical line at the station cuts the sea-level below.

$$\alpha_{47}=68^\circ{\cdot}94,\ \phi_{47}=10^\circ{\cdot}33;\ \phi=\log^{-1}\left\{\begin{array}{r}11{\cdot}0379639\\ \underline{9{\cdot}7799655}\\ 20{\cdot}8179294\\ \underline{19{\cdot}8041256}\\ 1{\cdot}0138038\end{array}\right\}=10^\circ{\cdot}33.$$

$$\alpha_{48}=79^\circ{\cdot}27,\ \phi_{48}=14^\circ{\cdot}03;\ \phi=\log^{-1}\left\{\begin{array}{r}11{\cdot}0379639\\ \underline{9{\cdot}8348646}\\ 20{\cdot}8728285\\ \underline{19{\cdot}7263656}\\ 1{\cdot}1464629\end{array}\right\}=14^\circ{\cdot}01.$$

$$\alpha_{49}=93^\circ{\cdot}30,\ \phi_{49}=23^\circ{\cdot}38;\ \phi=\log^{-1}\left\{\begin{array}{r}11{\cdot}0379639\\ \underline{9{\cdot}8995636}\\ 20{\cdot}9375275\\ \underline{19{\cdot}5685648}\\ 1{\cdot}3689627\end{array}\right\}=23^\circ{\cdot}38.$$

$\alpha_{50}=116^\circ{\cdot}68$, and ϕ_{50} reaches beyond the antipodes. For let us find α, supposing that ϕ just reaches to the antipodes; then $\alpha+\phi=180^\circ$, $\frac{1}{2}\alpha+\frac{1}{4}\phi=90^\circ-\frac{1}{4}\phi$, and equation (3.) gives

$$\phi=\frac{4}{21}\,\frac{\cos\frac{1}{4}\phi}{\sin^2\frac{1}{4}\phi},$$

an equation which is satisfied by $\phi=82^\circ\ 30'$, and therefore the corresponding value of α would be $180^\circ-82^\circ\ 30'=97^\circ\ 30'$. But this is *less than* α_{50} as above determined. Hence ϕ_{50} will reach, as I have said, beyond the antipodes. The reason why the compartments increase in length with such excessive rapidity when they are more than 90° from A, is, not only the increasing distance of the attracting mass, but their convergency towards the antipodes and the consequent contraction of their width, and also the great angle at which the attraction acts with the tangent at A, and the consequent smallness of its resolved part along that line.

28. The following list of values, then, of α and ϕ will form the continuation of those in Art. 25:—

	Distance from A of the nearer ends of the compartments.	Lengths of the compartments.	Distance from A of the middle points of the compartments.
43.	$\alpha_{43}=43^\circ{\cdot}17$	$\phi_{43}=4^\circ{\cdot}980$	$45^\circ{\cdot}66$
44.	$\alpha_{44}=48{\cdot}15$	$\phi_{44}=5{\cdot}783$	$51{\cdot}04$
45.	$\alpha_{45}=53{\cdot}93$	$\phi_{45}=6{\cdot}800$	$57{\cdot}33$
46.	$\alpha_{46}=60{\cdot}73$	$\phi_{46}=8{\cdot}210$	$64{\cdot}83$
47.	$\alpha_{47}=68{\cdot}94$	$\phi_{47}=10{\cdot}330$	$74{\cdot}10$
48.	$\alpha_{48}=79{\cdot}27$	$\phi_{48}=14{\cdot}030$	$86{\cdot}28$
49.	$\alpha_{49}=93{\cdot}30$	$\phi_{49}=23{\cdot}380$	$104{\cdot}99$
50.	$\alpha_{50}=116{\cdot}68$	ϕ_{50} is imperfect.	

29. These distances should be laid down and the circles drawn on a map or globe; and nothing remains to be done, but to ascertain the average heights of the masses standing on the compartments thus drawn.

If the surface of any of these masses is very irregular, several vertical sections should be taken in directions most favourable for giving a fair average. One convenient method of using such sections is, after laying them down on a scale on good paper, to cut them out, weigh them, and compare the weight with that of a portion of the same kind of paper of known dimensions on the same scale. The resulting number divided by the aggregate length of all the sections will give the average height. It may be convenient to use different scales for the vertical and horizontal measures: this may be done if it be carefully attended to in carrying out this method.

30. It will be evident that mountain ranges will assume a less importance in this calculation as they are more distant from A, since they will stand on a larger compartment; and therefore when in imagination levelled down to cover the whole compartment and give the average height, they will stand at a much less altitude. It is for this reason that a knowledge of extensive table-lands of considerable elevation, and of the elevated channels of rivers, is of far more importance in this calculation—especially in the remoter parts—than of mountain peaks and mountain ridges.

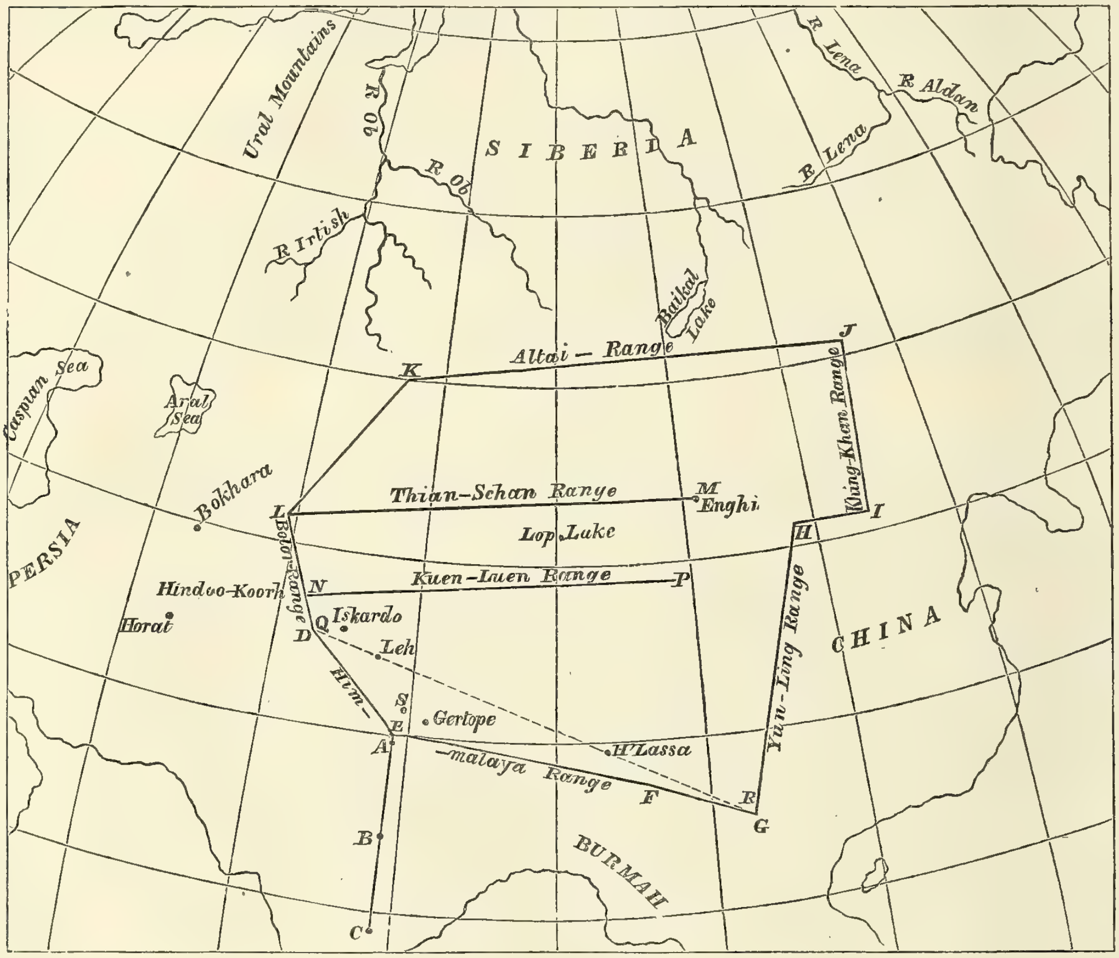
II. *Approximation to the amount of mountain attraction at the two extremities and the middle station of the Great Indian Arc of the Meridian between latitudes* 18° 3′ 15″ *and* 29° 30′ 48″.

31. The complete application of the method I have developed requires a full survey of the earth's superficies. In the absence, however, of sufficiently accurate and extensive information to make an exact calculation, I propose now to use such data as I have been able to gather—chiefly from books on geography and HUMBOLDT's works, as well as the published Maps of the Indian Survey—to obtain an approximation to the amount of attraction on the plumb-line on the Indian arc. I regret that absence from India—which is the occasion of my finding leisure to draw up this paper—prevents my consulting HUMBOLDT's Map of Central Asia (published in 1842), although extracts from his 'Aspects of Nature' will in part supply the want.

32. Fig. 4 represents an outline of the continent of Asia. A, B, C are the northern, middle, and southern stations of the Great Arc I am about to consider. These stations are Kaliana in latitude 29° 30′ 48″, Kalianpur in 24° 7′ 11″, and Damargida in 18° 3′ 15″. The longitude of the arc is about 77° 42′.

The polygonal figure DEFGHIJKLD, which for the convenience of a name I shall call the *Enclosed Space*, marks out the boundary of an irregular mass, which is the only part of the earth's surface that appears to have a sensible effect on the plumb-line in India. DEF is the Himalaya range, having a bend at E from north-west on the left to east by south on the right. FG is a range running to the table-land of Yu-nan in lat. 25° and long. 103°. GH is the range of the Yun-Ling mountains, in which there are many peaks of perpetual snow. HI is the Inshan range. IJ is the Khing-Khan range, very steep on the east side, not so on the west: the passes are said to be 5525 feet above the sea. JK is the Altai range, the highest peak of which is 10,800 feet, the average height is 6000: the range declines towards the east. KL was once thought to be a range of mountains, but it is now found to be

Fig. 4.



a line of broken country. LD is the Bolor range, rising to an elevation similar to that of the Hindoo Koosh. There are, besides these, two ranges of high mountains running east into the Enclosed Space, parallel to the Altai and Southern Himalayas, viz. LM, the Thian-Schan range, or Celestial Mountains, and NP the Kuen-Luen range, being a continuation of the Hindoo Koosh, which rises from an altitude of 2558 feet near Herat to about 20,000 near N, where it meets the Bolor range. It is with the elevation of the Enclosed Space itself that we are principally concerned, since, as observed in art. 30, ranges of mountains have not so important an influence, when distant, as table-lands of elevation.

33. Before describing the country within these limits, I will give a general sketch of the parts which lie outside, from which it will appear that we may confine our calculations to this Enclosed Space. As the elevation of the Station A. is about 1000 feet above the level of the sea, we shall take at this height the surface of reference parallel to the sea-level from which all altitudes are to be measured in our calculations. The stations Kalianpur and Damargida are higher than Kaliana. In making the Kaliana-level our basis, while we consider A to be the station Kaliana itself, B and C must be considered to be points vertically below Kalianpur and Damargida and situated on the Kaliana-level. So that our calculation of the attraction at the middle and south end of the arc will strictly speaking apply to these points. The difference, however, of the attraction at these points and the stations under which they lie will be utterly inappreciable, unless the country around B and C, which we have left out of the account as having no sufficient elevation, produce a sensible effect*.

34. Humboldt says in his 'Aspects of Nature,' that to the *lowlands* belong almost the whole of Northern Asia to the north-west of the volcanic range of the Thian-Schan, the steppes to the north of the Altai and of the Sayan range, the countries which extend from the mountains of Bolor or Bulyt-Tagh, from the upper Oxus to the Caspian, and from Tenghir or the Balkhash Lake through the Kirghis steppe, towards the sea of Aral and southern extremity of the Ural Mountains. "As compared," he adds, "with high plains of 6000 and 10,000 feet above the level of the sea, it may well be permitted to use the expression *lowlands* for flats of little more than 200 to 1200 feet of elevation." Again, "the plains through which the upper Irtish flows [rising near K in fig. 4 and running north towards the north sea] are scarcely raised 850, or at most 1170 feet above the level of the sea." The mountains about the river Aldan on the north-east and the Ural Mountains to the north-north-west—the former not averaging more than 2000 feet in height, and the latter 4000—can have no influence owing to their great distance, as well as small elevation. In short, the whole country to the north, north-west, and north-east of the Enclosed Space is of so inconsiderable a height above the sea, that it may be left out of our reckonings. The same may be said of all to the west. Thus Sir Alexander Burnes assigns to Bokhara an elevation of only 1190 feet. The level of the Caspian is 83 feet below the Black Sea, and is the centre of an extensive depression—the Caucasus excepted. The Hindoo Koosh runs off at N, a short distance to the west, but soon descends into the generally-plain country, and at Herat is only 2558 feet high. The only

* The actual heights above the sea are thus given in Colonel Everest's work:—

Height of Kaliana above the sea .... 942·3 feet.
Height of Kalianpur above the sea ...... 1878·2 feet.
Height of Damargida above the sea ...... 2090·5 feet.

great elevations in these parts are the Hindoo Koosh and the Caucasus; but these are of so small an extent and width, that when levelled down they will not sensibly raise the average height of the large compartments in which they stand. In Arabia I believe there are some moderately elevated table-lands. But their effect will be somewhat lessened by the intervening ocean, as its density is only two-fifths that of rock. Moreover the effect will be sensible, if sensible at all, only in the direction of the prime-vertical at the three stations, and not in the meridian, which is the only direction of importance to our calculation. The effect is also in part counterbalanced by those parts of the countries of Burmah, Malacca, Siam, and China, which lie outside the Enclosed Space, and are on the opposite side of the Indian Arc from Arabia. The effect of the Ghat Mountains on the west of India, and the table-land of Central and Southern India, will be counteracted in part by the extensive ocean beyond them. Their effect upon the northern Station A. will be inconsiderable; and with regard to B. and C., what effect they may have will be chiefly in the prime-vertical. So to the east of the Enclosed Space. The parts of China beyond it, in which there is only a mountain range on the sea-coast and of no considerable elevation or extent, will have but a feeble influence. Hence these regions around the Enclosed Space may be left out of the account. And those lying still further off and running to the antipodes may also be passed over, as the distance of their several parts is so great compared with the distances of A, B, and C from each other, that the resultant attraction of those regions, whatever high table-lands may occur in them, must be almost precisely the same at all three of the stations. It would not be difficult indeed to show that this resultant attraction is itself of imperceptible amount.

We may fairly conclude, then, that the disturbing cause lies wholly in the enormous mass included within the Enclosed Space, which I shall now describe.

35. The Himalayas rise abruptly from the plains of India to 4000 feet and more, and cover an extensively broken surface some 100 or 200 miles wide, rising to great heights,—perhaps 200 summits exceeding 18,000 feet, and the greatest reaching more than 28,000. The general base on which these peaks rest rises gradually to 9000 or 10,000 feet, where it abuts on the great plateau north of the range. The character of the country to the south of this plateau is much better known than that to the north. If a circle be drawn around A with a radius $=5^{\circ}\cdot046$ (the value of α_{21}), it will pass over the highest part of this plateau, just taking in Leh or Ladak within its compass. The general features of the country within this circle may be learnt from the Survey and other maps of India. This portion, then, of the Enclosed Space I shall call the *Known Region*, and the whole that lies beyond it the *Doubtful Region*. By introducing an arbitrary factor into the calculations I shall be able to separate the effects of these two divisions, and to gather what influence our uncertainty in the Doubtful Region has upon the total result.

36. These two divisions join on the great plateau. On this plateau are H'Lassa, according to HUMBOLDT 9590 feet above the sea; Gertrope, 10,500; and Leh or

Ladak, 9995; but this last is "in a hollow, the surrounding plateau rising to 13,430 feet." Iskardo is 6300 feet; but "south of Iskardo the plateau Deotsuh rises to 11,977 feet." "There are, properly speaking," HUMBOLDT observes, "very few plains [in this part of the Enclosed Space now under consideration]; the most considerable are those between Gertope, Daba, Schang-thung, the native country of the shawl goat, and Shipke (10,450 feet),—those around Ladak, which have an elevation of 13,430 English feet, and [as noticed above] must not be confounded with the depression in which the town is situated,—and lastly, the plateau of the sacred lakes of Manasa and Ravanahrada (probably 15,000 feet)...... From many carefully collected measurements of elevation I think I may say," he adds, "that the plateau of Thibet, between 73° and 85° east longitude, does not reach a mean height of 1800 toises, or 11,510 English feet." In addition to these observations, I may add, that the Survey Map of India, No. 65, shows that the bed of the Sutledge at the point marked S in fig. 4 is 10,792 feet.

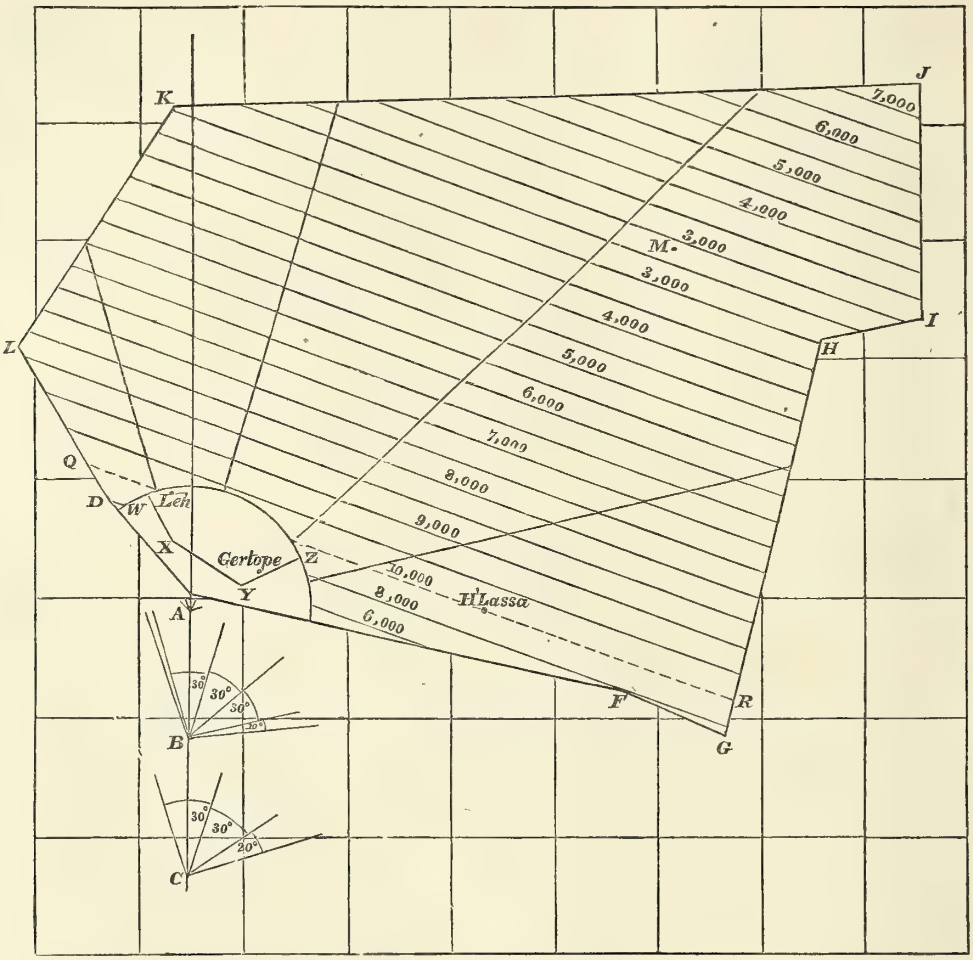
As the result of these data, I shall take the elevation of the dotted line QR, marked in fig. 4, and passing through Leh and H'Lassa, to be 10,000 feet, the greater portion being 11,000 or more, and the extremities somewhat less. This will be rather under the mark than above it.

37. With regard to the parts of the Enclosed Space further removed, HUMBOLDT observes,—"The mean height of this part of Gobi [between the sources of the river Selenga, lat. 50°, long. 102°, and the great wall of China, 600 geographical miles] is barely 4,264 English feet......Enghi is half-way, and is only about 2558 feet." I shall therefore take the point M, which represents the situation of Enghi, to be 2560 feet. I gather from the general account of the country between the great plateau of Thibet and the parts about Enghi and beyond that place, that, although it varies in its surface, it has a general slope down to Enghi, and then a rise again to the mountains. In describing the parts to the north of NP, the Kuen-Luen range, HUMBOLDT states, that between that and the Thian-Schan ranges there is a considerable depression. "CARL ZIMMERMAN," he remarks, "has made it appear extremely probable, that the Tarim depression, that is, the desert between the mountain chains of Thian-Schan and Kuen-Luen, where the steppe-river Tarim-gol empties itself into the Lake of Lop, which used to be described as an Alpine lake, is hardly 1279 English feet above the level of the sea." Again, he informs us that the line KL (in fig. 4), though once supposed to be a chain of mountains, is a line of broken country; and the west of this line we know to be lowland.

38. Guided, then, by such data as I have been able to gather, I assume, as the best general representation of the facts, that the Doubtful Region of the Enclosed Space to the north of QR, and outside the circle about A which bounds the Known Region, slopes gradually from 10,000 feet down to 2560 at M, and then rises again at the same angle to J. And I shall assume that the parts of the same region to the south of QR, and not included in the Known Region, slopes at four times that rate. In

fig. 5 I have laid down the Enclosed Space on a scale, and, for convenience of reference, drawn parallel lines over the whole of the Doubtful Region, marking the elevations of the various parts according to the assumed law.

Fig. 5.



39. The width of the lunes by which the enclosed space is to be traversed I shall make 30°, as this will lead to no material error by making the compartments too wide to obtain a fair average of their height (see art. 16.). In a closer approximation, however, it would be desirable to make the lunes more numerous and narrower. At each of the stations A, B, C, I shall make the middle line of one lune coincide with the meridian. On the right and left limits of the Enclosed Space there will be lunes of a less width than 30°. In short, the lunes will be as follows:—

At A, lune I., width $\beta=24°$, azimuth of middle line $=27°$ west.
At A, lune II., width $\beta=30°$, azimuth of middle line $=0°$.
At A, lune III., width $\beta=30°$, azimuth of middle line $=30°$ east.
At A, lune IV., width $\beta=30°$, azimuth of middle line $=60°$ east.
At A, lune V., width $\beta=20°$, azimuth of middle line $=85°$ east.

These will take in the whole Enclosed Space, both Known and Doubtful Regions. In like manner for the other stations:—

At B, lune I., width $\beta=8°$, azimuth of middle line $=19°$ west.
At B, lune II., width $\beta=30°$, azimuth of middle line $=0°$.
At B, lune III., width $\beta=30°$, azimuth of middle line $=30°$ east.
At B, lune IV., width $\beta=30°$, azimuth of middle line $=60°$ east.
At B, lune V., width $\beta=10°$, azimuth of middle line $=80°$ east.

At C, lune I., width $\beta=30°$, azimuth of middle line $=0°$.
At C, lune II., width $\beta=30°$, azimuth of middle line $=30°$.
At C, lune III., width $\beta=20°$, azimuth of middle line $=55°$ east.

40. By spherical trigonometry and an examination of the maps the following results have been obtained, from which it will be seen which of the quantities $h_0\, h_1\, h_2 \ldots.$ in the several lunes fall within the Enclosed Space (and therefore have any sensible effect), and which in the surrounding low country.

STATION A.

In lune I., from h_5 to h_{20} occur in Known Region, from h_{20} to h_{31} in Doubtful Region.
In lune II., from h_4 to h_{20} occur in Known Region, from h_{21} to h_{35} in Doubtful Region.
In lune III., from h_2 to h_{20} occur in Known Region, from h_{21} to h_{37} in Doubtful Region.
In lune IV., from h_1 to h_{20} occur in Known Region, from h_{21} to h_{40} in Doubtful Region.
In lune V., from h_3 to h_{20} occur in Known Region, from h_{21} to h_{34} in Doubtful Region.

STATION B.

In lune I., only h_{27} occurs in Known Region, from h_{28} to h_{34} in Doubtful Region.
In lune II., from h_{23} to h_{28} occur in Known Region, from h_{29} to h_{37} in Doubtful Region.
In lune III., from h_{22} to h_{26} occur in Known Region, from h_{27} to h_{39} in Doubtful Region.
In lune IV., none occur in Known Region, from h_{27} to h_{39} in Doubtful Region.
In lune V., none occur in Known Region, from h_{32} to h_{37} in Doubtful Region.

STATION C.

In lune I., from h_{30} to h_{33} occur in Known Region, from h_{34} to h_{40} in Doubtful Region.
In lune II., none occur in Known Region, from h_{32} to h_{40} in Doubtful Region.
In lune III., none occur in Known Region, from h_{33} to h_{38} in Doubtful Region.

41. The following is the method I have pursued to determine the heights of the masses standing on the successive compartments which fall within the Known and Doubtful Regions. For the Known Region, I marked out the lunes and their middle lines upon the Survey Maps, and such others as I could procure for regions beyond them, by stretching threads upon them; and for the Doubtful Region I used my own diagram, fig. 5, where maps were not available. I then marked on a slip of paper for each map or diagram, according to the scale of each, the distances from the

attracted station of the middle points of the compartments of each lune, these distances being taken from the third column of numbers in art. 25 and 28. By laying the slip along the middle line of the lune, with one end at the station, with the greatest ease I dotted down on the map the centres of the compartments, and then by an inspection of the heights, recorded on the map, of ranges and especially of the beds of rivers, determined the average height of the mass on each compartment above the level of Station A. In using the diagram fig. 5, I counted the number of centres of compartments which occur, and marked the heights of the first and last lying on the slope, whence by summing an arithmetic series I obtained the sum of the whole heights, without taking the trouble of noting each of them down. The result of these measures is given in the following Tables. H_1 and H_2 represent the sums of the heights in *miles* in the Known and Doubtful Regions. The line E at the bottom of each gives the correction which should be applied to the values of $H \sin \frac{1}{2}\beta \cos Az$, in any given lune, for either the Known or Doubtful Region, if the heights in that lune are all increased or diminished by 100 feet.

Table I.—Station A. (Kaliana), *Known Region.*

Heights of the masses in the compartments above A.	Lune I.	Lune II.	Lune III.	Lune IV.	Lune V.
h_1	....	....	....	600 feet	....
h_2	....	....	500 feet	600 feet	....
h_3	....	....	600 feet	1200 feet	300 feet
h_4	....	500 feet	1500 feet	800 feet	300 feet
h_5	300 feet	800 feet	1500 feet	1500 feet	300 feet
h_6	500 feet	1000 feet	1200 feet	1600 feet	300 feet
h_7	800 feet	2000 feet	600 feet	1600 feet	800 feet
h_8	1300 feet	3000 feet	2000 feet	1800 feet	1000 feet
h_9	2300 feet	3000 feet	2500 feet	2500 feet	1500 feet
h_{10}	1800 feet	5000 feet	3000 feet	6000 feet	2000 feet
h_{11}	300 feet	2500 feet	5500 feet	7000 feet	2500 feet
h_{12}	800 feet	4000 feet	7000 feet	7000 feet	3000 feet
h_{13}	800 feet	4000 feet	8500 feet	9000 feet	2500 feet
h_{14}	800 feet	5000 feet	9500 feet	9000 feet	2500 feet
h_{15}	800 feet	5500 feet	9000 feet	9000 feet	2500 feet
h_{16}	800 feet	6000 feet	9500 feet	9000 feet	2500 feet
h_{17}	1300 feet	7000 feet	9500 feet	9500 feet	2500 feet
h_{18}	1800 feet	7500 feet	9500 feet	9500 feet	3000 feet
h_{19}	2800 feet	8300 feet	9500 feet	9500 feet	3000 feet
h_{20}	4800 feet	9500 feet	9500 feet	9500 feet	3000 feet
Sum = ...	22,000 feet	74,600 feet	100,400 feet	106.200 feet	33,500 feet
H_1=	4·167 miles	14·129 miles	19·015 miles	20·114 miles	6·344 miles
$H_1 \sin \frac{1}{2}\beta$=	0·866 mile	3·656 miles	4·921 miles	5·202 miles	1·163 mile
$H_1 \sin \frac{1}{2}\beta \cos Az.$ =	0·772 mile	3·656 miles	4·262 miles	2·602 miles	0·095 mile
$H_1 \sin \frac{1}{2}\beta \sin Az.$ =	−0·393 mile	0·000 mile	2·460 miles	4·223 miles	1·057 mile
E=....................	0·05614 mile	0·08331 mile	0·08066 mile	0·04901 mile	0·00515 mile

TABLE II.—Station A. (Kaliana), *Doubtful Region.*

<table>
<tr><th>Heights of the masses in the compartments above A.</th><th>Lune I.</th><th>Lune II.</th><th>Lune III.</th><th>Lune IV.</th><th>Lune V.</th></tr>
<tr><td>h_{21}</td><td rowspan="3">Three heights, from 7000 feet to 9000 feet</td><td rowspan="15">Fifteen heights, from 8700 feet to 2000 feet</td><td rowspan="15">Fifteen heights, from 8500 feet to 1500 feet</td><td>7000 feet</td><td rowspan="9">Nine heights, from 3000 feet to 9000 feet</td></tr>
<tr><td>h_{22}</td><td rowspan="17">Seventeen heights, from 9000 feet to 1500 feet</td></tr>
<tr><td>h_{23}</td></tr>
<tr><td>h_{24}</td><td rowspan="8">Eight heights, from 9000 feet to 7000 feet</td></tr>
<tr><td>h_{25}</td></tr>
<tr><td>h_{26}</td></tr>
<tr><td>h_{27}</td></tr>
<tr><td>h_{28}</td></tr>
<tr><td>h_{29}</td></tr>
<tr><td>h_{30}</td><td rowspan="5">Five heights, from 9000 feet to 6000 feet</td></tr>
<tr><td>h_{31}</td></tr>
<tr><td>h_{32}</td><td>.........</td></tr>
<tr><td>h_{33}</td><td>.........</td></tr>
<tr><td>h_{34}</td><td>.........</td></tr>
<tr><td>h_{35}</td><td>.........</td><td>.........</td></tr>
<tr><td>h_{36}</td><td>.........</td><td>.........</td><td>1500 feet</td><td>.........</td></tr>
<tr><td>h_{37}</td><td>.........</td><td>.........</td><td>3500 feet</td><td>.........</td></tr>
<tr><td>h_{38}</td><td>.........</td><td>.........</td><td>.........</td><td>.........</td></tr>
<tr><td>h_{39}</td><td>.........</td><td>.........</td><td>.........</td><td>1500 feet</td><td>.........</td></tr>
<tr><td>h_{40}</td><td>.........</td><td>.........</td><td>.........</td><td>4000 feet</td><td>.........</td></tr>
<tr><td>Sum = ...</td><td>88,000 feet</td><td>80,250 feet</td><td>80,000 feet</td><td>101,750 feet</td><td>91,500 feet</td></tr>
<tr><td>H_2= ....</td><td>16·667 miles</td><td>15·199 miles</td><td>15·152 miles</td><td>19·271 miles</td><td>17·329 miles</td></tr>
<tr><td>$H_2 \sin\frac{1}{2}\beta$= ....</td><td>3·383 miles</td><td>3·933 miles</td><td>3·981 miles</td><td>4·986 miles</td><td>3·008 miles</td></tr>
<tr><td>$H_2 \sin\frac{1}{2}\beta \cos \text{Az.}$ =</td><td>3·009 miles</td><td>3·933 miles</td><td>3·395 miles</td><td>2·493 miles</td><td>0·235 mile</td></tr>
<tr><td>$H_2 \sin\frac{1}{2}\beta \sin \text{Az.}$ =</td><td>−1·536 mile</td><td>0·000 mile</td><td>1·960 mile</td><td>4·318 miles</td><td>2·989 miles</td></tr>
<tr><td>E= ....</td><td>0·03633 mile</td><td>0·06577 mile</td><td>0·07217 mile</td><td>0·04901 mile</td><td>0·00458 mile</td></tr>
</table>

TABLE III.—Station B. (Kalianpur), *Known Region.*

Heights of the masses in the compartments above A.	Lune I.	Lune II.	Lune III.	Lune IV.	Lune V.
h_1	....	....	....	....	....
h_2	....	....	....	....	....
h_3	....	....	....	....	....
h_4	....	....	....	....	....
h_5	....	....	....	....	....
h_6	....	....	....	....	....
h_7	....	....	....	....	....
h_8	....	....	....	....	....
h_9	....	....	....	....	....
h_{10}	....	....	....	....	....
h_{11}	....	....	....	....	....
h_{12}	....	....	....	....	....
h_{13}	....	....	....	....	....
h_{14}	....	....	....	....	....
h_{15}	....	....	....	....	....
h_{16}	....	....	....	....	....
h_{17}	....	....	....	....	....
h_{18}	....	....	....	....	....
h_{19}	....	....	....	....	....
h_{20}	....	....	....	....	....
h_{21}	....	....	....	....	....
h_{22}	....	....	500 feet	....	....
h_{23}	....	500 feet	2500 feet	....	....
h_{24}	....	2500 feet	7000 feet	....	....
h_{25}	....	4500 feet	9000 feet	....	....
h_{26}	....	6500 feet	9500 feet	....	....
h_{27}	1200 feet	8200 feet	....	....	....
h_{28}	....	9500 feet	....	....	....
Sum = ...	1200 feet	31,700 feet	28,500 feet	....	....
$H_1=$	0·227 mile	6·186 miles	5·398 miles	....	....
$H_1 \sin\frac{1}{2}\beta=$	0·016 mile	1·600 mile	1·387 mile	....	....
$H_1 \sin\frac{1}{2}\beta \cos \text{Az.} =$	0·015 mile	1·600 mile	1·211 mile	....	....
$H_1 \sin\frac{1}{2}\beta \sin \text{Az.} =$	−0·005 mile	0·000 mile	0·698 mile	....	....
$E=$	0·00125 mile	0·02941 mile	0·02123 mile	....	....

TABLE IV.—Station B. (Kalianpur), *Doubtful Region.*

<table>
<tr><th>Heights of the masses in the compartments above A.</th><th>Lune I.</th><th>Lune II.</th><th>Lune III.</th><th>Lune IV.</th><th>Lune V.</th></tr>
<tr><td>h_{27}</td><td>.........</td><td>.........</td><td rowspan="11">Eleven heights, from 9000 feet to 1500 feet</td><td rowspan="4">Four heights, from 4000 feet to 9000 feet</td><td>.........</td></tr>
<tr><td>h_{28}</td><td>6000 feet</td><td>.........</td><td>.........</td></tr>
<tr><td>h_{29}</td><td>9000 feet</td><td rowspan="9">Nine heights, from 9000 feet to 2500 feet</td><td>.........</td></tr>
<tr><td>h_{30}</td><td rowspan="5">Five heights, from 9000 feet to 6700 feet</td><td>.........</td></tr>
<tr><td>h_{31}</td><td rowspan="8">Eight heights, from 9000 feet to 3000 feet</td><td>.........</td></tr>
<tr><td>h_{32}</td><td rowspan="3">Three heights, from 5000 feet to 9000 feet</td></tr>
<tr><td>h_{33}</td></tr>
<tr><td>h_{34}</td></tr>
<tr><td>h_{35}</td><td>.........</td><td>8250 feet</td></tr>
<tr><td>h_{36}</td><td>.........</td><td>7500 feet</td></tr>
<tr><td>h_{37}</td><td>.........</td><td>.........</td></tr>
<tr><td>h_{38}</td><td>.........</td><td>.........</td><td>2000 feet</td><td>.........</td></tr>
<tr><td>h_{39}</td><td>.........</td><td>.........</td><td>3000 feet</td><td>.........</td><td>.........</td></tr>
<tr><td>Sum= ....</td><td>54,250 feet</td><td>51,750 feet</td><td>62,750 feet</td><td>74,000 feet</td><td>36,750 feet</td></tr>
<tr><td>H_2= ....</td><td>10·275 miles</td><td>9·801 miles</td><td>11·794 miles</td><td>14·015 miles</td><td>6·960 miles</td></tr>
<tr><td>$H_2 \sin \frac{1}{2}\beta$= ....</td><td>0·716 mile</td><td>2·536 miles</td><td>3·052 miles</td><td>3·626 miles</td><td>0·060 mile</td></tr>
<tr><td>$H_2 \sin \frac{1}{2}\beta \cos$ Az. =</td><td>0·677 mile</td><td>2·536 miles</td><td>2·643 miles</td><td>1·813 mile</td><td>0·011 mile</td></tr>
<tr><td>$H_2 \sin \frac{1}{2}\beta \sin$ Az. =</td><td>— 0·238 mile</td><td>0·000 mile</td><td>1·526 mile</td><td>3·141 miles</td><td>0·059 mile</td></tr>
<tr><td>E= ....</td><td>0·00874 mile</td><td>0·04412 mile</td><td>0·05519 mile</td><td>0·02941 mile</td><td>0·00143 mile</td></tr>
</table>

Table V.—Station C. (Damargida), *Known Region.*

Heights of the masses in the compartments above A.	Lune I.	Lune II.	Lune III.
h_1	....	....	....
h_2	....	....	....
h_3	....	....	....
h_4	....	....	....
h_5	....	....	....
h_6	....	....	....
h_7	....	....	....
h_8	....	....	....
h_9	....	....	....
h_{10}	....	....	....
h_{11}	....	....	....
h_{12}	....	....	....
h_{13}	....	....	....
h_{14}	....	....	....
h_{15}	....	....	....
h_{16}	....	....	....
h_{17}	....	....	....
h_{18}	....	....	....
h_{19}	....	....	....
h_{20}	....	....	....
h_{21}	....	....	....
h_{22}	....	....	....
h_{23}	....	....	....
h_{24}	....	....	....
h_{25}	....	....	....
h_{26}	....	....	....
h_{27}	....	....	....
h_{28}	....	....	....
h_{29}	....	....	....
h_{30}	600 feet	....	....
h_{31}	4000 feet	....	....
h_{32}	7300 feet	....	....
h_{33}	9500 feet	....	....
Sum = ...	21,400 feet	....	....
$H_1=$	4·530 miles	....	....
$H_1 \sin \frac{1}{2}\beta=$	1·173 mile	....	....
$H_1 \sin \frac{1}{2}\beta \cos \text{Az.}=$ ...	1·173 mile	....	....
$H_1 \sin \frac{1}{2}\beta \sin \text{Az.}=$ ...	0·000 mile	....	....
E=......................	0·01961 mile	....	....

Table VI.—Station C. (Damargida), *Doubtful Region.*

Heights of the masses in the compartments above A.	Lune I.	Lune II.	Lune III.
h_{31}	....	6000 feet	....
h_{32}	....	Eight heights, from 9000 feet to 1500 feet	....
h_{33}	....		7000 feet
h_{34}	Seven heights, from 9000 feet to 2500 feet		Five heights, from 9000 feet to 5500 feet
h_{35}			
h_{36}			
h_{37}			
h_{38}			
h_{39}			....
h_{40}		2000 feet	....
Sum = ...	40,250 feet	50,000 feet	43,250 feet
$H_2=$	7·623 miles	9·470 miles	8·192 miles
$H_2 \sin\frac{1}{2}\beta=$	1·973 mile	2·451 miles	1·422 mile
$H_2 \sin\frac{1}{2}\beta \cos \text{Az.}=$ ...	1·973 mile	2·103 miles	0·816 mile
$H_2 \sin\frac{1}{2}\beta \sin \text{Az.}=$ ...	0·000 mile	1·225 mile	1·165 mile
E=	0·03432 mile	0·04220 mile	0·01143 mile

42. The following results are gathered from these Tables:—

			Known Region.	Doubtful Region.
Station A.	$H \sin\frac{1}{2}\beta \cos \text{Az.}$,	Lune I.	0·772	3·009
	"	Lune II.	3·656	3·933
	"	Lune III.	4·262	3·395
	"	Lune IV.	2·602	2·493
	"	Lune V.	0·095	0·235
		Totals	11·387	13·065
	E	Lune I.	0·05614	0·03633
	"	Lune II.	0·08331	0·06577
	"	Lune III.	0·08066	0·07217
	"	Lune IV.	0·04901	0·04901
	"	Lune V.	0·00515	0·00458
		Totals	0·27427	0·22786
	$H \sin\frac{1}{2}\beta \sin \text{Az.}$,	Lune I.	−0·393	−1·536
	"	Lune II.	0·000	0·000
	"	Lune III.	2·460	1·960
	"	Lune IV.	4·223	4·318
	"	Lune V.	1·057	2·989
		Totals	7·347	7·731

Multiplying these several totals by $1''\cdot1392$ (see formula 6. in art. 22.), we have the following results:—

Deflection of plumb-line in meridian	$12''\cdot972$	$14''\cdot881$
Correction of same for every 100 feet of change in heights ...	0·312	0·260
Deflection of plumb-line in prime vertical	8·136	8·806

			Known Region.	Doubtful Region.
STATION B.	$K \sin \frac{1}{2}\beta \cos Az.$,	Lune I.	0·015	0·677
	"	Lune II.	1·600	2·536
	"	Lune III.	1·211	2·643
	"	Lune IV.	0·000	1·813
	"	Lune V.	0·000	0·011
		Totals	2·826	7·680
	E	Lune I.	0·00125	0·00874
	"	Lune II.	0·02941	0·04412
	"	Lune III.	0·02123	0·05519
	"	Lune IV.	0·00000	0·02941
	"	Lune V.	0·00000	0·00143
		Totals	0·05189	0·13889
	$H \sin \frac{1}{2}\beta \sin Az.$,	Lune I.	−0·005	−0·238
	"	Lune II.	0·000	0·000
	"	Lune III.	0·698	1·526
	"	Lune IV.	0·000	3·141
	"	Lune V.	0·000	0·059
		Totals	0·693	3·488

Multiplying the several totals by $1''\cdot 1392$ as before, we have—

Deflection of the plumb-line in meridian	3″·219	8″·749
Correction of the same for every 100 feet of change in heights...	0·059	0·158
Deflection of the plumb-line in prime vertical	0·789	3·974

STATION C.	$H \sin \frac{1}{2}\beta \cos Az.$,	Lune I.	1·173	1·973
	"	Lune II.	0·000	2·103
	"	Lune III.	0·000	0·816
		Totals	1·173	4·892
	E	Lune I.	0·01961	0·03432
	"	Lune II.	0·00000	0·04220
	"	Lune III.	0·00000	0·01143
		Totals	0·01961	0·08795
	$H \sin \frac{1}{2}\beta \sin Az.$,	Lune I.	0·000	0·000
	"	Lune II.	0·000	1·225
	"	Lune III.	0·000	1·165
		Totals	0·000	2·390

Multiplying the several totals by $1''\cdot 1392$, we have—

Deflection of plumb-line in meridian	1″·336	5″·573
Correction of same for every 100 feet of change in heights ...	0·022	0·100
Deflection of plumb-line in prime vertical	0·000	2·723

43. Adding together the results of the last article, we have—

Deflection of plumb-line in meridian at A . . . $=27''.853$
Deflection of plumb-line in meridian at B . . . $=11.968$
Deflection of plumb-line in meridian at C . . . $=\ 6.909$
Deflection of plumb-line in prime vertical at A$=16.942$
Deflection of plumb-line in prime vertical at B$=\ 4.763$
Deflection of plumb-line in prime vertical at C$=\ 2.723$

∴ Difference of meridian deflections at A and B$=15.885$
Difference of meridian deflections at A and C$=20.944$
Difference of meridian deflections at B and C$=\ 5.059$

The first of these quantities is considerably greater than $5''.236$, the quantity brought to light by the Indian Survey. And the values of the deflections at B and C bear a far higher ratio to those at A than has been generally supposed. For even Kaliana was selected by Colonel Everest in the expectation that it would be beyond the sensible influence of mountain attraction; whereas even at C the deflection in the meridian$=6''.909$, if the heights have been rightly assigned in this approximation. In the following articles I shall examine these values more minutely, and consider the effect of various hypotheses for reducing them. In the mean time I will write down the following results from these values for the deflections:—

Total deflection at A$=32.601$, and in azimuth . . . . $31^\circ\ 18'$ East
Total deflection at B$=12.880$, and in azimuth . . . . 21 42 East
Total deflection at C$=\ 7.426$, and in azimuth . . . . 21 31 East*

44. In attempting to reduce the amount of deflection deduced by these calculations, the first thought that comes to mind is, that the density of the attracting mass may have been chosen too large. I have made it 2·75 of distilled water, which is that which was assigned as the mean density of the mountain Schehallien in the calculations of Maskelyne. This can hardly be too great; for, as I shall soon show, a very large share of the deflection is produced by the attraction of the elevated plateau which lies in Thibet and south of that country; and as this is on an average 10,000 feet or more high, the lower part of the materials must be denser rather than lighter than those of a mountain of inconsiderable altitude. If, however, we do reduce the density, say to 2·25, which is yielding much, still the deflections and their differences are reduced by only one-fifth part, and therefore this will not solve the difficulty.

45. The next thought is, that I may have assigned too great a mass to the Doubt-

* Some idea may be formed of the amount of these deflections from the following representation. Conceive three hemispherical mountains of granite to exist close to the three stations A, B, C, their bases being horizontal and just coming up to the stations, and the centres of the bases bearing respectively 31° 18′, 21° 42′, 21° 31′ east of the north meridian. That the horizontal attraction of these hemispherical mountains on the plumb-line at the three stations may be equal to the attraction of the Himalayas and the regions beyond, the diameters of their bases must be respectively 5, 2, $1\frac{1}{6}$ miles very nearly.

ful Region. This I will now examine. All the results which depend upon this part I will multiply by a factor $1-x$. If $x=0$, the results will stand as they do now. If $x=1$, this will amount to supposing the mass standing on the whole Doubtful Region to be non-existent, an hypothesis clearly impossible. By giving x any intermediate fractional value we shall be supposing that all the heights, and therefore the whole mass, are reduced in that ratio. Let A, B, C represent the deflections in meridian at the three stations A, B, and C. Then from art. 42. we gather—

$$\begin{aligned}
A &= 12''{\cdot}972+(1-x)\,14''{\cdot}881 \\
&= 27{\cdot}853- \quad 14{\cdot}881\,x \\
B &= 3{\cdot}219+(1-x)\,8{\cdot}749 \\
&= 11{\cdot}968- \quad 8{\cdot}749\,x \\
C &= 1{\cdot}336+(1-x)\,5{\cdot}573 \\
&= 6{\cdot}909- \quad 5{\cdot}573\,x; \\
\therefore \quad A-B &= 15{\cdot}885- \quad 6{\cdot}132\,x \\
A-C &= 20{\cdot}944- \quad 9{\cdot}308\,x \\
B-C &= 5{\cdot}059- \quad 3{\cdot}176\,x.
\end{aligned}$$

These show that the extravagant hypothesis of supposing $x=1$, or that the whole mass on what we have called the Doubtful Region is non-existent, will not reduce the difference of deflections at A and B lower than $9''{\cdot}753$, which is greater than $5''{\cdot}236$ in the ratio of 13 : 7. Nor will this even come down sufficiently if we reduce also the density of the remaining mass, that on the Known Region.

46. A third means of reduction may be looked for in the Known Region. By examining the results gathered together in art. 42, it will be seen that the chief part of the meridian attraction of the mass on the Known Region arises from the lunes II. III. and IV. for A, and lunes II. and III. for B. By attentively examining these columns in Tables I. and III. in art. 41, we see that a large portion of the attraction arises from the Great Plateau. The result I arrive at is, that of the deflection $12''{\cdot}972$ at A, as much as $8''{\cdot}772$ arises from this plateau; and of the $3''{\cdot}219$ at B, as much as $2''{\cdot}010$ arises from the same cause. Hence if $1-y$ be an arbitrary factor,

$$\begin{aligned}
&A \quad (\text{Known Region}) = 4''{\cdot}200+(1-y)\,8''{\cdot}772 \\
&B \quad (\text{Known Region}) = 1{\cdot}209+(1-y)\,2{\cdot}010 \\
&A-B \ (\text{Known Region}) = 2{\cdot}991+(1-y)\,6{\cdot}762.
\end{aligned}$$

It will be necessary, then, to cut down the height of the plateau as much as 6000 feet, to make this come down to $5''{\cdot}236$; or, if we suppose that all the heights in the other part of the Known Region are twice too large, and if we therefore replace $2''{\cdot}991$ by its half, $1''{\cdot}496$, even then y must equal 0·55, and the elevation of the plateau* above the sea be reduced from 10,000 feet to 6000 feet. And all this *in addition to* the hypothesis of the non-existence of the whole mass on the Doubtful Region!

* I should mention to what extent I assume that this plateau is comprised within what I have called the Known Region. Let four points be marked down on the map, viz. W in lat. 34° and long. 76°, X in lat. 32° 45′

It appears, in short, to be quite hopeless by any admissible hypothesis to reduce the calculated deflection so as to make it tally with the error brought to light by the Survey. In the conclusion of this paper, however, it will appear that such a reduction is not necessary for reconciling the discrepancy.

47. I will here write down the formulæ in their most general shape. Let $1-z$ be a factor (similar to $1-x$ and $1-y$) for the part of the Known Region not including the Plateau. Suppose also the whole of the heights of the Known Region are reduced by a hundreds of feet, and those of the Doubtful Region by b hundreds (in this way I bring in the correction E at the foot of the six Tables). Then

$$A=(1-z)4''\cdot200+(1-y)8''\cdot772+(1-x)14''\cdot881-0''\cdot312a-0''\cdot260b.$$
$$B=(1-z)1''\cdot209+(1-y)2''\cdot010+(1-x)\ 8''\cdot749-0''\cdot059a-0''\cdot158b.$$
$$C=(1-z)0''\cdot729+(1-y)0''\cdot607+(1-x)\ 5''\cdot573-0''\cdot022a-0''\cdot100b.$$

The method of using these arbitrary symbols is this. If it appear on examining the contour of the earth's surface more carefully, that the heights in the Known Region and south of the space I call the Plateau, ought to be reduced in a certain ratio, we have but to give the value of that ratio to z: the same is the case, as I have already shown, with y and the Plateau itself. If, on the other hand, we do not wish to reduce the heights in the Known Region in a certain ratio, but by the same given quantity, z and y must be put $=0$, and $a=$ the number of hundreds of feet by which we wish to reduce. Thus if we wish to reduce the whole Known Region by 1000 feet in altitude, we must put $a=10$. We may, moreover, combine these methods of reduction, and both reduce the general ratio of the heights, and afterwards cut these reduced heights down by a constant quantity by giving z, y, and a the proper values. The same things may be done for the Doubtful Region by assigning proper values to x and b: so that the formulæ here given admits of adaptation to various hypotheses of reduction.

48. We may use these last formulæ for comparing the masses which stand on the Known and Doubtful Regions with each other, and with the mass of the earth. In doing this I shall suppose that the heights have been rightly assigned in the present paper. Therefore $z=0$, $y=0$, $x=0$. Then if $a=41\cdot58$, the part of A which arises from the Known Region is reduced to zero. Hence 4158 feet is the average height of the mass standing on that part. In the same manner, by making $b=57\cdot30$, the part of A which arises from the Doubtful Region vanishes; and therefore the average height of the whole mass standing on that portion of the Enclosed Space is 5730 feet. It will be observed that this is greater than the former. The obvious reason of this is, that the mass of the Known Region is highest at its furthest parts from A, whereas the reverse is the case with the mass of the Doubtful Region. The superficial extent of each

and long. 76° 45′, Y in lat. 30° 30′ and long. 80°, and Z in lat. 31° 30′ and long. 83°. Join W and Z by a circle about Kaliana as centre, and join these and the other points by arcs of great circles of the sphere. I take the average height of the enclosed mass to be 10,000 feet above the sea, a mean altitude which I conceive is rather under than over the mark.

of these two regions is found by the following formula of spherical trigonometry:—
Area of portion of a lune

$$=\frac{\pi\beta r^2}{180}(\sin\alpha_n-\sin\alpha_m),$$

the portion being bounded by arcs of great circles at distances α_m and α_n from A, β the angle of the lune expressed in degrees, and r being the radius of the earth. The values of α_m and α_n for the several lunes are obtained from arts. 40. and 25. The results are, that the superficial extents of the Known and Doubtful Regions equal respectively $0{\cdot}1679616\,r^2$ and $0{\cdot}7555430\,r^2$ square miles, $r=4000$; and therefore putting the density = half that of the earth's mean density, and the respective heights =4158 feet or 0·7761 mile, and 5730 feet or 1·0852 mile, we have the following results:—

$$\frac{\text{mass on Known Region}}{\text{mass of the earth}}=\frac{3}{8\pi}\,\frac{0{\cdot}1679616\times0{\cdot}7761}{r}$$

$$=0{\cdot}000003890$$

$$\frac{\text{mass on Doubtful Region}}{\text{mass of the earth}}=\frac{3}{8\pi}\,\frac{0{\cdot}7555430\times1{\cdot}0852}{r}$$

$$=0{\cdot}000024367.$$

Hence, also, $$\frac{\text{mass on Doubtful Region}}{\text{mass on Known Region}}=6{\cdot}264,$$

or the mass on the Doubtful Region is greater than that on the Known Region in a ratio higher than 25 : 4.

Also $$\frac{\text{mass on whole Enclosed Space}}{\text{mass of the earth}}=0{\cdot}000028257.$$

49. By means of this last result we can determine at what distances from A, B, and C, the whole attracting mass must be imagined concentrated in a point, so as to produce the deflections at those three stations found in art. 43. If, instead of multiplying the totals in art. 42. by $1''{\cdot}1392$, we multiply them by $0{\cdot}000005523\,g$ (see art. 21.), we shall have the attractions in terms of gravity. Hence

$$\begin{aligned}&\text{attraction at A in meridian} &&=24{\cdot}452\times0{\cdot}000005523\,g;\\ &\text{attraction at A in prime vertical} &&=15{\cdot}078\times0{\cdot}000005523\,g;\\ \therefore\quad &\text{total attraction on A}\ \ldots &&=28{\cdot}72\ \times0{\cdot}000005523\,g\\ & &&=0{\cdot}00015862\,g.\end{aligned}$$

In the same way I find

$$\begin{aligned}&\text{Total attraction on B}\ \ldots &&=0{\cdot}00006245\,g.\\ &\text{Total attraction on C}\ \ldots &&=0{\cdot}0000360\,g.\end{aligned}$$

Hence $$\frac{\text{distance from A of point of concentration}}{\text{radius of earth}}$$

$$=\sqrt{\frac{\text{attracting mass}}{\text{mass of earth}}\times\frac{1}{0{\cdot}00015862}}$$

$$=\sqrt{\frac{0{\cdot}000028257}{0{\cdot}00015862}}={\cdot}422;$$

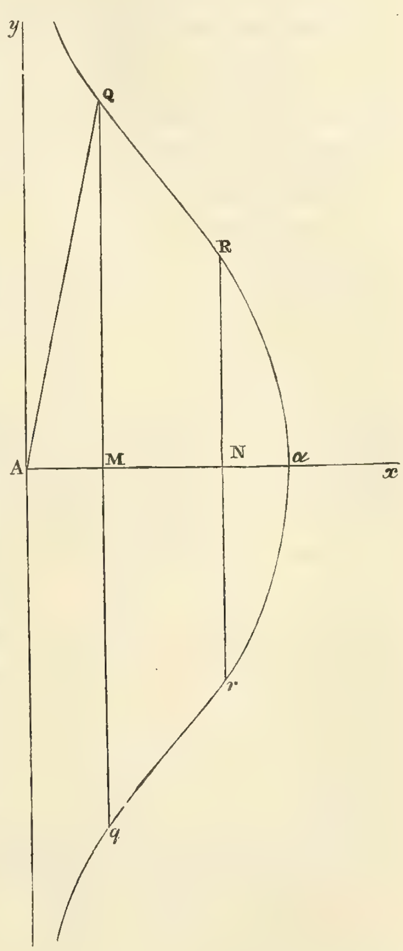
$\therefore$ distance from A of point of concentration =1688 miles.

In the same manner it may be shown that

distance from B of the point of concentration $=2692$ miles;
distance from C of the point of concentration $=3544$ miles.

The differences of these two last from the first are far greater than the distances of B and C from A, viz. $5° 23' 37''$ and $11° 27' 33''$. From this it is easily inferred, what indeed did not need this proof, that the mass in no sense whatever, even an approximate one, attracts as if concentrated in a fixed point.

Fig. 6.



50. It is extremely difficult to obtain a simple law of attraction, even an approximate one, of such a mass as that under consideration. I have, however, arrived at one which appears to represent the facts with a considerable degree of exactness, and by which we can interpolate for the amount of deflection at any station of the arc intervening between Kaliana and Damargida. It depends upon the properties of the curve of which the equation is $y^2=\frac{a^4}{x^2}-x^2$. Let Ax, Ay be the axes of x and y, and Qaq be the curve; the axis of y is an asymptote, and the curve cuts the axis of x at right angles at a distance A$a=a$. The property of this curve which I am about to use is as follows. The attraction upon A of any slender prism of matter Qq, parallel to the axis of y and terminated at both ends by the curve, is the same as if the mass of the prism were concentrated in the point a. This property is easily demonstrated. For by the process pursued in the proof of the Lemma (art. 13.), it can be shown that the attraction on A of the half prism QM in the direction AM

$$=\frac{\text{mass of QM}}{\text{AM.AQ}};$$

$$\therefore \text{ attraction of Q}q\text{ on A in AM}=\frac{\text{mass of Q}q}{\text{AM.AQ}}$$

$$=\frac{\text{mass of Q}q}{x\sqrt{x^2+y^2}}=\frac{\text{mass of Q}q}{a^2}$$

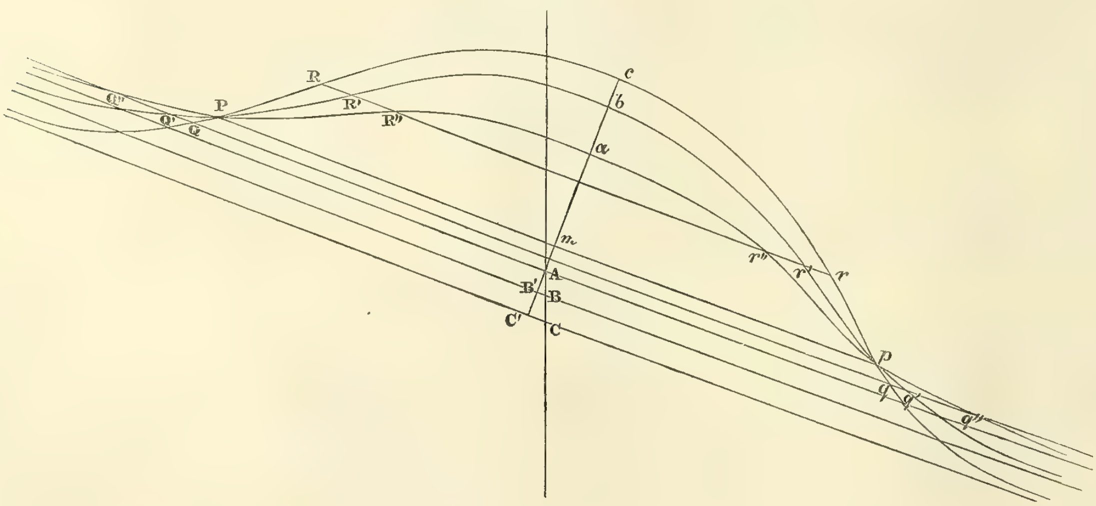
by the equation to the curve. This demonstrates the property. The same, then, is true of any line parallel to y: and it follows easily that a mass of the form QRrq and of uniform thickness will attract the point A as if concentrated in a. But more than this; the property is true also for any prisms parallel to y and lying above the plane xy: so that the mass lying on QRrq need not be of uniform thickness, but may vary in any manner whatever, so long as its height is always small and the same at the same distance from the axis of y.

The property, then, in its most general form is this. If a mass lie on the space QRrq, always of small comparative altitude and having its section perpendicular to the axis of y always the same (so far as it falls within the curve), the section itself being of any conceivable form whatever, then the attraction of the mass on A is precisely the same as if it were concentrated in the point a (I should mention that in drawing the figure I have for convenience made the dimensions in y one-third of the size they should be compared to those of x).

51. In the previous articles we have seen that by far the largest part of the attraction, at the three stations, arises from the elevated regions lying parallel to the line QR in figs. 4 and 5, running from west-north-west to east-south-east. If, then, we take A in the figure of last article to be Kaliana, and Ax in the north-north-east direction, we may make the parts of QRrq nearest A coincide as nearly as possible with the attracting mass of the Himalayas and the regions beyond, by giving the vertical section MN the right form: and the parts of this mass towards the extremities, and therefore furthest from A, will have a less and less effect in the direction Ax, both on account of their increasing distance and the larger angle at which they act with Ax. Hence we can easily conceive that such a form can be given to the transverse section MN as to make the mass on QRrq *in its effect in the direction* Ax a very fair representative of the actual mass, producing the deflection of the plumb-line at Kaliana. This being the case, the property I have demonstrated, together with the result arrived at in art. 49, shows that Aa, or a in the equation to the curve, $=1688$ miles.

52. By taking different values for a in the curve we may find curves differing in their dimensions, but possessing the same property.

Fig. 7.



In the accompanying figure (fig. 7) three such curves are drawn having a common ordinate Pp, so that the three masses on QRrq, Q$'$R$'r'q'$, and Q$''$R$''r''q''$ are of the same length and coincide precisely with each other, except in a very trifling degree at the two extremities. AB$'$C$'$ are the three attracted points for the three curves: *abc* are the three points where the attracting mass may be conceived to be concentrated, corresponding respectively to AB$'$C$'$. ABC is the meridian through the three stations Kaliana, Kalianpur, and Damargida. It is evident that the attractions of the mass on A, B, and C, in the direction parallel to Aa, will not differ from the attractions on A, B$'$, and C$'$ by any appreciable quantities.

Let A$a=a$, B$'b=b$, C$'c=c$. By art. 49, we have shown that $a=1688$ miles, $b=2692$, $c=3544$. I will now show the consistency of these results, as flowing from the property I have enunciated.

The Trigonometrical Survey shows that AB$=371$ miles, and BC$=430$. AB makes $22°\frac{1}{2}$ with AB$'$, and $\cos 22°\ 30'=0{\cdot}92388$. Hence AB$'=343$ miles, and AC$'=740$. Taking A as the origin of coordinates, and the axes as before, the equations to the three curves give (A$m=x$)

$$\mathrm{P}m^2=\frac{a^4}{x^2}-x^2,\ \mathrm{P}m^2=\frac{b^4}{(x+343)^2}-(x+343)^2$$

and

$$\mathrm{P}m^2=\frac{c^4}{(x+740)^2}-(x+740)^2.$$

If we put $a=1688$ and $b=2692$ in the first and second of these and equate them, the equation is satisfied by $x=222$ miles. And when this value of x is put in the equation formed by equating the first and third values of Pm^2, we obtain

$$c=\sqrt[4]{962^2\left\{\frac{1688^4}{222^2}-222^2+962^2\right\}}$$

$$=3559 \text{ miles.}$$

The very close agreement of this result with that obtained from the calculations of this paper, and shown in art. 49. to be 3544 miles, shows how exactly the law here deduced represents the facts of the case. The value of x, viz. 222 miles, places the line Pp on the part we have called the Plateau, running W.N.W. and E.S.E. about thirty miles north of Gertope.

53. The law thus developed by aid of the curve enables us to interpolate the amount of the deflection of the plumb-line at any station of the arc between Kaliana and Damargida. Thus suppose X is the distance of the station from the fixed line Pp, and A the distance of the centre of concentration for that station. Then

$$\frac{\mathrm{A}^4}{\mathrm{X}^2}-\mathrm{X}^2=\mathrm{P}m^2,\ \text{and this}=\frac{a^4}{222^2}-222^2\ ;$$

$$\therefore\ \frac{\mathrm{A}^4}{a^4}=\frac{\mathrm{X}^2}{222^2}\left\{1+\frac{(\mathrm{X}^2-222^2)222^2}{a^4}\right\}.$$

Since X is never greater than 962, and $a=1688$, the second term within the bracket will always be an extremely small fraction, which may be neglected;

$$\therefore \quad \frac{A^2}{a^2}=\frac{X}{222};$$

$$\therefore \quad \frac{\text{deflection of plumb-line at proposed station}}{\text{deflection of plumb-line at Kaliana}}$$

$$=\frac{\text{attraction at the station}}{\text{attraction at Kaliana}}=\frac{a^2}{A^2}=\frac{222}{X}.$$

But by art. 43. deflection at Kaliana $=32''601$;

$$\therefore \quad \text{deflection at the station in question} =\frac{7237''\cdot422}{X}.$$

If we had used the second and third values of Pm^2 above, we should have obtained from the results for Kalianpur and Damargida given in art. 43., the following:—

$$\text{deflection at the station in question} =\frac{7277''\cdot200}{X}.$$

$$\text{deflection at the station in question} =\frac{7143''\cdot812}{X}.$$

These three formulæ are very nearly the same. I shall adopt the mean of them; and in taking the mean I shall give the three their respective "weights," which are as the numbers 76, 30, 17, as the deflections in art. 43. show. This leads to the following formula:—

Deflection of plumb-line, at a station the distance of which from Pp is X miles,

$$=\frac{7235''}{X}.$$

54. We may obtain X in terms of the latitude. Let L be the latitude of the place, and l the latitude of Kaliana; then $X=222+(l-L)\cos 22^\circ\ 30'\times$ the number of miles in a degree of latitude at the centre of the Indian Arc. The value of the multiplier of $l-L$ which best suits my purpose is 63·06465. As this $=69 \cos 23^\circ\ 56'$, using this value shows that if the length of a degree in this latitude $=69$ miles, then the line Pp must run about the one-eighth of a point further from east and west than I have placed it, viz. W.N.W. and E.S.E. If the length of the degree be less than sixty-nine miles, then Pp will need a still smaller shift of position.

Substituting the above value, we have deflection at any station of which the latitude is $l-L$ degrees south of Kaliana

$$=\frac{7235''}{63\cdot06465(l-L)+222}=\frac{114''\cdot712}{l-L+3\cdot520}.$$

If we put $l-L=0$, $5^\circ\ 23'\ 37''$, and $11^\circ\ 27'\ 33''$ corresponding with the latitudes of Kaliana, Kalianpur, and Damargida, this formula gives for the three deflections at those stations $32''\cdot590$, $12''\cdot870$, and $7''\cdot668$. These so nearly accord with the values in art. 43, that the above formula will represent very approximately the deflection at any station on the whole arc.

55. It may easily be shown from the results of the foregoing articles, that the formula above amounts to supposing, that the Himalayas and the regions beyond attract the three stations and all intermediate stations on the arc between Kaliana and Damargida, as a bar of matter would, running about W.N.W. and E.S.E. at a distance of 222 miles from Kaliana, the length of the bar being infinite, its transverse section small, and its density such that the mass of the Himalayas and attracting regions beyond shall equal the mass of 1284 miles of its length.

56. In order to deduce from this formula for the total deflection the deflection in the meridian, which is the part of most importance, we ought to know the azimuth of the vertical plane in which the total deflection takes place. This is not the same for the three stations A, B, and C. By article 43. it appears that the azimuths at those three stations are 31° 18′, 21° 42′, and 21° 31′. After various trials I find that the following formula represents the law with sufficient exactness. If θ be the azimuth (measured from the north), then

$$\cos\theta = \frac{\cos 31^\circ\ 18'}{1-\frac{1}{10}\sin 10(l-\mathrm{L})}.$$

When $l-\mathrm{L}=0$, 5° 23′ 37″, and 11° 27′ 33″, this gives $\theta=31^\circ\ 18'$, 21° 35′ and 19° 58′. This last differs by 1° 33′ from the value in art. 43; but the second only by 7′. These are sufficiently near the truth, as the cosine in the extreme case will differ from the truth by only about $\frac{1}{100}$th part. The formula for the azimuth departs most from the truth when $l-\mathrm{L}=9^\circ$, that is, at a point about half a degree south of the middle point between Kalianpur and Damargida. But the form of the function is so chosen, that it does not vary much along the whole arc between those stations; and the above-mentioned departure amounts to only one-seventieth part of the proper value. Between Kaliana and Kalianpur I think the formula will represent the azimuth very exactly; and although below that not with the same exactness, yet to a degree of approximation which will introduce no error of importance in the value of the deflection in the meridian.

57. We may show this by combining this formula with that deduced for the total deflection in art. 54. Thus

Deflection in the meridian at any place of which the latitude is $l=$ total deflection $\times\cos\theta$

$$=\frac{114''\cdot712}{l-\mathrm{L}+3\cdot520}\times\frac{\cos 31^\circ\ 18'}{1-\frac{1}{10}\sin 10(l-\mathrm{L})}$$

$$=\frac{98''\cdot016}{(l-\mathrm{L}+3\cdot520)\left\{1-\frac{1}{10}\sin 10(l-\mathrm{L})\right\}}.$$

When $l-\mathrm{L}=0$, 5° 23′ 37″, and 11° 27′ 33″, this gives for the meridian deflections 27″·845, 11″·965, and 7″·207. These quantities, as calculated from the attracting mass in art. 43, are $27''\cdot853$, $11''\cdot968$ and $6''\cdot909$, which show how good an approximation the formula of this article gives.

58. Before proceeding to the conclusion of this paper I will gather together the formulæ which I have arrived at.

The deflections of the plumb-line at Kaliana, Kalianpur, and Damargida have been found to be as follows:—

In the meridian	$27''\cdot853$,	$11''\cdot968$,	$6''\cdot909$
In the prime vertical . .	$16''\cdot942$,	$4''\cdot763$,	$2''\cdot723$
Total deflections	$32''\cdot601$,	$12''\cdot880$,	$7''\cdot426$
In azimuths	$31^\circ\ 18'$,	$21^\circ\ 42'$,	$21^\circ\ 31'$.

The general formulæ including these results and the deflections for intermediate stations are,—

$$\text{total deflection} = \frac{114''\cdot712}{l-\mathrm{L}+3\cdot520};$$

and the azimuth in which its acts is given by

$$\cos\theta = \frac{\cos 31^\circ\ 18'}{1-\frac{1}{10}\sin 10(l-\mathrm{L})},$$

$$\text{deflection in meridian} = \frac{98''\cdot016}{(l-\mathrm{L}+3\cdot520)\left\{1-\frac{1}{10}\sin 10(l-\mathrm{L})\right\}}.$$

The formulæ for altering the deflections in meridian for any change in the heights of the attracting mass are brought together in art. 47. Similar formulæ might easily be calculated for the deflections in the prime vertical. If any change be made in the heights of the attracting mass, these formulæ will show what corrections must be introduced into the expressions for the total deflections and their azimuths given above, and also into the constants in the general formulæ.

Conclusion.

59. Before an arc can be made use of in the problem of the figure of the earth, we must know correctly two things concerning it,—its length and its amplitude. Of the two arcs I have been considering, viz. from Kaliana to Kalianpur, and from Kalianpur to Damargida, the correct lengths are known from the Survey; and, as shown in art. 6, these are altogether unaffected by mountain attraction. The same cannot be said of their amplitudes; and till they can be obtained correctly, the arcs can render no service to the great problem. But the amount of deflection in the plumb-line caused by mountain attraction having been determined, the amplitudes obtained astronomically may be corrected, and the arcs may take their place—and a very important place, owing to their length and the accuracy of the geodetic operations—in the investigation of the earth's form.

60. When the length and amplitude of an arc are known, the formula which I give

in art. 62. establishes a relation between the two quantities on which the figure of the earth, supposed to be a spheroid of revolution, depends, viz. the semi-axis major and the ellipticity. As there are two quantities to be determined, a single arc is not sufficient to enable us to find them, unless the lengths and amplitudes of portions of the arc, as well as of the whole, are known. Bessel, in a paper which has been translated in vol. ii. of Taylor's 'Scientific Memoirs,' has shown from ten arcs measured on various parts of the earth and from portions of five of them, by an application of the principle of least squares, that the mean ellipticity $=\frac{1}{300 \cdot 7}$; and that the semi-axis major $=3271953 \cdot 854$ toises, the length of the toise being to that of the fathom as $1 \cdot 06576542 : 1$. This result coincides almost exactly with the ellipticity which theory assigns, upon the supposition that the earth was once a fluid mass, its strata increasing in density from the surface to the centre, the density of the surface being that of granite, and the mean density being $5\frac{2}{3}$rds that of water—a fact which is generally considered to be a strong argument in favour of the original fluidity of the earth's mass.

61. But by the process described above, the peculiarities of the several arcs are all merged in the mean result. When the calculations for the separate arcs are examined, the values are found to vary on either side of the mean. This variety indicates that the several parts of the earth are not curved precisely according to the same elliptic law. Some may think that this militates against the original fluidity of the earth's mass. I do not think this is a fair inference. If the earth's surface ever were fluid, the science of geology shows us that it must have ceased to be so for many ages: and the interval affords time enough for the operation of that well-established law—that gradual changes of elevation and depression are unceasingly taking place in the surface, arising no doubt from chemical and mineralogical changes in the mass—to modify the original curvature of the various parts, making some greater and others less than before. The argument of the earth's original fluidity lies in the fact, that the *present mean form* is that which the earth must have had when it was fluid.

62. I will conclude this paper by calculating the form of the Indian arc between Kaliana and Damargida and its two subdivisions. The result affords, I think, the only explanation of the discordance between the difference of amplitude as brought out in Colonel Everest's work, and by my calculation of the amount of deflection.

The lengths of the three arcs, Kaliana—Kalianpur, Kalianpur—Damargida, and their sum, Kaliana—Damargida (which I shall call arcs I., II., III.), are shown by the Survey to be

$$326859 \cdot 52,\ 367154 \cdot 37, \text{ and } 694013 \cdot 89 \text{ fathoms.}$$

From these the amplitudes may be deduced by assuming a form of the meridian. Colonel Everest assumes

$$\text{semi-axis major} = 20922931 \cdot 8 \text{ feet, and ellipticity} = \frac{1}{300 \cdot 8}.$$

The formula for thus calculating the amplitude is derived from the usual relation,

$$\frac{\text{arc}}{\frac{1}{2}\text{ axis major}}=(1-e^2)\int_{\text{L}}^{l}\frac{dl}{\{1-e^2\sin^2 l\}^{\frac{3}{2}}},$$

where l and L are the latitudes of the extremities of the arc, and e is the eccentricity of the ellipse.

Let a be the semi-axis major;

ε the ellipticity;

λ the amplitude of the arc;

μ the latitude of the middle point of the arc.

Then $e^2=2\varepsilon-\varepsilon^2$, $\lambda=l-\text{L}$, $2\mu=l+\text{L}$; and the above form leads to the following,

$$\frac{\text{arc}}{a}=\lambda\left\{1-\frac{1}{2}\varepsilon\left(1+3\frac{\sin\lambda}{\lambda}\cos 2\mu\right)-\frac{1}{16}\varepsilon^2\left(1-15\frac{\sin\lambda}{\lambda}\cos\lambda\cos 4\mu\right)\right\};$$

$$\therefore\quad \lambda=\frac{\text{arc}}{a}\left\{1+\frac{1}{2}\varepsilon\left(1+3\frac{\sin\lambda}{\lambda}\cos 2\mu\right)\right.$$
$$\left.+\frac{1}{4}\varepsilon^2\left(\left(1+3\frac{\sin\lambda}{\lambda}\cos 2\mu\right)^2+\frac{1}{4}\left(1-15\frac{\sin\lambda}{\lambda}\cos\lambda\cos 4\mu\right)\right)\right\}.$$

If the values for arcs I., II., III. and a are put in the expression $\frac{\text{arc}}{a}$, we obtain the three first approximate values of the computed amplitudes of the three arcs. They are

0·0937324, 0·1052876, and 0·1990201;

or the same expressed in angles are

5° 22′ 13″·715, 6° 1′ 57″·145, and 11° 24′ 10″·869.

These must be put in the small terms on the right-hand side of the form for λ above, and we have the second approximation to the three values, viz.—

$\lambda'=5° 23' 41''·796$, $\lambda''=6° 3' 53''·286$, $\lambda'''=11° 27' 36''·300$.

A third approximation will not affect these results.

Now the astronomically determined amplitudes are, as I gather from Colonel Everest's work *,

5° 23′ 37″·058, 6° 3′ 55″·973, and 11° 27′ 33″·032.

And in art. 43. I have found the differences of deflection of the plumb-line at the extremities of the arcs I., II., III., to be

15″·885, 5″·059, and 20″·944.

* These will be found at pp. lxx, lxxi. Expressed in seconds they are 19417″·058, 21835″·973, and 41253″·031. These are the amplitudes, given under the symbol Δl in the Table of final results at the close of his work, and which form the data for the comparison of the Indian arc with arcs in other parts of the world, for the determination of the mean ellipticity of the earth. It is obvious that these values of Δl are not correct as final data, if mountain attraction is to be taken account of. They should be increased by the differences in deflection in the plumb-line caused by attraction at the extremities of the arcs.

I shall use arbitrary factors with these to give the opportunity of introducing any changes in the deflections that may be thought necessary on a further examination of the contour of the surface of the attracting mass; so that

$$15''\cdot885(1-u),\quad 5''\cdot059(1-v),\quad \text{and } 20''\cdot944(1-w)$$

will be the differences of deflection in the meridian at the extremities of the three arcs I., II., III. As the third of these must be the sum of the other two, we have the relation

$$20\cdot944w=15\cdot885u+5\cdot059v,$$

or

$$w=0\cdot7584u+0\cdot2415v.$$

63. Also, by comparing the above values of the deflections with those in art. 47, we have the relations

$$u=0\cdot1883z+0\cdot4257y+0\cdot3861x,$$
$$v=0\cdot0949z+0\cdot2773y+0\cdot6278x,$$
$$w=0\cdot1657z+0\cdot3898y+0\cdot7044x.$$

64. Before proceeding, I will remark that the amplitude of the arc II. determined astronomically, as given above, is somewhat *greater than* that deduced by calculation from the length of the arc. Unless this can be accounted for by the form of the assumed ellipse, it intimates that there is some disturbing cause north of Damargida which increases the inclination of the plumb-line to that at Kalianpur. Should this be the case, the correction for it may be effected by adding a small quantity to the deflection between Kalianpur and Damargida; that is, by increasing $5''\cdot059(1-v)$, or by diminishing v by some quantity v'.

65. We must now add the meridian deflections to the astronomical amplitudes. The results are the true amplitudes of the three arcs I., II., III., viz.—

$$5°\ 23'\ 52''\cdot943-15''\cdot885u,\quad 6°\ 4'\ 1''\cdot032-5''\cdot059v,\quad \text{and } 11°\ 27'\ 53''\cdot976-20''\cdot944w.$$

A comparison of these with the three values of λ, viz. λ', λ'', λ''' in art. 62. deduced by computation, shows that ε and a have been chosen so as to make λ in each case too small. Let $d\lambda'$, $d\lambda''$, $d\lambda'''$ be the three errors of λ', λ'', λ'''; and suppose $\varepsilon+d\varepsilon$ and $a+da$ are the values of ε and a which will by computation bring out the true amplitudes;

$$\therefore\ \frac{d\lambda'}{\lambda'}=\frac{11''\cdot147-15''\cdot885u}{5°\ 23'\ 41''\cdot796}=0\cdot0005739-0\cdot0008179u,$$

$$\frac{d\lambda''}{\lambda''}=\frac{7''\cdot746-5''\cdot059v}{6°\ 3'\ 53''\cdot286}=0\cdot0003548-0\cdot0002317v,$$

$$\frac{d\lambda'''}{\lambda'''}=\frac{17''\cdot676-20''\cdot944w}{11°\ 27'\ 36''\cdot300}=0\cdot0004284-0\cdot0005076w.$$

By the formula of art. 62. we have

$$\text{arc}=a\lambda'\left\{1-\frac{1}{2}\varepsilon\left(1+3\frac{\sin\lambda'}{\lambda'}\cos 2\mu'\right)\right\}.$$

If we differentiate the logarithm of this, we have

$$0=\frac{da}{a}+\frac{d\lambda'}{\lambda'}-\frac{1}{2}\left(1+3\frac{\sin\lambda'}{\lambda'}\cos 2\mu'\right)d\varepsilon.$$

There will be two similar equations in λ'' and λ'''. Put $\frac{da}{a}=\alpha$, and substitute the value found above, and this becomes, after reduction,

$$0=\alpha+0{\cdot}0005739-0{\cdot}0008179u-1{\cdot}3880d\varepsilon.$$

In a similar manner we obtain the following equations:

$$0=\alpha+0{\cdot}0003548-0{\cdot}0002317v-1{\cdot}6095d\varepsilon,$$
$$0=\alpha+0{\cdot}0004284-0{\cdot}0005076w-1{\cdot}5055d\varepsilon.$$

Eliminating α from the 1st and 2nd, the 1st and 3rd, and the 2nd and 3rd, we have the three following equations in $d\varepsilon$:—

$$0=0{\cdot}0002191-0{\cdot}0008179u+0{\cdot}0002317v+0{\cdot}2215d\varepsilon,$$
$$0=0{\cdot}0001455-0{\cdot}0008179u+0{\cdot}0005076w+0{\cdot}1175d\varepsilon,$$
and $$0=0{\cdot}0000736+0{\cdot}0002317v-0{\cdot}0005076w+0{\cdot}1040d\varepsilon.$$

These are the same as

$$d\varepsilon=-0{\cdot}000989+0{\cdot}003693u-0{\cdot}001046v,$$
$$d\varepsilon=-0{\cdot}001238+0{\cdot}006961u-0{\cdot}004320w,$$
$$=-0{\cdot}001238+0{\cdot}003685u-0{\cdot}001043v, \text{ substituting for } w \text{ by art. 62,}$$
and $$d\varepsilon=-0{\cdot}000708-0{\cdot}002228v+0{\cdot}004881w,$$
$$=-0{\cdot}000708+0{\cdot}003702u-0{\cdot}001049v.$$

The method of least squares shows that the arithmetic mean of these is the nearest value;

$$\therefore \quad d\varepsilon=\frac{1}{3}(-0{\cdot}002935+0{\cdot}011080u-0{\cdot}003138v)$$
$$=-0{\cdot}000978+0{\cdot}003693u-0{\cdot}001046v.$$

It is worthy of remark, that the terms depending upon the calculated deflection, viz. those in which u and v enter, are very nearly exactly the same in the mean value and the three separate values of $d\varepsilon$. Adding the mean value of $d\varepsilon$ to ε or $\frac{1}{300{\cdot}8}$ (which equals 0·003324), we have

$$\text{corrected ellipticity }=0{\cdot}002346+0{\cdot}003693u-0{\cdot}001046v;$$

and by adding the three equations in α together, substituting for $d\varepsilon$ its mean value, and dividing by 3, we have

$$\alpha=-0{\cdot}0039737-0{\cdot}0051426u+0{\cdot}0016881v;$$

$$\therefore \quad \text{corrected semi-axis major }=a(1+\alpha)$$
$$=a\{0{\cdot}9960263-0{\cdot}0051426u+0{\cdot}0016881v\}.$$

66. With regard to these results, I will first observe, that if mountain attraction be neglected altogether, or $u=1$ and $v=1$,

$$\text{ellipticity }=0{\cdot}002346+0{\cdot}003693-0{\cdot}001046$$
$$=0{\cdot}005093=\frac{1}{196{\cdot}3}.$$

This value coincides almost exactly with Colonel Everest's determination of the ellipticity from a comparison of the arcs I. and II. He makes it $\frac{1}{191\cdot6}$ (see the Table at the end of his work).

67. Let us next see whether, by altering the altitudes of the attracting mass, we can make the curvature of the Indian arc equal to that corresponding with the mean ellipticity. The above form will then give

$$\frac{1}{300\cdot7} \text{ or } 0\cdot003324 = 0\cdot002346 + 0\cdot003693u - 0\cdot001046v\,;$$

$$\therefore\ 3693u - 1046v = 978$$

$$u = \frac{978 + 1046v}{3693} = 0\cdot265 + 0\cdot283v.$$

As v cannot be less than 0, u cannot be less than 0·265, or somewhat more than $\frac{1}{4}$, to satisfy this equation. Or the heights laid down in the Tables of art. 41. must be so far cut down as to diminish the difference of deflection at Kaliana and Kalianpur by one-fourth of its amount.

Should it be necessary to allow for the larger astronomical amplitude of the arc II. (noticed in art. 64.), by diminishing v by a quantity v', then

$$u = 0\cdot265 - 0\cdot283v' + 0\cdot283v.$$

The astronomical amplitude of arc II. is 2″·687 larger than that obtained by computation from the arc; whereas it should be 1″·509* smaller than that determined from computation, if it bear the same relation to the difference of deflections as in the arc I. If then we put

$$v' = \frac{2''\cdot687 + 1''\cdot509}{5''\cdot059} = \frac{4\cdot196}{5\cdot059} = 0\cdot8294$$

$$u = 0\cdot030 + 0\cdot283v,$$

and a reduction of the difference of deflections at Kaliana and Kalianpur by something more than $\frac{1}{30}$th part, the difference in arc II. remaining unaltered, will bring the ellipticity out $\frac{1}{300\cdot7}$. There is, however, no reason for supposing either that the curvature of the Indian arc is precisely equal to the mean curvature of the whole quadrant; nor that the heights of the attracting masses have been made so much too great.

68. If these heights have been rightly assigned in this paper, that is, if $u=0$ and $v=0$, then the ellipticity of the Indian arc

$$= 0\cdot002346 = \frac{1}{426\cdot2},$$

which shows that the arc is more curved than it would be if it had the mean ellipticity.

* 15″·885 : 5″·059 : : 4″·738 : 1″·509, the quantity used in the text.

The degree of increased curvature may be judged of from this. The height of the middle point of an arc of which the amplitude is λ, above the chord of the arc,

$$=\frac{1}{8}a.\lambda^2\left\{1-\varepsilon\left(\frac{1}{2}+\frac{3}{2}\cos 2\mu\right)\right\}$$

μ being the latitude of the middle point, and λ sufficiently small to allow λ^4 to be neglected. In the arc between Kaliana and Damargida (which is about 800 miles long), $\lambda=0\cdot2$, $\cos 2\mu=0\cdot67473$, and $a=4000$ miles. Hence height of middle point above the chord $=20(1-1\cdot512\varepsilon)$ miles.

For the mean ellipticity, this $=19\cdot8992$ miles.

For the ellipticity $\frac{1}{426\cdot2}$, this $=19\cdot9290$ miles.

For the ellipticity $\frac{1}{196\cdot3}$, this $=19\cdot8460$ miles.

The ellipticity, therefore, which results from taking account of mountain attraction raises the middle point of the arc by 0·0298 of a mile, or 157 feet; whereas the ellipticity when mountain attraction is neglected depresses the arc through 0·0532 of a mile, or 281 feet. These quantities are nearly in the ratio of 5 : 9. Hence the consideration given to mountain attraction in this paper brings the curvature of the Indian arc nearer to the mean curvature than the neglect of mountain attraction does in the ratio of 5 : 9 *. This is, as far as it goes, in favour of these calculations.

69. The conclusion, then, to which I come is, that there is no way of reconciling the difference between the error in latitude deduced in Colonel Everest's work and the amount I have assigned to deflection of the plumb-line arising from attraction—and which, after careful re-examination, I am decidedly of opinion is not far from the truth, either in defect or in excess—but by supposing, that the ellipticity which Colonel Everest uses in his calculations, although correct as a mean of the whole quadrant, is too large for the Indian arc. This hypothesis appears to account for the difference most satisfactorily. The whole subject, however, deserves careful examination; as no anomaly should, if possible, remain unexplained in a work conducted with such care, labour, and ability, as the measurement of the Indian arc has exhibited.

* An *increased* curvature is, moreover, more in accordance with what might be expected, as the effect of the upheaving of the enormous mass of the Himalayas and neighbouring regions, than a diminished curvature.

Deep River, Cape of Good Hope,
July 12, 1854.

III. *On the Computation of the Effect of the Attraction of Mountain-masses, as disturbing the Apparent Astronomical Latitude of Stations in Geodetic Surveys.* *By* G. B. Airy, *Esq., Astronomer Royal.*

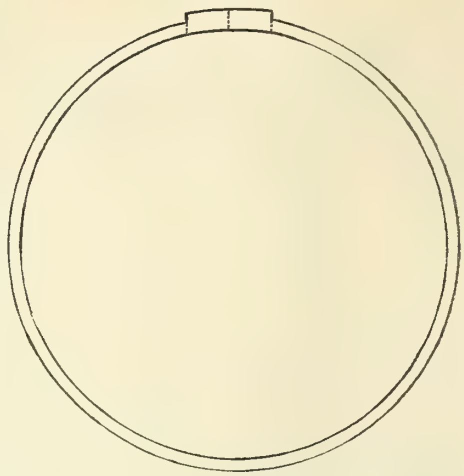
Received January 25,—Read February 15, 1855.

A PAPER of great ability has lately been communicated to the Royal Society by Archdeacon Pratt, in which the disturbing effects of the mass of high land north-east of the valley of the Ganges, upon the apparent astronomical latitudes of the principal stations of the Indian Arc of Meridian, are investigated. It is not my intention here to comment upon the mathematical methods used by the author of that paper, or upon the physical measures on which the numerical calculation of his formulæ is based, but only to call attention to the principal result; namely, that the attraction of the mountain-ground, thus computed on the theory of gravitation, is considerably greater than is necessary to explain the anomalies observed. This singular conclusion, I confess, at first surprised me very much.

Yet, upon considering the theory of the earth's figure as affected by disturbing causes, with the aid of the best physical hypothesis (imperfect as it must be) which I am able to apply to it, it appears to me, not only that there is nothing surprising in Archdeacon Pratt's conclusion, but that it ought to have been anticipated; and that, instead of expecting a positive effect of attraction of a large mountain mass upon a station at a considerable distance from it, we ought to be prepared to expect no effect whatever, or in some cases even a small negative effect. The reasoning upon which this opinion is founded, inasmuch as it must have some application to almost every investigation of geodesy, may perhaps merit the attention of the Royal Society.

Although the surface of the earth consists everywhere of a hard crust, with only enough of water lying upon it to give us everywhere a *couche de niveau*, and to enable us to estimate the heights of the mountains in some places, and the depths of the basins in others ; yet the smallness of those elevations and depths, the correctness with which the hard part of the earth has assumed the spheroidal form, and the absence of any particular preponderance either of land or of water at the equator as compared with the poles, have induced most physicists to suppose, either that the interior of the earth is now fluid, or that it was fluid when the mountains took their present forms. This fluidity may be very imperfect; it may be mere viscidity; it may even be little more than that degree of yielding which (as is well known to miners) shows itself by changes in the floors of subterraneous chambers at a great depth when their width exceeds 20 or 30 feet; and this yielding may be sufficient for my present explanation. However, in order to present my ideas in the clearest form, I will suppose the interior to be perfectly fluid.

Fig. 1.



In the accompanying diagram, fig. 1, suppose the outer circle, as far as it is complete, to represent the spheroidal surface of the earth, conceived to be free from basins or mountains except in one place; and suppose the prominence in the upper part to represent a table-land, 100 miles broad in its smaller horizontal dimension, and two miles high. And suppose the inner circle to represent the concentric spheroidal inner surface of the earth's crust, that inner spheroid being filled with a fluid of greater density than the crust, which, to avoid circumlocution, I will call *lava*. To fix our ideas, suppose the thickness of the crust to be ten miles through the greater part of the circumference, and therefore twelve miles at the place of the table-land.

Now I say, that this state of things is impossible; the weight of the table-land would break the crust through its whole depth from the top of the table-land to the surface of the lava, and either the whole or only the middle part would sink into the lava.

In order to prove this, conceive the rocks to be separated by vertical fissures at the places represented by the dotted lines; conceive the fissures to be opened as they would be by a sinking of the middle of the mass, the two halves turning upon their lower points of connexion with the rest of the crust, as on hinges; and investigate the measure of the force of cohesion at the fissures, which is necessary to prevent the middle from sinking. Let C be the measure of cohesion; C being the height, in miles, of a column of rock which the cohesion would support. The weight which tends to force either half of the table-land downwards, is the weight of that part of it which is above the general level, or is represented by 50×2. Its momentum is $50 \times 2 \times 25 = 2500$. The momenta of the "couples," produced at the two extremities of one half, by the cohesions of the opening surfaces and the corresponding thrusts of the angular points which remain in contact, are respectively $C \times 12 \times 6$ and $C \times 10 \times 5$; their sum is $C \times 122$. Equating this with the former, $C = 20$ nearly; that is, the cohesion must be such as would support a hanging column of rock twenty miles long. I need not say that there is no such thing in nature.

If, instead of supposing the crust ten miles thick, we had supposed it 100 miles thick, the necessary value for cohesion would have been reduced to $\frac{1}{5}$th of a mile nearly. This small value would have been as fatal to the supposition as the other. Every rock has mechanical clefts through it, or has mineralogical veins less closely connected with it than its particles are among themselves; and these render the cohesion of the firmest rock, when considered in reference to large masses, absolutely insignificant. The miners in Cornwall know well the danger of a "fall" of the firmest granite or killas* where it is undercut by working a lode at an inclination of 30° or 40° to the vertical.

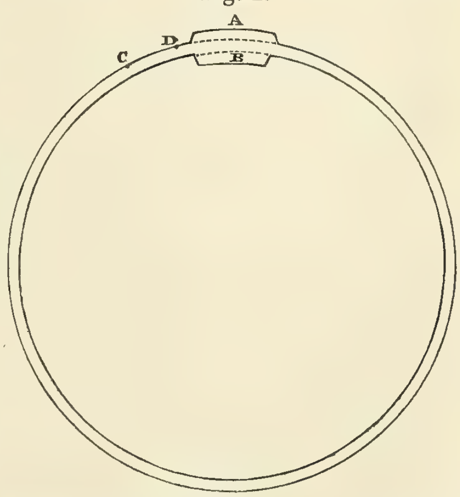
We must therefore give up the supposition that the state of things below a table-land of any great magnitude can be represented by such a diagram as fig. 1. And we may now inquire what the state of things really must be.

The impossibility of the existence of the state represented in fig. 1 has arisen from the want of a sufficient support of the table-land from below. Yet the table-land does exist in its elevation, and therefore it *is* supported from below. What can the nature of its support be?

I conceive that there can be no other support than that arising from the downward projection of a portion of the earth's light crust into the dense lava; the horizontal extent of that projection corresponding rudely with the horizontal extent of the table-land, and the depth of its projection downwards being such that the increased power of floatation thus gained is roughly equal to the increase of weight above from the prominence of the table-land. It appears to me that the state of the earth's crust lying upon the lava may be compared with perfect correctness to the state of a raft of timber floating upon water; in which, if we remark one log whose upper surface floats much higher than the upper surfaces of the others, we are certain that its lower surface lies deeper in the water than the lower surfaces of the others.

This state of things then will be represented by fig. 2. Adopting this as the true representation of the arrangement of masses beneath a table-land, let us consider what will be its effect in disturbing the direction of gravity at different points in its proximity. It will be remarked that the disturbance depends on two actions; the positive attraction produced by the elevated table-land; and the diminution of attraction, or negative attraction, produced by the substitution of a certain volume of light crust (in the lower projection) for heavy lava.

Fig. 2.



The diminution of attractive matter below, produced by the substitution of light crust for heavy lava, will be sensibly equal to the increase of attractive matter above. The difference of the negative attraction of one and the positive attraction of the other, as estimated in the direction of a line perpendicular to that joining the centres of attraction of the two masses (or as estimated in a horizontal line), will be proportional to the difference of the inverse cubes of the distances of the attracted point from the two masses.

* A "fall" occurred in the Dolcoath mine, while I was engaged there with Messrs. WHEWELL, SHEEPSHANKS, and other friends, on pendulum-experiments, in 1828.

Suppose then that the point C is at a great distance, where nevertheless the positive attraction of the mass A, considered alone, would have produced a very sensible effect on the apparent astronomical latitude, as ten seconds. The effect of the negative attraction of B will be $10'' \times \frac{CA^3}{CB^3}$; and the whole effect will be $10'' \times \frac{CB^3 - CA^3}{CB^3}$, which probably will be quite insensible.

But suppose that the point D is at a much smaller distance, where the positive attraction of the mass A would have produced the effect n''. The whole effect, by the same formula, will be $n'' \times \frac{DB^3 - DA^3}{DB^3}$, or $n'' \times \left(1 - \frac{DA^3}{DB^3}\right)$; and as in this case the fraction $\frac{DA}{DB}$ is not very nearly equal to 1, there may be a considerable residual disturbing attraction. But even here, and however near to the mountains the station D may be, the real disturbing attraction will be less than that found by computing the attraction of the table-land alone.

The general conclusion then is this. In all cases, the real disturbance will be less than that found by computing the effect of the mountains, on the law of gravitation. Near to the elevated country, the part which is to be subtracted from the computed effect is a small proportion of the whole. At a distance from the elevated country, the part which is to be subtracted is so nearly equal to the whole, that the remainder may be neglected as insignificant, even in cases where the attraction of the elevated country itself would be considerable. But in our ignorance of the depth at which the downward immersion of the projecting crust into the lava takes place, we cannot give greater precision to the statement.

In all the latter inferences, it is supposed that the crust is floating in a state of equilibrium. But in our entire ignorance of the *modus operandi* of the forces which have raised submarine strata to the tops of high mountains, we cannot insist on this as absolutely true. We know (from the reasoning above) that it will be so to the limits of *breakage* of the table-lands; but within those limits there may be some range of the conditions either way. It is quite as possible that the immersion of the lower projection in the lava may be too great, as that the elevation may be too great; and in the former of these cases, the attraction on the distant stations would be negative.

Again reverting to the condition of *breakage* of the table-lands, as dominating through the whole of this reasoning, it will be seen that it does not apply in regard to such computations as that of the attraction of Schehallien and the like. It applies only to the computation of the attractions of high tracts of very great horizontal extent, such as those to the north of India.

Royal Observatory, Greenwich,
January 19, 1855.

IV. *An Account of some recent Researches near Cairo, undertaken with the view of throwing light upon the Geological History of the Alluvial Land of Egypt.—Instituted by* LEONARD HORNER, *Esq., F.R.SS. L. & E., F.G.S.*

PART I.

Received January 25th,—Read February 8th and 15th, 1855.

CONTENTS :—

INTRODUCTION.

THE progress of geology has demonstrated, that the portion of the crust of the globe which is accessible to us, has been formed by a series of successive operations, and that each member of the series of great changes must have required a period of vast duration for its development. We learn from the astronomer that the mean distance from our earth to the sun is ninety-five millions of miles, and that the distance which separates us from the 61st star of the Swan is 412,000 times ninety-five millions. Although he thus describes an extent of distance of which it is scarcely possible for us to form a just conception, still he expresses himself in definite terms. Not so the geologist: while the astronomer with his telescope penetrates into the remotest regions of Space, and in the known velocity of light has a scale by which he can estimate the vast distance, the geologist looks into an unfathomable abyss of Time; for no power of sounding its depth has yet been discovered. If he attempts to assign a definite term, in time, for the period of the formation of any particular series of strata, even among those that belong to the most recent of the tertiary deposits, he has hitherto sought in vain for any reliable scale of measurement; and he speaks of thousands, or millions, or myriads of years or ages, just as imagination leads him to give a form to his ideas of vast immeasurable antiquity.

It is scarcely within the range of possibility that the absolute age of the earth's crust, reckoned in years backward from some historical epoch, will ever be discovered, because its formation has been progressive, and the several stages of its growth must each have been so modified by a variety of causes, irregular in their extent, duration, and recurrence, that there would exist no uniformity in the rate of progression.

Although it be thus highly improbable that we can ever form an approximate estimate in years of the age even of the most modern strata, we are not cut off from all hope of being able to assign an amount in years to the duration of some of the great geological changes which, in past ages, the present surface of the earth has undergone, by causes that are still in operation. It has been estimated that the delta of the Mississippi must have required not less than 100,000 years for its formation, and that the recession of the Falls of Niagara to their present position has been the work of many thousand years*. But even here we have probability only to rest upon, strong though it be; may we not hope to arrive at the knowledge of some instances when our estimates may possess some degree of precision, where we may find a link connecting historical and geological time†?

If in a country in which a certain alteration in the land has occurred, we know that such alteration has taken place in part within historical time, and if the entire change under consideration presents throughout a tolerable uniformity of character, shall we not be justified in holding the portion that has taken place within the historical period to afford a measure of the time occupied in the production of the antecedent part of the same change? If a region exists where such a blending, as it were, of geological and historical time occurs, we may then be able to estimate in definite terms, the time that has elapsed since the change in the form and structure of the land under examination first began.

Of the various agencies which modify the earth's surface, rivers are the most constant, the most uniform in their operation within given periods, and the most appreciable in their effects. The materials which they transport from the higher parts of their course are frequently spread over an extensive surface in the lower lands near their mouths, and encroach upon the sea, leaving far inland towns that at one time stood on the shore. But when the foundations of such towns are on detrital travelled materials, they show that similar geological changes had been in progress before the first buildings in these towns were erected. If the date be known when such towns were last frequented as sea-ports, we can judge of the extent of geological change brought about between that period and the present time. But as the more recently transported materials, those which have accumulated during the historical period,

* Sir Charles Lyell, Second Visit to the United States, vol. ii. p. 250, and Travels in North America, vol. i. p. 34.

† Strictly speaking, the present day is "geological time;" for not an hour passes without the crust of the earth, externally and internally, undergoing a change. Meteoric forces are ever acting on the rocks, rivers are transporting to distant parts the loosened particles, and springs and volcanic forces are bringing up from the interior materials that are spread over the outer surface of the earth.

may have been brought down at irregular intervals, and by unequal increments, they afford no data by which we can estimate the rate of increase of the detritus brought down by the same river, previously to the foundation of the sea-port town; nor can we discover whether the formation of the land, composed of travelled materials, on which the town was built, and which stretches far inland from it, was the operation of a brief period of time or of one continued through a long series of ages.

Egypt affords the earliest authentic evidence of the existence of the human race, recorded in works of art; in its monuments we find the dawn of the historical period and of civilization; and that land alone, of all parts of the world as yet known to us, offers an instance of a great geological change that has been in progress throughout the whole of the historical period, down to the present day; and which, we have very reasonable grounds for believing, had been going on with the same uniformity for ages prior to that period when our reckoning of historical time begins. I refer to the annual inundation of the Nile, and the sediment that falls from its waters on the surface of the land it overflows.

The question of the raising of the valley of Upper Egypt and the formation of the Delta by the deposit from the Nile, has been a subject of controversy from the days of HERODOTUS to our own time: there is scarcely a writer on Egypt who does not allude to it. HERODOTUS, who visited Egypt about 455 years before our era, says * that "the soil of Egypt is a black earth, cracked and friable, as if it had been formed by the mud brought down by the Nile from Ethiopia, and which has been accumulated by its overflowings. The greater part of the land is a present from the Nile, as the priests informed me, and it is the conclusion to which I have myself arrived. It seemed to me, in truth, that the whole extent of country lying between the mountains above Memphis was formerly an arm of the sea." Thus far his conclusions are in the main correct; but he goes farther, conceiving that Lower Egypt was wholly formed within historical time; for he says, "in proportion as the land extended from Upper Egypt by the deposits of the Nile, a part of the inhabitants migrated into Lower Egypt." This latter theory of HERODOTUS, which an examination of the geological features of the country has shown to be erroneous, was even recently adopted by the acute and learned NIEBUHR, as we learn from the lectures he delivered at Bonn, a very short time before his death †.

The theory of HERODOTUS, supported by ARISTOTLE, DIODORUS SICULUS, SENECA, STRABO, PLINY and PLUTARCH, was combated in a learned disquisition, in the Memoirs of the Academy of Inscriptions and Belles Lettres, by FRERET, bearing the date of 1742 ‡, and this memoir was replied to fifty years afterwards by the eminent geologist DOLOMIEU §. The latter author observes, that the question of the effects of the inundations of the Nile on the formation of the Delta had been treated of by the

* Book ii. 10 and 15. † Alte Geschichte, vol. i. pp. 50. 56. 79. ‡ vol. xvi. p. 333.

§ Sur la Constitution Physique de l'Egypte, par M. DEODAT de DOLOMIEU, Journal de Physique, 1793, tome xi. ii. p. 41.

learned, by an examination of, and quotations from, ancient writers; whereas the problem to be solved was more one of physical geography, and belonged more to the geologist than to the man of letters. Admitting the theory of Herodotus as to the mode of formation of the Delta, he combats that part of it in which he refers the operation to historical time.

That there has been an annual inundation of the Nile, of greater or less amount, from the earliest period to which history or tradition reaches, does not admit of a doubt; and it is equally certain that the river so flooded was loaded with solid materials, for the fertilizing effect of a sediment left upon the ground is recorded equally with the fact of an annual inundation*. Unless therefore this same addition was wholly washed away again from the surface between the fall of the water of one year and the rise of the next, there must have been an accumulation from year to year. That the fertilizing effect of the inundation is exhausted, or nearly so, is true; but in a country where there are no streams tributary to the main river, where rain is almost unknown in the greater portion of it, that is in Upper Egypt, and with an inclination so slight as that of the land over which the inundation spreads, the solid insoluble matter must in great part remain where it is deposited. That the fertilizing effect of the inundation is increased in proportion to the depth of the water, is shown by the unequivocal proof that the taxes, from time immemorial, have been levied upon the land according to the height to which the river rises†. It was to regulate this impost that the Nilometer on the island of Elephantine near the First Cataract was erected in ancient times, and that a similar instrument was set up in the island of Rhoda, near Cairo, a thousand years ago.

To investigate the formation of the alluvial land in the valley of the Nile in Upper and Lower Egypt is therefore an object of the highest interest to the geologist and the historian. Nowhere else on the face of the earth can we hope to find such a link connecting the earliest historical with the latest geological time; for in Egypt we have accurate records of the earliest periods of the human race, in which any trace of civilization has been discovered, combined with records, of scarcely less accuracy, of geological changes contemporaneous with history, and these last having such a degree of uniformity as to warrant us in carrying back the dates of changes of a like nature beyond that of the earliest historical documents.

Having been long impressed with a conviction that this geological problem could only be solved by having shafts and borings made in the alluvial deposits to the greatest practicable depth, and concurring in the opinion, long ago expressed by Cuvier, that it was a matter of regret that the depth of these deposits between the surface and

* How much the fertilizing effect of the inundation belongs to the solid matter held in suspension and deposited on the land, and how much belongs to the matter held in solution in the water, is a question that, so far as I know, has not been solved.

† Beyond a certain height, it is disadvantageous, as the water has not drained off sufficiently to leave the land in a proper state at the right sowing time of the following year.

the rock on which they may rest had not been ascertained, I determined to make an effort to have the experiment made, even to the limited extent within my means, as the results thus obtained might lead the way to other researches on a greater scale.

The ground upon which I hoped to be able to form a chronometric scale by which the total depth of sediment reached might be measured, was the same as that on which the French engineers in 1800 had proceeded, viz. the accumulation of Nile sediment around monuments of a known age. Certain works of art of a very early age exist near the Nile, the approximate dates of whose erection have been established upon reliable evidence; and we know also that the sediment has accumulated to a considerable height above their base. If that depth of sediment be divided by the number of centuries that have elapsed since the date of the erection of the monument, we obtain a scale of the secular increase of which the base of the monument is the zero, assuming, as we are entitled to do, that the average increase from century to century has been uniform within an area of some extent. If the excavation be continued below the base stone, and the sediment passed through exhibits similar characters as to composition with that above the base line of the monument, it would be fair to apply the same graduation below the zero-point of the scale as above it; and, if we reached so far, we should be able to estimate the time that has elapsed since the first layer of sediment was deposited on the rock forming the channel over which the water spread when it first flowed northward from its source in the interior of Africa; subject, however, to correction for causes that might make a difference in the rate of increase between earlier and later periods; an investigation of which causes forms a necessary, but a very difficult part of such an inquiry.

I submitted my project to the President and Council of the Royal Society, stating, that it appeared to me to be a scientific inquiry of sufficient importance to justify my asking for a grant from the Donation Fund under their control, for the purpose of defraying the expense of the proposed excavations. My proposal was favourably received, and the Council were pleased to place a liberal grant of money at my disposal.

I have thought it advisable, before entering upon the narrative of the researches carried on towards the accomplishment of the object of this inquiry, to give a brief sketch of the physical geography of Egypt and of its geological structure; and a somewhat more detailed account of the annual inundations of the Nile and of the sediment it deposits.

Physical Geography and Geological Structure of Egypt.*

Egypt is separated from Nubia by a low hilly region, fifty miles in breadth from north to south, which is a part of a range extending from the Red Sea in an east and

* The principal authorities are,—Russegger, Reisen in Europa, Asien und Africa, Stuttgart, 1843; Brocchi, Giornale delle Osservazioni fatte ne' Viaggi in Egitto, &c., Bassano, 1841; Newbold, Quarterly Journal of the Geological Society, Nov. 1848; and Talabot, Mémoire de la Société d'Etudes de l'Isthme de Suez, 1846–47,—the latter not published.

west direction until it gradually sinks in the desert of Libya. This part of the range nowhere rises to a greater height than about 214 feet above the bed of the Nile, and 546 feet above the Mediterranean. Granite is the predominant rock, of different varieties, sometimes passing into gneiss, sometimes having an admixture of hornblende, when it gets the name of syenite, from its occurrence near the ancient town of Syene.

The granites and other unstratified rocks are associated in the district near Assouan, or the First Cataract, with two sedimentary rocks, both sandstones, and very similar in mineral structure, but very different in point of age; for the one belongs to the lower members of the cretaceous period, and the other covers in several places, farther north, a tertiary nummulite limestone.

Through a labyrinth of these granites and sandstones, extending from the island of Philæ to the neighbourhood of Assouan, the Nile enters Egypt in a succession of rapids, having a descent of about 85 feet in a distance of about five miles and a half from Philæ to Assouan, forming what is called the First Cataract. There is no waterfall, as commonly understood by the term cataract, for Russegger and his companions were dragged up in a boat the whole distance in two hours, during the time of low water, that is, towards the end of January.

The valley of Upper Egypt is flanked by two parallel ranges of hills, the Arabian on the east, the Libyan on the west. At Assouan, the southern extremity of the valley, they each approach close to the Nile, the bed of which is strewed with rocky islands, the most northernly being the celebrated Elephantine. Both ranges are divided by rents of various magnitudes, forming valleys, some of them running north and south, others crossing the ranges from east to west. One of the great north and south valleys gave a passage to the waters of the Nile, in a somewhat tortuous course; the appearance of the boundaries on either side, and the very gentle fall of the land, from south to north, excluding all idea that the valley has been excavated by the action of running water. The Nile valley varies considerably in breadth, its widest part between Minieh* and Benisuef being about eighteen miles, frequently contracting to two miles; and at Gebel Silsilis, about forty-five miles below Assouan, the hills approach so close to each other, that the river, for three-quarters of a mile, runs through a pass about 1200 feet wide†, and there is scarcely a yard of alluvial deposit on either bank for a considerable distance. At the apex of the Delta the valley is six and a half miles wide‡.

The Libyan range falls with a slope towards the valley, the rocks of which it is composed appearing to extend under the valley, forming a solid basin covered with sand and detritus that had accumulated before the alluvial matter brought down by the Nile began to be deposited over it.

The Arabian hills, except where broken by transverse valleys, present cliffs towards

* In the spelling of the proper names, I follow Lepsius in his Briefe aus Ægypten, 1852.

† According to the atlas that accompanies the Description de l'Egypte, 200 Toises.

‡ Bonomi, Trans. of Roy. Soc. ot Lit., 2nd Series, ii. 297.

the river, nearly throughout their whole length. In the neighbourhood of Assouan the hills are little more than 200 feet above the Nile, but they go on increasing in height to the parallel of Thebes, where they attain an elevation of 1065 feet above the Nile; and from that point, northwards, they have a gradual fall, but rise again in some parts of their northern extremity to nearly 1000 feet.

The Libyan and Arabian ranges are nearly identical in mineral composition. The lowest sedimentary rock exposed to day, that which comes in contact with the igneous rocks above Assouan, is the lower sandstone above described. It constitutes the chief composition of the hills on both sides of the valley as far north as the neighbourhood of Esneh, about eighty-five miles below Assouan. Here it is covered by a limestone which both Russegger and Newbold identify with the chalk of Europe, the former considering it to belong to the period of the upper chalk. It occupies both sides of the valley as far as Siut, a distance of nearly 130 miles, in a direct line.

At Siut the chalk is covered by nummulite limestone, a part of that vast tertiary formation which extends through Southern Europe, Asia and Northern Africa. The hills on both sides of the valley from Siut to Cairo, a distance of more than 200 miles, are composed of it, and it extends from the Nile to the Red Sea, and from the left bank of the river into the Libyan desert.

In the latitude of Cairo the Arabian and Libyan ranges of hills no longer run parallel; the former terminating in a line from W.N.W. to E.S.E., between Cairo and Suez, so that near Cairo the hills on the right bank of the Nile appear to turn abruptly, nearly at right angles to the course they have held from Assouan northwards. Near Cairo the group of Gebel Mokattam rises to the height of 448 feet above the Mediterranean. The hills composing the northern part of the Libyan range, from the latitude of Cairo take a N.W. direction, being a continuation of the hills that extend from the Red Sea. They in no part rise to a greater elevation than about 320 feet above the Mediterranean.

North and east of the Mokattam, the different members of the tertiary nummulite limestone formation are covered by a sandstone, identical in character with the upper sandstone near Assouan. No fossils have been found in it hitherto, but its overlying the nummulite limestone clearly determines it to be at least a tertiary deposit. This sandstone is the prevailing rock throughout the Isthmus of Suez, wherever it rises above the desert sand or other alluvial covering.

The engineers who accompanied the French army in Egypt in 1799, having made a survey of the Isthmus of Suez, came to the conclusion that the level of the Red Sea was 9 metres (29 feet 5 inches) above that of the Mediterranean, and their high reputation gave currency in the scientific world to this result for many years. Circumstances, however, having in more recent times thrown doubts on the accuracy of the survey, which had been made in a very short time, and under almost every disadvantage, a private association of French, English and Austrian civil engineers was formed in 1846, under the patronage of the Viceroy of Egypt, MEHEMET ALI, for the express purpose of making a careful survey of the isthmus, in order to determine the question of the level of the two seas. The association consisted of Messieurs TALABOT and BOURDALOUE on the part of the French, Mr. ROBERT STEPHENSON on the part of the English, M. NEGRETTI on the part of the Austrians, and M. LINANT DE BELLEFONDS (Linant Bey) on the part of the Pacha. Mr. STEPHENSON undertook to observe the levels of the tides at Suez, M. NEGRETTI those at Tineh on the Mediterranean, near the ancient Pelusium, and the survey of the land was undertaken by M. TALABOT and M. BOURDALOUE with several assistants. A report of their operations, accompanied by maps on a large scale, and detailed tables, was printed at Nismes in 1847, under the title of "Société d'Etudes de l'Isthme de Suez, Travaux de la Brigade Française, Rapport de l'Ingénieur," but has not been published*. The results obtained were as follows:—

That the low-water mark of ordinary tides in the two seas, at Suez and at Tineh, is very nearly on the same level, the difference being, that at Suez it is three centimetres lower, that is, rather more than 1 inch;

That the mean rise of ordinary tides in the Red Sea is somewhat higher than in the Mediterranean, but that the maximum difference is not more than 80 centimetres, or $31\frac{1}{2}$ inches;

That the rise of the equinoctial spring tides at Suez over the low-water mark in the Mediterranean at Tineh at the same period is 2·38 metres, or 7 feet 9 inches; and

That the deepest low-water mark at the same period at Suez is 0·45 metre, or 17·7 inches under the deepest low-water mark at Tineh.

The highest point of the isthmus between Pelusium and Suez is 12·74 metres, or 41 feet above the Mediterranean, the distance between the two places being $12\frac{5}{8}$ myriametres, or about 78 English miles.

The north-westernly prolongation of the Libyan range of hills, which form the western boundary of Upper Egypt, is composed of the same nummulite limestone, covered by the upper sandstone. The sandy desert at the foot of the range, like that on the eastern or Arabian side, contains pebbles of agate and flint, and masses of fossil wood, stems of which have been found 40 feet in length changed into hornstone. Parallel to the direction of the Libyan hills, and on their eastern side, are two depressions of great extent, one of which there is every reason to consider as a former channel of the Nile, and goes by the name of the Valley of the Waterless River; the other is the Valley of the Natron Lakes.

Lower Egypt, geologically considered, is formed of the low and almost level land included between the Mediterranean and the hilly regions which form what may be termed the natural boundary of Upper Egypt. The central portion formed by the divergence of the Nile about sixteen miles below Cairo, into two branches that fall into the sea at Rosetta and Damietta, constitutes the present Delta. The distance

* I have had it in my possession through the kindness of Mr. ROBERT STEPHENSON.

from Cairo to the sea in a direct line is 106 miles, and from Rosetta to Damietta, or the base of the triangle, in a direct line, eighty-two miles, but following the sinuosities of the coast about ninety miles. In earlier times, when the Nile flowed in the Valley of the Waterless River, and when a branch entered the sea at Pelusium, near the modern Tineh, the base of the Delta must have been about 170 miles, but the low land extends beyond each of these limits*.

Lower Egypt is thus a vast plain of alluvial land, with scarcely any natural elevations except the sand-hills near the coast; it is furrowed in every direction by a multitude of natural and artificial canals. The central part is composed of the mud deposited by the Nile, and of sand brought down by the inundations, or blown from the desert on either side; and all around the plain the boundary of Lower Egypt is composed of quartzose sands, that are generally white on the east, and reddish-white on the west, and the ground which these sands cover is at a higher level than that of the Nile at its highest inundations.

Two great shallow lakes, Burlos and Menzaleh, occupy the greater part of the base of the modern Delta, besides smaller lakes, lagoons and swamps, behind the sand-hills that line the coast. These sand-hills rest upon a reef which forms a powerful dam against the encroachments of the sea, and which Russegger describes as being in a continual state of formation and waste; as being a calcareous stone of a dirty grey colour, composed of sand mixed with worn fragments of ordinary marine testacea, mingled with microscopic shells, many of the latter being of freshwater and land origin, brought down by the Nile, thrown up again by the sea and mingled with marine shells. In structure the stone is not usually very coherent, but in some places it is hard enough to be used for building, and in ancient times numerous catacombs were excavated in it, some of which are the so-called baths of Cleopatra.

At the island of Philæ, about five miles above Assouan, may properly be placed the first entrance of the Nile into Egypt. The mighty stream has here a breadth of nearly one mile†, but soon after it is divided into several branches, by the rocks that rise up in its bed to form the most northern of the rapids, the First Cataract, of which many occur in the higher parts of its course. The breadth of the river is here contracted to about a third of a mile. From the junction of the Atbara in latitude 17·38 N. until it reaches the sea in latitude 31·25, or nearly fourteen degrees of latitude, the Nile does not receive a single tributary, with the exception of torrents after heavy rains in the lower parts of its course in Egypt, from the hills on either side. Assouan, according to the barometrical measurements of Russegger, is 300 feet

* "Although it has been usual to commence Egypt at Tineh (Pelusium), some geographers have restored it to the ancient point El Arish (Rhinocorura), the southern boundary of Syria. Between this and Tineh are the moving sands called by the Hebrews Shúr, and by the Arabs Al Jofár, bordered by the Serbonian Pool. From this notable landmark the shores of Egypt extend to Rasal Kanáis, about 115 leagues to the westward. The central portion is the Delta."—The Mediterranean, by Admiral Smyth, pp. 83, 84.

† 1500 metres=1640 yards, by the Atlas of the Description de l'Egypte.

above Cairo, and the distance between the two places by the Nile being 556 miles, the average fall of the river is little more than half a foot in a mile, viz. 0·54 ft.; and Assouan being 365 feet above the Mediterranean, and 696 miles distant from it, the average fall of the Nile from the foot of the First Cataract to the sea is 0·524 ft. in a mile*. Russegger does not give the place in Cairo from which he measured the above 300 feet, but by the careful measurements of the French Brigade in 1847, before referred to, M. Talabot states the lowest water of the Nile in the Nilometer of Rhoda near Cairo in that year to have been 14·08 metres, or 46 feet 2 inches above low-water mark of the Mediterranean at Tineh; and the distance to the mouth of the Damietta branch, following the course of the river, being 149 miles, the average fall is thus only $3\frac{3}{4}$ inches in a mile.

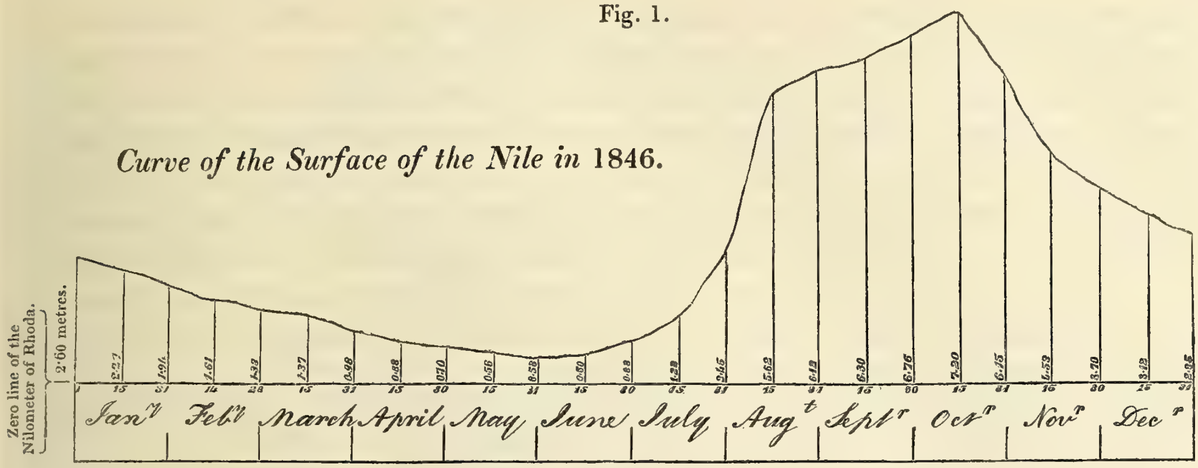
The Inundations of the Nile.

The commencement of the rise of the Nile, immediately below Assouan, is about the summer solstice. The first rise at Cairo, indicated by an increasing motion in the stream, is usually in the first week of July. The rise is scarcely perceptible for six or eight days, and it then becomes more rapid. About the middle of August it has obtained two-thirds of the height between the lowest ebb and highest rise. At this period the water enters the great side branch on the left bank, the Bahr el Jusef, Joseph's Canal, called also the Magrour, and now is the time when the artificial branches or canals are opened, the commencement of the inundation over the parched plains. The rise attains its maximum between the 20th and 30th of September, and this state of the inundation is called the *Salibe*. The water remains pretty stationary for fourteen days; it then begins to fall, at first at a more rapid rate than that with which it rose, but after it has fallen one-half, the decrease is very gradual. About the 10th of November it has usually fallen one-half, and it goes sinking slowly until somewhat beyond the following May. The rise of the river continues therefore about ninety days (from 1st of July to 28th of Sept.), but it continues falling about 230 days (12th of October to end of May). The changes of level are well illustrated by the annexed diagram, given to M. Talabot in 1847 by Mougel Bey, the engineer for the Barrage of the Nile at the apex of the Delta.

* Russegger, Reisen, ii. 271. The fall of the Thames from Chertsey to Teddington Lock, a distance of $13\frac{1}{2}$ miles, is nearly $17\frac{1}{2}$ inches in a mile. See Rennie, Report to the British Association in 1834, p. 487.

"Colonel Cautley, the projector of the Ganges Canal (recently constructed), decided after careful thought and due regard to the experience gained on canals previously opened, that a fall of fifteen inches in every mile of length would best secure the desired ends."—Short Account of the Ganges Canal, p. 7.

Fig. 1.



0·50 metre=18,685 English inches. 7·20 metres=23½ feet.

From the increased fertility of the land by the overflowing of their river, the earliest inhabitants of Egypt must have been led to watch the rise and fall of the water with anxious care. At an early period in the history of the country they contrived an instrument to measure its diurnal changes, a Nilometer, which was erected on the island of Elephantine near Assouan. That instrument is now only an object of interest to the antiquarian, as a ruin is all that is left of it; but it was in a state of considerable preservation a little more than fifty years ago when visited by the French engineers*. I describe it now, as it then existed, because special reference will be made hereafter to this ancient standard of the Nile's annual increase. It is near the south end of the island, and is described by Strabo from personal observation, he having twice visited the spot.

The French engineers have given the following description of it as they found it. It was in a building constructed of regular horizontal layers of sandstone, having two flights of steps, at right angles to each other. One of the walls of the staircase was marked with a vertical groove, crossed by horizontal lines, at regular distances, each of these divisions being a cubit. Three of these divisions were marked with the Greek numerical letters, the highest being ΚΔ, or 24; the second ΚΓ, or 23; the fifth Κ, or 20. The engineers assumed that at the time this Nilometer was constructed, the number 24 marked the greatest rise of the Nile then known†. M. Girard and his companions made an exact measurement of the cubits from 24 to 18, and the result gave 527 millimetres for each cubit, =20·75 English inches. Sir Gardner Wilkinson, from personal observation and examination of the different

* Among the scientific men who accompanied the French army in 1799, we find the following celebrated names:—Berthollet, Monge, Fourier, Malus, Girard, and M. Cordier the geologist, still living. M. Girard occupied the rank of Ingénieur en Chef des Ponts et Chaussées. Dolomieu also went, but did not remain.

† Girard, Observations sur la Vallée d'Egypte, et sur l'exhaussement seculaire du Sol qui la recouvre: Mémoires de l'Acad. Roy. de l'Institut, 1817, tom. ii. p. 261, and Description de l'Egypte Antique, sur le Nilomètre de l'Ile Elephantine.

Egyptian cubits, informs us, that every cubit is divisible into fourteen parts, each of two digits, and the length of the cubit being 20·625 inches, we have 0·736 inch for each digit. It will thus be seen that the measurements of Sir G. Wilkinson and those of the French engineers very nearly agree, the difference being only between 20·75 and 20·625.

Upon the island of Rhoda near Cairo a Nilometer was erected more than a thousand years ago, and is the only measure of the rise and fall of the Nile referred to in the present day. It is known by the name of the Mekyas (instrument of measure). For a reason that will afterwards appear, I give the description of it by the French engineers in 1800. They describe it as an octagonal pillar, having a scale divided into 16 cubits, and each cubit into 24 digits. Each cubit of this scale they found to be equal to 541 millimetres, or 21·3 English inches*.

Between the first entrance of the Nile into Egypt and its mouth, the mass of water must be vastly diminished from the following causes: it receives no tributary; spread over so wide a surface under a burning sun and a cloudless sky, the evaporation must be very great; the water is drawn off from the main channel by numberless canals, and there is a further absorption by infiltration through the soil, for several miles inland, on the left bank in Upper Egypt, and on both sides in Lower Egypt. Thus, while the rise of the river at the island of Rhoda on an average of years is 24 feet, near Ramanyeh about sixty-five miles in a direct line north of the apex of the Delta, the difference between highest and lowest water is about 13 feet, and at Rosetta and Damietta not more than 42 inches †.

From observations made in 1799 at Siut, about midway between Assouan and Cairo, the French engineers estimated the volume of water in the Nile to be as follows:—that every second of time a mass of water passes a given line across the river equal to 678 cubic metres at low Nile, and 10,247 cubic metres at high Nile‡. Linant Bey states that the volumes near Cairo are 414 cubic metres at low Nile, and 9440 at high Nile. M. Talabot makes on this subject the following observations:—"En partant des données qui fournit M. Girard dans son Mémoire sur la Vallée d'Egypte, j'ai cherché à calculer le produit moyen du Nil; il resulte de ce calcul, que la hauteur moyenne du Nil, d'après les quatre années d'observations certaines qui nous ont été fournies, soit par l'Expedition, soit par M. Mougel, étant de 3^{m}·23, correspondrait à un débit moyen de 2860 mètres cubes par seconde, ou d'environ 90,000 millions de mètres cubes par an." But the Rosetta branch, at its mouth, is not more than about 600 metres (656 yards) wide, and at lowest water the depth is only 1·60 metre (5 feet 3 inches); in the Damietta branch the width is about 300 metres (328 yards), and the depth, at the same period, 2·50 metres (8 feet 2½ inches)§.

* Description de l'Egypte, vol. xviii. p. 603.

† Lancret et Chabrol, Descr. de l'Egypte, Etat Moderne, tome ii. p. 187.

‡ Girard, *loc. cit.* 208.

§ Girard.

When we consider, therefore, the large amount of earthy matter held suspended in the water, as will presently appear, and how much the volume of water is diminished before the Nile reaches the sea, it is evident that a vast amount of sediment must be annually left upon the land which the inundation overspreads, that a much larger proportion must be deposited in Upper Egypt than in the Delta, and, from the greater surface of the latter, that the depth of the annual accumulation there must be greatly less than in the more contracted part of the valley.

The solid matter conveyed by the Nile, to form its sedimentary deposits.

When the inundations commence the Nile comes down of a reddish colour, loaded with sand and mud. From the small amount of the fall between the cataract at Wadi-Halfa (the second) and that at Assouan (the first), a distance of 214 miles, and the difference of height between the two places being only 157 feet*, thus making the average fall of the river not quite 9 inches in a mile, it is not to be expected that much coarse gravel can be carried forward, and that which arrives at the island of Philæ must be much sifted and comminuted in its passage through the rocks forming the rapid of Assouan. Below that place, the fall of the land goes on diminishing, so that the transporting power of the stream is small. The greater proportion of the heavier detritus thus falls down in the higher parts of Upper Egypt, and from the very gentle slope of the Delta, it might be concluded that only a small amount of the solid matter suspended in the water can reach the mouths of the river. But very fine particles of earthy matter, as is well known, are long in subsiding, and much is carried out to sea†. HERODOTUS notices that during the inundation the sea is rendered turbid, and BRUCE observed the same thing. NEWBOLD states, that he found the sea coloured at a distance of forty miles from the shore.

A modern traveller thus describes the appearance of the water in the Nile opposite to Thebes on the 7th of November:—"Ce sera curieux pour nous de revoir ce Nil lorsque l'eau en sera transparente, au lieu d'être, comme maintenant, de la couleur de café-au-lait très noir;" and he says of it three weeks before at Cairo, "Maintenant que l'eau est à son plus haut niveau, elle est d'une épaisseur inconcevable, presque comme du chocolat, et plus foncée‡."

The sediment is slightly modified in character at various localities, according to the nature of the formation near to which the river flows. Its composition and texture are also subject to variation from its proximity to or distance from the main channel of the stream, where the coarser and heavier siliceous particles are usually

* See a paper by the author in the Edinburgh Philosophical Journal for July 1850.

† "The out-pouring of the Nile during the inundation is so powerful that fresh water may be skimmed off the surface of the sea at the distance of two or three miles out in the offing. During the full surcharge, potable water may be baled on the surface of the Mediterranean even out of sight of land."—The Mediterranean, by Admiral SMYTH, p. 84 and 169.

‡ MELLY, Lettres d'Egypte et de Nubie, printed for private circulation, 1852.

found, whereas the finer and more argillaceous and calcareous portions are held in suspension and carried out laterally by the gently overspreading waters.

A transverse section of the valley often presents the following appearance.

Fig. 2.

In the middle we see the Nile, and on both sides of it elevations of the ground like two dams. These run parallel to the river and form its banks. Beyond these, the ground again sinks and forms depressions, which, for the most part, are deeper than the present bed of the river, so that it flows, as it were, on a great dam. The explanation of this is, that the Nile accumulates more alluvium in its immediate neighbourhood, and this chiefly consists, though not always, of gravel and sand, whereas in places more distant, to which the water never reaches except during the inundation, or is conducted by canals, less alluvium is deposited; but as the water remains long there in a tranquil state, it lets fall the more fertilizing mud, and thus the land near the desert is the most productive. But in some places the banks of the river consist of from 23 to 33 feet of pure mud, sometimes divided by layers of sand. In numerous places, beds of mud may be seen rising from the level of low-water to the summit of the bank, and in digging below the lower level, the mud is frequently found to be continued*. Newbold found some banks exhibiting what he considered to be stratified annual layers, varying from an inch to a few lines in thickness, in the same situation, the upper part of each layer being usually of a lighter colour than the lower part, and each separable from that immediately above or below it. But, as will hereafter be explained, this appearance was local and the effect of a secondary cause, and was not produced by the regular annual inundation.

The height of the banks of the river generally diminishes from Assouan to the Delta, and thence in a greater ratio to the mouths of the river, owing to the wider extent over which the mud-charged waters spread below the point of the Nile's bifurcation below Cairo. About the time of the medium rise of the river the banks below Thebes are usually from 20 to 30 feet above the surface of the water, at Cairo from 15 to 25 feet, at Rosetta from 3 to 12 feet.

The annual deposit is variable in thickness in different parts of Upper Egypt and in the Delta from a variety of causes, and that both in the vicinity of the river and at a distance from it. In the vicinity of the river, at particular places, where the stream is retarded by the comparative flatness of the country, the deposition is greater than in other localities. The deposit of one year is in some places stripped off by the flood of the next, and the quantity of earthy matter held in suspension in the water is sometimes augmented by portions of the mud cliff falling into the river. M. Melly, writing from Tahtah on the 3rd of November 1850, says, "Le Nil est déjà

* Rozière, Description de l'Egypte, vol. xx.

déscendu de $1\frac{1}{2}$ à 2 pieds, et partout il dégrade les bords à tel point, qu'on voit à chaque moment une série d'avalanches du plus beau terroir noir imaginable:" and on the following day he says, "Nous sommes arrivés à Girgeh, une des villes de la haute Egypte, qui était trois fois plus grande il y a quelques années qu'aujourdhui, mais un courant du Nil qui s'est determiné contre la colline friable sur laquelle elle est bâtie, en enlève des quartiers entiers de tems à autre. Une portion des maisons et une mosquée, dont la moitié est déjà tombées dans le Nil, ressemblent aux gravures du Diable Boiteux, seulement que ces débris ne sont pas habités." The sediment of one year is also carried back into the river the following year from another cause:—"Pendant trois ou quatre mois de l'année, la surface de l'Egypte, dénuée de végétation, sèche et pondreuse, est balayée par des vents violents, qui soulèvent dans les airs la poussière du sol, en laissant précipiter une partie dans le fleuve, qui l'entraine à la mer, et en dispersent une autre partie dans les déserts, ou l'accumulent sur d'autres portions de l'Egypte*." At a distance from the river, especially in those parts which are in the vicinity of valleys or gorges, in the lateral ranges of hills, blown sand from the desert is often mixed with the mud of the river that is spread upon the land.

In all calculations, therefore, as to the secular increase of the deposit, by measurements of its depth, we must take into our consideration whether the pits or the borings have been made in places least liable to these irregularities; whether the solid matter held in suspension may not have been augmented by portions of former deposits washed or falling into the stream; and whether the mud deposited by the river has had no intermixture of blown sand.

The following observation of the traveller Rüppell is a remarkable indication of an accumulation of the Nile sediment in the Faiûm, at a distance of about twenty-five miles from the left bank of the river:—"I had a desire to visit Lake Mœris and its islands, and quitted Medina in a north-east direction, travelling over very fertile plains. In the neighbourhood of a large village called Fedimin, we passed the dried-up bed of a very deep canal, in the side of which I saw, to my great surprise, horizontal beds of the mud of the Nile, having a depth of sixty feet†."

THE RECENT RESEARCHES.

The first step which I had to take in this inquiry was to decide upon the situation in which the proposed vertical shafts should be sunk. As the neighbourhood of Cairo might afford great facilities for prosecuting the work, as the standing obelisk of Heliopolis is one of the most ancient of the existing monuments, and as the time of its erection has been made out on very reliable grounds, I chose that spot.

Having obtained an introduction to A. C. Harris, Esq., of Alexandria, well known by his active and long-continued researches in Egyptian antiquities, I requested him

* De la Constitution physique de l'Egypte, Hist. Nat. ii. 493.

† Letter in Baron de Zach's Correspondance Astronomique, vol. vii. p. 245.

to endeavour to find some one resident in Cairo who might be capable of conducting the contemplated operations, explaining my views to him in the following terms:—

"I am anxious to investigate in a more satisfactory manner than has yet been done, the frequently agitated question, how far the sedimentary deposits of the Nile afford a chronometric scale that will carry us back beyond what may be termed the known zero of authentic historical time. There is every reason to believe, that, reckoning from century to century, the average increments of the deposits are pretty regular, due care being taken to make the observations in a part of Egypt where there is not likely to exist any abnormal state of the solid contents held in suspension in the Nile water, from the breaking down of a part of its banks. But I have not been able to discover that any borings have been made with much care since those by the French in Upper Egypt in 1799, and those recently at the Barrage of the Nile by Mougel Bey. What is wanted is this: to have a pit sunk in a situation where the Nile deposit has accumulated over or close to some of the most ancient works of art known to exist, and the date of the foundation of which is known with tolerable certainty, such as the Obelisk at Heliopolis; the strata in such a pit being regularly marked as to their several thickness and their composition, and specimens of each variety being taken. This being done, to the lowest part of the foundation of the monument, the excavation to be continued as far downwards as any sediment is found having the known characters of the Nile deposit, carefully noting the dimensions of the several layers gone through below the lowest part of the artificial structure, or foundation of the same. Such an examination could not be carried on with trustworthy value except by or under the immediate superintendence of some one capable of directing it, and of ensuring accuracy in all the successive excavations."

Mr. Harris applied to his friend Hekekyan Bey, an Armenian gentleman resident in Cairo, who had been educated and long resident in England, and who, as a Civil Engineer, had occupied some important positions in the service of the Viceroy Mehemet Ali, especially as Chief of the Polytechnic School in Cairo. Hekekyan Bey most readily accepted the proposal, evincing an earnest desire to be employed in a scientific inquiry of this nature. How fortunate I have been in obtaining such valuable cooperation will fully appear in the sequel. But nothing could be done without the previous consent of the then Viceroy Abbas Pacha, the more especially as the spot where I wished the excavation to be made, close to the Obelisk of Heliopolis, is in a garden belonging to the Pacha, into which the site of the renowned city has been converted. Through the active intervention of the Hon. Charles Augustus Murray, at that time Her Majesty's Agent and Consul-General in Egypt, and who, during the remainder of his stay in the country, took a warm interest in the inquiry, and continued to give me his powerful support, the consent of the Viceroy was obtained. His Highness not only acceded to the request, but directed his ministers to place at the disposal of Hekekyan Bey, whom he appointed to conduct the operations, whatever was necessary to carry them out in the most complete manner, but with truly royal munificence told Mr. Murray that the whole expense of them should be defrayed by his Treasury.

When I submitted my proposal to the Council of the Royal Society, I did not contemplate the accomplishment of anything beyond the sinking of a few pits; but I had now the prospect of researches being made on a great scale; how widely they were afterwards extended, by the continued exertions of Mr. Murray and of his successor the Hon. Frederick Bruce, and by the unabated liberality of the Viceroy, will appear in the course of this memoir.

It may appear remarkable to many, as it had done to Cuvier, that researches of this nature had not been undertaken before. With the exception of the corps of scientific men appointed by the French Government to accompany the Egyptian expedition under General Buonaparte, very few of those who have visited Egypt have turned their attention to geological researches; most travellers have been attracted by the interesting objects of art and the history of the people. In the introduction to his memoir 'Sur la vallée d'Egypte,' M. Girard* observes,—"Parmi les nombreux voyageurs qui ont donné des descriptions de l'Egypte, il n'en est aucun qui se soit proposé d'examiner la Vallée où coule le Nil, avec assez de détails pour conclure, de son état present, les changemens successifs qu'elle a subis et ceux qu'elle doit éprouver dans la suite" (p. 185); and at p. 251 he goes on to say, "La question de l'exhaussement du sol de l'Egypte, et de l'accroissement du Delta, avait été traitée, jusque dans ces derniers temps, ou par des voyageurs qui ne faisaient pas de cette question un objet particulier de recherche, ou par des érudits qui prétendaient l'éclaircir en essayant de concilier certains passages d'auteurs anciens contradictoires entre eux, ou du moins que leur obscurité rend susceptibles d'interprétations différentes. On ne pouvait espérer d'obtenir une solution complète de cette question, que lorsque les géologues et ceux qui ont fait une étude particulière de la théorie des cours des fleuves s'en seraient emparés."

But the operations of which I am about to give an account are of a nature and extent that scarcely any individual traveller could undertake; for they have required a large body of men, and some of them practised in the art of surveying; and as they could only be carried on after the waters of the inundation have subsided for some time, and therefore at a season of the year when the heat is excessive, those only inured to the climate could undertake such a work.

I explained my views as to the manner in which I desired that the researches should be carried on in the following directions to Hekekyan Bey:—

"Mr. Harris has communicated to me your most obliging letter to him, in which you so warmly enter into the subject of my correspondence with him, viz. the institution of experiments to measure the depth of the alluvial deposits of the Nile, with

* Mémoires de l'Académie Royale des Sciences de l'Institut de France, année 1817, p. 185.

reference to the great question of the duration of its secular increase. I rejoice to think that I am to have so able a coadjutor on the spot, for in the prosecution of this inquiry you may solve some of the most interesting questions in the history of ancient Egypt. Your suggestions as to the spots where the shafts should be sunk around the Obelisk of Heliopolis are excellent.

"In carrying the plans into execution, it is important that the following particulars should be attended to:—

"1. To make a ground plan of the space around the obelisk within which the shafts are to be sunk, marking the exact spot of each shaft and its distance from the obelisk.

"2. In digging the shaft A, to mark the depth from the highest point upon or near to the obelisk which the alluvial deposit now reaches to the base of the masonry on which the obelisk stands.

"3. If the above masonry rests on rubbish or on Nile mud; if the former, the nature of the rubbish and its thickness.

"4. If the rubbish rests on Nile mud.

"5. The sinking of the shaft to continue so long as it passes through characteristic Nile mud, marking the depth of the mud.

"6. If the mud be completely penetrated through, the nature of the ground on which the lowest layer of mud rests.

"7. In all the shafts sunk, to note every change in the nature of the soils passed through, and to preserve a specimen of each variety of soil, carefully marking the specimen with a number referring to a catalogue descriptive of the sinking.

"8. To examine carefully whether there are any shells or other organic bodies in the soils passed through; and if so, to preserve them, marking each specimen in the way mentioned in No. 7.

"9. If any fragments of human art be found in the soils passed through; and, unless they be brick or other rude material, to preserve them, marking each specimen in the way mentioned in No. 7.

"10. To note the thickness of the layers of Nile deposit, and the number of them in a given space, say a foot."

On the 3rd of June 1851, Hekekyan Bey wrote to me as follows:—"His Highness the Viceroy has been pleased to grant every aid and means required for the execution of the works of research at Heliopolis, consequently I trust that your plans will be carried out to your complete satisfaction. Arrangements have been made to commence working at Heliopolis on the 10th of this month. During the operations I shall most scrupulously adhere to the general directions with which you have favoured me."

The Excavations at Heliopolis.

At a distance of about five and a half miles N.N.E. of Cairo, and less than four miles, at the time of low water, from the right bank of the Nile, the traveller discovers the solitary Obelisk of Heliopolis, all that remains above ground of that once renowned city of the Pharaohs, the On of Scripture. This obelisk, the oldest known, was erected by Sesurtesen (Sesortosis I. of Manetho) of the Old Monarchy, and the twelfth Dynasty, about 2300 years before Christ, according to Lepsius, and has thus stood at least 4000 years, which, according to the *marginal* chronology printed in the latest editions of our Bibles, is about 300 years before the death of Noah*.

The obelisk is a single block, measuring from the pedestal on which it rests, 67 feet 6 inches, its faces being 6 feet 6 inches at the base, and tapering to 3 feet 10 inches at the lowest line of the pyramidal summit. It is of red granite, such as is found in the district of Assouan, and was doubtless transported from the quarries in that locality.

The walls which surrounded the city are still to be traced by long mounds of earth, covering the unbaked bricks of which they were constructed, in some places from 60 to 65 feet in width, and from 13 to 16 feet in height, enclosing an area of about 1540 by 1100 yards. These mounds are now much more than sufficiently high to keep out the greatest inundation; formerly the water entered by gaps and converted the interior area into a marsh; but in the time of Mehemet Ali embankments were raised to keep out the inundation water and render the ground cultivable. The land immediately eastward of the obelisk rises abruptly; it is composed of coarse sand and marl, and is out of the reach of the inundation. It appears probable that the site originally chosen for the temple and city of Heliopolis was a portion of the desert line somewhat raised above the level of the rest of the skirt of the desert, and advancing into the low grounds then inundated by the Nile.

On the 8th of June 1851, forty labourers, under the direction of Omar Effendi Adjutant of Artillery, were on the ground. The next day was devoted to arrangements, the men being shown where and how the works were to be commenced, and explanations given, that they might have some idea of the nature and object of the operations. A party of young engineers from the Polytechnic School in

* "Eben so wenig findet sich bei den Ægyptern irgend eine Andeutung einer grossen Fluth. Dass im ganzen Verlauf ihrer Geschichte keine grosse Naturveränderung, keine unheilvolle Katastrophe stattgefunden habe, wurde dem Herodot ausdrücklich von den Priestern bezeugt, und es möchte nach seinen Worten fast zu vermuthen sein, dass er diese bestimmte Versicherung erst auf Nachfragen erhielt, welche gerade durch die ihm wohl bekannten Fluthsagen anderer Völker bei ihm veranlast waren. Seit Menes (3892 B.C.), hätten ihm die Priester gesagt, habe sich nichts auf Ægypten bezugliches geändert, weder in Bezug auf ihr land, noch auf ihren Fluss, noch in Bezug auf Krankheiten, noch auf Sterbefälle. Herodot. ii. 142."—Lepsius, Chronologie der Ægypter, Einleitung, s. 24.

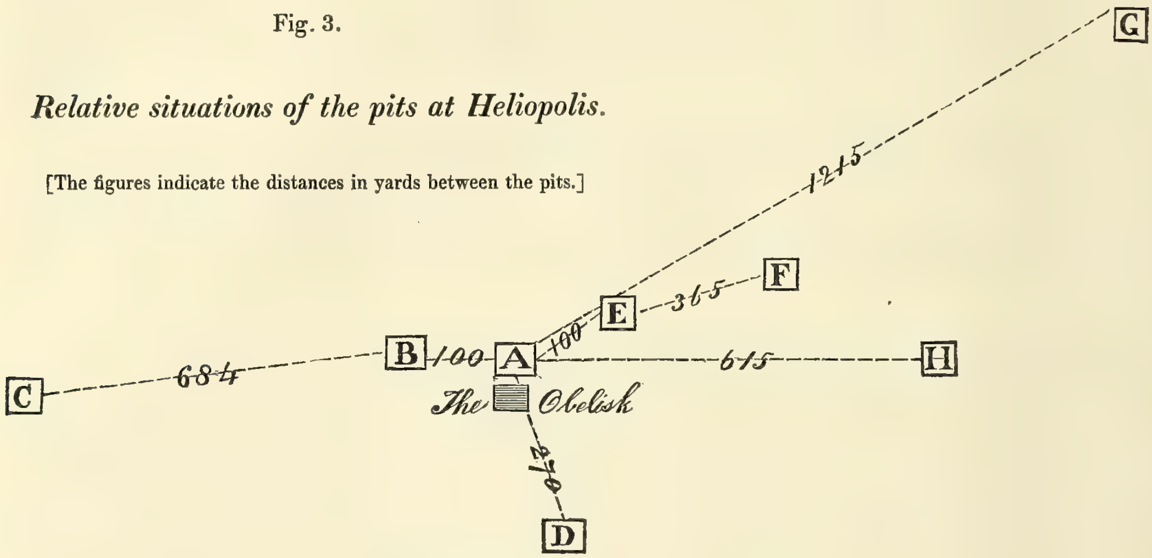
Cairo was afterwards added, with the necessary instruments, to make plans and sections and to take the levels of the ground. Twenty Bedouin labourers were also engaged.

Eight pits or excavations were sunk around the obelisk at different distances, in the situations indicated in the annexed ground plan, and they were carried down to the lowest level of the waters of infiltration in the Heliopolis district on the 16th of July.

Fig. 3.

Relative situations of the pits at Heliopolis.

[The figures indicate the distances in yards between the pits.]



Before beginning to describe the several excavations, it will make the descriptions of them more intelligible and will save repetitions, if I give an account of the nature of the soils sunk through.

As it was to be expected that, on the same level, and within a space of moderate extent, there would be an identity of composition, I requested Hekekyan Bey to send me specimens of all the varieties of soil he met with in sinking the pits, he himself keeping corresponding specimens; thus establishing a standard to refer to in his reports, and saving the necessity of sending specimens of identical alluvia. My request however was not made until he was carrying on similar researches in the district of Memphis, and they were selected from his excavations there; but they have equally served as a standard of comparison for the soils sunk through at Heliopolis, samples of which were in my possession when I made the request.

All the Nile mud, properly so called, has at one time or other been suspended in its water. I was therefore desirous that an experiment should be made to ascertain the quantity of solid matter held in suspension in the water, at a given place near Cairo. Having communicated my wish to Mr. Murray, he prevailed upon Dr. Abbott, a physician long resident at Cairo, to undertake the inquiry. I then described the process and apparatus by which I had in the year 1832 ascertained the amount of solid matter held in suspension in the water of the Rhine, and requested that a similar process should be followed*. Dr. Abbott's account of his experiment, contained in a letter to me dated Cairo the 12th of December 1850, is as follows:—"I began your experiment on the 1st of October, and on that day took an imperial gallon of water from the Nile at the depth of 20 feet, and at that part of the river opposite the Transit Wharf at Boulac. The current is there very strong, and the water is not likely to have any of the dirt or filth that might possibly be mixed with it lower down, where a large number of boats are collected. I took one gallon of water daily for ten days, which I put into an earthen filter, and left covered, until it became perfectly dry; and then put it into a paper and kept it until a week ago, when I weighed it and found the quantity to be 18½ drachms apothecaries weight (1110 grains). I am now endeavouring to dry it in a cake, or rather to bake it in the form of a small brick to send to you."

I weighed the little brick sent to me accurately on the 11th of May 1851, and found it to be 1106 grains, so that the solid matter held in suspension is 110·6 grains in an imperial gallon. An analysis of this solid matter was made at the Royal College of Chemistry in London, by Mr. Brazier, under the superintendence of Dr. Hofmann†, and yielded the following results:—

Silica	53·04
Sesquioxide of iron	18·43
Sesquioxide of alumina	8·76
Carbonate of lime	4·19
Sulphate of lime	0·75
Lime	2·25
Magnesia	0·66
Potassa	0·69
Soda	2·16
Chloride of sodium	0·04
Organic matter	9·03
	100·00

This hardened mass, when moistened, kneaded into a clay.

In future references to this specimen I distinguish it by the letter A.

B. *A specimen of Nile sediment from the lower part of the Delta, sent to me by Mr.* Murray.—It was collected at Damanhour, about six miles and a half from the left bank of the Rosetta branch, from a branch canal which connects the Nile with the Mahmudieh Canal near Bastié, and was a part of the sediment deposited by the inundation of 1849. This deposit is carefully collected on account of its ferti-

* Edinburgh New Philosophical Journal, January 1835.

† That the accuracy of the analyses might be relied on, I requested and obtained the aid of my distinguished friend Dr. Hofmann, who kindly undertook the superintendence of this and of the other analyses hereafter mentioned.

lizing properties, and brought to improve the gardens near Alexandria. It is a fine-grained blackish-grey, loosely coherent earth. It was analysed by Mr. Johnson at the Royal College of Chemistry, under the superintendence of Dr. Hofmann, and yielded the following results:—

Silica	56·86
Sesquioxide of iron	13·19
Alumina	12·11
Carbonate of lime	3·12
Sulphate of lime	0·38
Lime	3·53
Magnesia	2·73
Potassa	0·90
Soda	0·89
Chloride of potassium	0·57
Organic matter	5·53
Loss	0·19
	100·00

On the application of Mr. Murray, Mougel Bey, the French Engineer of the Barrage of the Nile, was so obliging as to send me ten specimens of the soils penetrated at different depths in sinking the foundations. These were analysed at the Royal College of Chemistry by Mr. Brazier, and gave the following results:—

C.—A greenish grey, smooth, fine-grained earth, which when moistened kneads into a somewhat gritty clay. The exact locality was not given, but Mougel Bey describes it thus: "Dans les couches d'argile on trouve des nids de limon ferrugineux que les Fellahs emploient comme amendement sur les terres."

Silica	49·77
Sesquioxide of iron	22·25
Sesquioxide of alumina	5·49
Alumina	7·38
Carbonate of lime	3·37
Lime	1·53
Magnesia	0·14
Potassa	0·77
Soda	0·37
Sulphuric acid	0·15
Phosphoric acid	traces
Organic matter	8·78
	100·00

D.—A blackish brown earth, very much resembling A. except in colour. Like it, when moistened, it kneads into a clay. From the apex of the Delta, on the right bank of the Damietta branch, and from a depth of nearly 20 feet from the surface.

Silica	54·99
Sesquioxide of iron	21·04
Sesquioxide of alumina	11·14
Carbonate of lime	4·20
Lime	3·08
Magnesia	0·17
Potassa	0·69
Soda	0·46
Sulphuric acid	0·12
Phosphoric acid	trace
Chlorine	trace
Organic matter	4·11
	100·00

E.—A blackish brown earth, undistinguishable in external characters from D. From the left bank of the Rosetta branch, and behind the dyke, at a depth of about 20 feet from the surface.

Silica	55·66
Sesquioxide of iron	15·94
Sesquioxide of alumina	4·60
Alumina	5·80
Carbonate of lime	5·26
Lime	3·42
Magnesia	1·03
Potassa	0·74
Soda	0·55
Sulphuric acid	0·33
Phosphoric acid	traces
Chlorine	traces
Organic matter	6·67
	100·00

F.—A blackish brown earth, similar to D. and E. in external characters, but indurated: when reduced to powder, not distinguishable from them, but when moistened it kneads into a more plastic clay than they do. From the same locality as D, but at a depth of only 5 feet 3 inches.

Silica	52·76
Sesquioxide of iron	24·94
Sesquioxide of alumina	13·84
Carbonate of lime	2·95
Sulphate of lime	0·83
Lime	0·94
Magnesia	trace
Alkalies	trace
Chlorine	trace
Organic matter	3·74
	100·00

G.—Very similar to F. in external characters, but the colour inclining more to red. From the same locality as E, but at a depth of only 5 feet 3 inches from the surface.

Silica	57·96
Sesquioxide of iron	19·04
Alumina	11·85
Carbonate of lime	6·65
Lime	0·56
Magnesia	1·37
Alkalies	trace
Chlorine	trace
Organic matter	2·57
	100·00

H.—Almost identical in external characters with E. in all respects. From the same locality as D. and F, but at a depth of 9 feet 10 inches from the surface.

Silica	55·64
Sesquioxide of iron	26·89
Sesquioxide of alumina	7·52
Alumina	4·76
Carbonate of lime	trace
Organic matter	5·19
	100·00

Composition of the eight specimens of Nile mud, taken from different localities and at different depths.

	A.	B.	C.	D.	E.	F.	G.	H.	Average.
Silica	53·04	56·86	49·77	54·99	55·66	52·76	57·96	55·64	54·585
Sesquioxide of iron	18·43	13·19	22·25	21·04	15·94	24·94	19·04	26·89	20·215
Sesquioxide of alumina	8·76	......	5·49	11·14	4·60	13·84	......	7·52	6·418
Alumina	......	12·11	7·38	......	5·80	......	11·85	4·76	5·237
Carbonate of lime	4·19	3·12	3·37	4·20	5·26	2·95	6·65	trace	3·717
Sulphate of lime	0·75	0·38	......	......	......	0·83	......	......	0·245
Lime	2·25	3·53	1·53	3·08	3·42	0·94	0·56	......	1·912
Magnesia	0·66	2·73	0·14	0·17	1·03	trace	1·37	......	0·762
Potassa	0·69	0·90	0·77	0·69	0·74	trace	trace	......	0·473
Soda	2·16	0·89	0·37	0·46	0·55	trace	trace	......	0·553
Chloride of potassium	......	0·57	......	......	......	......	......	......	......
Chloride of sodium	0·04	......	......	......	......	......	......	......	......
Chlorine	......	......	......	trace	trace	trace	trace	......	......
Sulphuric acid	......	......	0·15	0·12	0·33	......	......	......	......
Phosphoric acid	......	......	trace	trace	trace	......	......	......	......
Organic matter	9·03	5·53	8·78	4·11	6·67	3·74	2·57	5·19	5·701
Loss	......	0·19	......	......	......	......	......	......	......
	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	

I.—Grey sand, composed of rounded and angular transparent quartz, some opake, and minute white, brown, and black particles. I could not discover with a strong magnifier any quartz crystals with their faces entire, nor any scales of mica. As it effervesces briskly with acid, the white particles are probably carbonate of lime. From the apex of the Delta, on the left bank of the Rosetta branch, behind the dyke, at a depth of 16 feet 4 inches from the surface.

Silica	71·76
Alumina	10·40
Sesquioxide of iron	7·81
Carbonate of lime	6·77
Magnesia	0·76
Organic matter	2·50
	100·00

K.—Sand very similar to I, but more exclusively quartzose and with a few scales of mica, with some hard concretions of sand. From the right bank of the Damietta branch, at a depth of 31 feet from the surface.

Silica	85·31
Sesquioxide of iron	5·98
Sesquioxide of alumina	1·72
Alumina	1·04
Carbonate of lime	3·64
Organic matter	2·31
	100·00

L.—Brown quartzose sand, very similar to K. Taken from the surface, near the river, at the time of low water, on the left bank of the Rosetta branch.

Silica	83·21
Sesquioxide of iron	6·01
Sesquioxide of alumina	4·09
Carbonate of lime	4·72
Organic matter	1·97
	100·00

M.—A light brown sand, consisting almost exclusively of transparent quartz with a few minute brown and green particles interspersed, effervescing slightly with acid. From the same locality as L, but from a depth of 67 feet 3 inches from the surface.

Silica	96·93
Sesquioxide of iron	2·05
Sesquioxide of alumina	1·02
Lime	trace
	100·00

It will be seen that the two specimens of the present superficial Nile mud, B and C, taken from localities widely apart, although consisting of nearly the same ingredients, differ considerably in the proportions. But in a deposit of this mechanical nature, it is probable that no two specimens, although taken from localities near to each other, would yield the same proportions in a chemical analysis. It will be observed how very similar they are in composition to the solid matter suspended in the water of the Nile, collected by Dr. ABBOTT, Specimen A.

The samples of the alluvial soils penetrated in the excavations, forwarded to me by HEKEKYAN BEY, are divisible into two classes, each offering several varieties, viz. into blackish brown argillaceous sandy earths, and into sands. As all of them closely resemble in external characters the specimens analysed, as above described (A. to M.), it was not deemed necessary for the object of this inquiry to have *the samples* analysed also; but that it would be sufficient to refer them to the analysed standard specimens. They may be thus described:—

No.	Nature of the soil.	Standard specimen it closely resembles in external characters.
	EARTHS.	
I.	A blackish brown, fine-grained indurated earth, not distinguishable in external characters from	F.
II.	A blackish brown, fine-grained indurated earth, not distinguishable from No. I., except in being a shade lighter in colour...............	F.
III.	A blackish brown, fine-grained indurated earth, identical with No. I., except in having interspersed some white calcareous concretions...	F.
IV.	A blackish brown, fine-grained indurated earth, containing the same white concretions as No. III., and some small rounded fragments of burnt brick. It closely resembles No. I., but is more argillaceous	F.
V.	A blackish brown earth, the indurated parts of which, when reduced to powder, are not to be distinguished from B. It contains fragments of brick...........	B.
VI. and VII.	A brown, very friable sandy earth, with minute scales of mica, closely resembling L.	L.
VIII. and IX.	A brown, fine-grained indurated earth, not to be distinguished from No. II..	F.
	SANDS.	
X.	Quartzose sand, containing rounded quartz pebbles, and fragments of burnt brick, desert sand, closely resembling M.	M.
XI.	Quartzose sand, with a few scales of mica; desert sand. Identical with K.	K.
XII., XIII., XIV.	A brownish white quartzose sand, with some aggregated portions, and fragments of bone and burnt brick. Similar to L.	L.
XV.	Fine quartzose sand, nearly identical with	K.
XVI.	A brown argillaceous earth, mixed with quartzose sand, partially aggregated, and containing white calcareous concretions	H. and B.
XVII.	Yellowish white, pulverulent, argillo-calcareous sandy earth, kneading into clay when moistened, partially aggregated. It is called in Egypt "Fine Potter's Earth."	

Besides the above samples, I. to XVII., Hekekyan Bey distinguishes in his sections of the pits layers of soils which are mixtures of the above varieties; thus we have

No.	Nature of the soil.
XVIII.	A mixture of black mud with clay, as if they were kneaded together.
XIX.	A mixture of compact black mud, clay and river-sand in nearly equal quantities.
XX.	Nearly equal proportions of compact black mud, river-sand and fine rubbish.
XXI.	River-sand mixed with a little black mud.

It will be seen that in all the excavations the downward progress was interrupted by filtration water. This is mainly derived from the Nile, but occasionally from side torrents after rain. As the river rises, the level of the water absorbed by the soil on its banks does not keep pace with the rise, for the water takes time to spread laterally, according as the soil is more or less pervious; and should its descent be impeded by a compact layer, it will continue to spread until it is exhausted at a considerable distance from the river. When the Nile falls, that portion of the filtration water which has not penetrated the soil to a depth below the river's ebb level, returns into the channel; but the amount returned will also depend upon the more or less pervious nature of the soil; and when retained by a compact layer, it will remain for some time at a higher level than the falling surface of the Nile.

Excavation A.

Ten men were set to work at this spot. A trench was commenced from a point opposite to and 40 feet distant from the north face of the obelisk, and carried southwards, descending by steps, so as to form an inclined plane downwards. In two days the upper surface of a red sandstone block was reached, being the pedestal upon which the granite obelisk immediately rests. The upper surface of this block was 5 feet 6 inches below the surface of the ground immediately round the obelisk.

A trench was opened opposite to the southern face of the obelisk, 35 feet distant from it, and when carried forward laid bare the pedestal to a depth of 4½ feet from its upper surface. The filtration water having been reached, it was baled out, and it was discovered that the pedestal rests immediately on two layers composed of several blocks of limestone, and under the lower of these white sand was found. The pedestal is 6 feet 10½ inches in height; the first layer of limestone on which it rests is 1 foot 4 inches, the lower layer 1 foot 3 inches in thickness. The limestone is the nummulite limestone of the country; the sandstone of the pedestal is identical with that of the neighbouring hill, Gebel Achmar, that is, the upper sandstone in an indurated state.

An iron bar, 16 feet long and 1½ inch in diameter, was worked perpendicularly by eight men into the sand close to the platform, to a depth of 3 feet $2\frac{1}{2}$ inches under the inferior line of the lower layer of limestone, without meeting with any obstacle from any solid matter, and when passed in a slanting direction 30 inches within the edge of the limestone, still no impediment was met with. Sixteen men were employed to bale out water and sand until a diver was enabled to extract some sand from under the limestone platform, which was fine-grained, white, and very clean.

The total depth of the soils sunk through in a vertical direction was 14 feet 6 inches to the level of the filtration water, which, together with the 8 feet 2 inches of sand penetrated beneath the filtration water, makes the total depth 22 feet 8 inches. The soils exhibit the following varieties, and it will be observed that the Nile mud is here not deeper than 9 feet 11 inches.

Specimen.	Feet.	Inches.	
		9	Disturbed ground mixed with rubbish.
1.	4	11	Undisturbed Nile sediment, with fragments of limestone about the size of a bean, rounded; also fragments of pottery. Slender vegetable fibres are scattered through the mass. It is an indurated earth, similar to the analysed specimen F, and to the sample No. I., but containing a small admixture of quartzose sand, and when moistened it kneads into a plastic clay.
2.	4	10	Termed by HEKEKYAN BEY "rubbish soil." It is a mixture of the blackish brown earth, sample No. I., with many calcareous particles and fragments of limestone, and some fragments of pottery. Near the bottom of the layer was found a band of 7 inches of sand.
3.	4	...	Coarse grey sand, with coral-shaped concretions of the sand, extending to the filtration water. In this layer was found a portion of an eight-sided column of dark green basalt. Near the surface of the water, the sand contained the right upper molar of a ruminant, of the size of a sheep, and the first molar, left side, of the upper jaw of an ass (*Equus Asinus*)*.
	14	6	Level of filtration water.
4.	1	6	Sand nearly identical with No. 3.
	6	8	Sand of the same kind from underneath the lowest limestone flag, containing coral-shaped concretions of sand.
	22	8	

EXCAVATION B.—100 *yards west of* A.

Ten men were set to make this excavation, across the ancient great western avenue leading to the obelisk†. The total thickness of the soils sunk through, from the

* All the fossil bones found in the excavations were submitted by me to Professor OWEN, and have been described by him.

† In the various excavations which have been made in the prosecution o this inquiry, many objects of art of historical interest have been discovered; but as these do not come within the province of the Royal Society, I propose to give an account of them in a memoir to be laid before another learned body.

surface of the ground to the level of the filtration water, was 12 feet 3 inches, and a vertical section exhibits the following varieties:—

Speci-men.	Feet.	Inches.	
1.	6	7	The surface layer of Nile mud. A blackish brown, fine-grained, indurated earth, not distinguishable from the sample No. I., and the analysed standard specimen F; neither is it distinguishable from layer 1. in Excavation A.
2.	1	10	A grey, sandy, pumiceous-looking earth, effervescing briskly with acid, and when moistened, kneading into a plastic clay. Except in being darker in colour, it closely resembles sample No. XVII. in all its characters.
3.	3	10	A dark brownish black sandy earth, closely resembling the analysed standard specimen B, which is a superficial Nile mud. Both effervesce with acid.
	12	3	Level of the filtration water.
4.	...	...	This specimen was obtained from under the surface of the filtration water, at a depth of about a foot. It is identical with the layer above, No. 3.

Here we have 13 feet 3 inches of Nile mud.

Excavation C.—684 *yards distant from* B, *and* 784 *yards due west of the obelisk.*

In this locality ten men were set to work, while others were employed at the excavations A, B and D. A trench was cut from north to south, across the line of the supposed avenue of Sphinxes, and in the progress of the excavation they came upon numerous blocks of stone, the remains of a pavement, and even fragments of a colossal Sphinx, which interrupted the regular vertical sinking. The excavations were carried round these obstructions, on the north of the trench to the depth of 12 feet 9 inches, on the south to the depth of 13 feet 10 inches from the surface to the level of the filtration water. Vertical sections of the soils sunk through exhibit the following varieties:—

North end.

Specimen.	Feet.	Inches.	
1.	5	11	Surface layer. A mixture of Nile mud, a blackish brown earthy sediment, with angular and rounded fragments of limestone*, bricks and pottery, and angular fragments of the sandstone of Gebel Achmar. In the inferior part of the layer were found fragments of a marrow-bone of a herbivorous animal. It is undistinguishable from layer No. 1. of the south end.
2.	2	0	A light brown, sandy, calcareous earth, very similar in appearance and characters to layer No. 2. in Excavation B, except in being somewhat darker in colour. It contains angular fragments of limestone. It is identical with layer No. 2. of the south end.
3.	1	5	A light brown, sandy earth, similar to the preceding layer, but more like layer No. 2. in Excavation B. It contains fragments of bricks and pottery. It is identical with the layer No. 3. of the south end.
4.	3	5	A light brown, sandy earth, very similar to layer No. 3. in Excavation B, except in being lighter in colour. It contains fragments of coarse pottery. It is identical with layers No. 3. and 6. of the south end.
	12	9	Level of the filtration water.
5.	1	6	Identical with the preceding, and containing fragments of chert, limestone and pottery.

South end.

Specimen.	Feet.	Inches.	
1.	3	$7\frac{1}{2}$	Surface layer. The same in all respects as layer No. 1. of north end. There was found in it a broken shell of *Murex trunculus*, a living Mediterranean species; remarkable in being found so far from the sea, but it may have been carried by a bird. The freshwater shell *Paludina impura* or *tentaculata* was also found †.
2.	3	0	The same as No. 2. in the north end. Containing also the same *Paludina* as in the layer above, but altered in the substance of the shell by having been long buried in the ground.
3.	0	7	Very similar to No. 2, and identical with Nos. 3. and 4. of the north end, except in being more sandy.
4.	0	4	Identical with No. 2. of this south end, with many fragments of limestone, bricks and pottery.
5.	0	9	Undistinguishable from the preceding, except fewer fragments. In this layer was found a portion of the pectoral fin of a fish.
6.	0	$7\frac{1}{2}$	Undistinguishable from Nos. 4. and 5. of this south end, except in being more sandy.
7.	0	6	Quartzose sand, with small concretions of Nile mud, and rounded fragments of opake quartz and portions of brick.
8.	1	8	Identical in all respects with layer No. 3. of this south end, and with layer No. 5. of north end.
9.	2	7	Identical with the preceding and with layer No. 5. of the north end. Contains fragments of pottery.
	13	10	Level of the filtration water.
10.	1	6	Identical with the above layers 8. and 9, and with No. 5. of the north end.

Here we have 14 feet 3 inches of Nile mud in the north part of the trench, and 15 feet 4 inches in the south end, with an interposed layer of sand of 6 inches.

* All the limestone fragments are the nummulite limestone of the neighbouring hills.

† On examining this shell, Sir C. Lyell observed, that it is rather a large variety and differs from the English ones in a slight degree, coming most nearly to the variety of the same species found in the Norwich Crag; that this is interesting, as the *Cyrena consobrina*, a species now recent in Egypt, is found fossil in the Norwich Crag.

EXCAVATION D.—270 *yards south by east of the obelisk.*

Ten men were set to work at this spot, to deepen an excavation which was made some time before, when a gateway of the time of TUTHMOSIS III. was discovered. A vertical section of the soils passed through exhibits the following varieties:—

Speci-men.	Feet.	Inches.	
1.	5	6	The surface layer. It is Nile mud, very closely resembling the sample No. V., and the standard specimen B, as well as layer No. 2. of Excavation C, north end, and layer No. 5. of Excavation C, south end. It is the substance of the crude bricks used in building.
2.	6	11	Similar to the preceding, but more sandy. It contains fragments of brick and pottery. It closely resembles layer No. 3. in Excavation C, south end. Near the surface of this layer were found, the lower end of the right humerus of a ruminant of the size of a sheep, part of the upper jaw of a dog, and the lower jaw of a dog with some loose teeth and the fang of a dog's tooth.
3.	2	10	Quartzose sand, very similar to layer No. 7. in the Excavation C, south end.
	15	3	Level of the filtration water.
4.	...	...	The same quartzose sand as No. 3.

EXCAVATION E.—100 *yards north-east of the obelisk.*

This pit was sunk in a mound. A vertical section of the soils sunk through exhibits the following varieties:—

Speci-men.	Feet.	Inches.	
1.	3	2	A brown sandy earth with fragments of limestone. It is identical with the layer No. 2. in Excavation D.
2.	9	4	A brown sandy earth, not distinguishable from the layer No. 1. in Excavation D.
3.	1	$7\frac{1}{2}$	Quartzose sand, scarcely distinguishable from the layer No. 7. in the Excavation C, south end.
4.	3	7	A brown sandy earth, scarcely distinguishable from the layer No. 1. of Excavation D.
	17	$8\frac{1}{2}$	Level of the filtration water.
5.	...	...	Identical with that immediately above. It was taken from the depth of a foot under the surface of the water.

EXCAVATION F.—365 *yards from Excavation* E, *and about* 383 *yards north-east of the obelisk.*

This excavation was made near an opening in a chain of mounds running N. and S., and parallel to that in which the Excavation E. was made. The opening answers to the entrance of an avenue leading towards the obelisk (the temple) from the east. The space between this chain of mounds and the present line of the desert is called by the Arabs the Bahr-il-Moussa, or River of Moses. A vertical section of the soils sunk through exhibits the following varieties:—

Speci-men.	Feet.	Inches.	
1.	3	6	Surface layer. A blackish brown earth, scarcely distinguishable from the sample No. III.
2.	3	9½	A grey sandy earth, very similar to layer No. 2. in Excavation B, and containing fragments of limestone and brick. In this layer was found a milk tooth (lower molar) of an ox, which had been worn down and shed.
3.	...	10	A brown sandy earth, identical with layer No. 2. in Excavation E.
4.	2	2½	Quartzose sand; the quartz nearly transparent, with rolled fragments of opake quartz and chert.
	10	4	Level of the filtration water.

Excavation G.—1215 *yards north-east of the obelisk.*

This excavation was made in the celebrated melon grounds of Heliopolis, which lie on the Bahr-il-Moussa. A vertical section of the soils sunk through exhibits the following varieties:—

1.	2	6	A brown sandy earth, very similar to layer No. 4. in the Excavation E, and the layer No. 3. in the Excavation F, and containing fragments of limestone.
2.	4	2	A brown sandy earth, very similar to layer No. 2. in the Excavation E, with numerous angular fragments of brick and pottery. The upper molar of an ox was found in this layer.
3.	8	4	Quartzose sand, undistinguishable from layer No. 4. in Excavation F.
	15	...	Level of the filtration water.
4.	...	...	The quartzose sand continued below the water.

Excavation H.—615 *yards east of the obelisk.*

This excavation was made in a line between the obelisk and a line of mounds which probably led to the necropolis of the city, which occupies a wide space of ground on the elevated skirts of the desert, due east of the obelisk. A vertical section of the soils sunk through exhibits the following varieties:—

1.	2	10	Surface layer. A blackish brown earth, similar to layer No. 1. in Excavation F, with fragments of brick.
2.	3	8	Chiefly quartzose sand, with numerous rounded fragments of quartz and chert, and a partial mixture of the brown earth of the Nile sediment.
3.	8	...	Quartzose sand, identical with layer No. 4. in Excavation F, and with layer No. 3. in Excavation G.
	14	6	Level of the filtration water.
4.	...	...	The same sand continued below the water.

Synopsis of the Soils sunk through in the nine excavations at Heliopolis.

By an examination of the preceding tables and the diagram, Plate IV., it will be seen that the soils consist of two principal kinds :—

I. Earths (1, 2, 3), more or less sandy and calcareous, varying in colour from a dark blackish brown to a light grey, but evidently so nearly allied, passing by such insensible shades into each other, and having, with slight variations, so great a resemblance to the modern Nile sediment, that they may all be classed as belonging to what is commonly called Nile mud, the earthy matter deposited by the river during the inundations; and

II. Sands (4, 5, 6, 7), partly mixed with indurated portions of Nile mud, but chiefly a pure quartzose sand, similar to that of the adjoining desert.

I have distinguished the chief varieties of each kind by different shadings in the accompanying Plate; whereby it will be more readily seen, that in the same horizontal plane, even in this limited space of half a square mile, there is a very considerable difference in the nature of the soil. Although it might, *à priori*, have been expected that fine earthy particles gradually subsiding from tranquil water, year after year, would form a series of thin layers, in none of the excavations was there an instance of the lamination of the sediment. To this remarkable fact, observed in all the excavations, both here and on the site of Memphis, I shall have occasion to refer in a subsequent part of this memoir.

When we consider the small amount of sediment left annually by the inundation in any one place, it is very difficult to conceive how there should be in any one spot so great a thickness of one kind of sediment without any lamination or other sign of successive deposition. For example, in the Excavation E. there is scarcely any perceptible difference in the nature of the soil to a depth of $12\frac{1}{2}$ feet, a thickness which, if accumulated by annual deposits, would be the work of a vastly long period*. But this great amount of thickness of one kind of soil becomes still more remarkable when we find other varieties at the same level in the immediate vicinity, as may be seen in the sections of these excavations. It is evident that other causes than the tranquil operation of annual inundations must have been at work in the formation of this portion of the alluvial land.

The crystalline quartzose sand, it will be seen, was found to the greatest amount in the pits nearest to the desert; and as it is not at all probable that matter so coarse would be suspended in the inundation water, especially in this locality, the layers of sand were most likely blown across the valley from the desert.

But further general remarks, and all inferences as to the secular increase of the alluvial deposits, the main object of this inquiry, I must defer, until I shall have had an opportunity of laying before the Society an account of the far more extensive researches that were carried on in the year 1852 in the district of Memphis, and

* Were we to adopt the estimate of secular increase given by M. Girard, viz. 5 inches in a century, it would amount to 3000 years.

during the last year in a series of seventy-two pits sunk across the valley, in the parallel of Heliopolis, from the Libyan to the Arabian chain of hills. For this great extension of the inquiry I am indebted to the unabated liberality of the Viceroy; to the continued warm interest taken in it by Mr. Murray, so long as he remained in the country, followed up as it has been, with equal zeal, by our present Consul-General in Egypt, the Honourable Frederick Bruce; and to the untiring energy of my very able coadjutor, Hekekyan Bey.

V. *Observations on the Respiratory Movements of Insects. By the late* William Frederick Barlow, *F.R.C.S. Arranged and communicated by* James Paget, *F.R.S.*

Received August 20,—Read November 16, 1854.

THE following essay contains the greater part of a series of observations, made between 1845 and 1850, by one whose recent death deprived physiology of one of its most earnest truth-loving students. The papers, as left by their author, and committed to me by his father, contained little more than a record of the observations. I have arranged them to illustrate certain general facts, and have added some of the conclusions which they plainly indicate. I have felt the more justified in making these additions, by the belief that my intimate friendship with Mr. Barlow would enable me to write what he would have written, had his life been spared. And in communicating his researches to the Royal Society, I believe I am fulfilling the design with which, not long before his death, he was preparing them for publication.—J. P.

Natural respiratory movements of the Dragon-fly (Libellula).

From nearly all the following observations it may be gathered that these movements constantly vary, in both rate and force. Volition and emotions, changes of temperature and of light, account for many of these differences; and it might have been inferred that, on these disturbing causes being removed, there would be a great uniformity in the mode of breathing, if care were taken that observations should be made with strict regard to sameness of circumstances. It is not so, however; the respiratory movements become much more equal, but they are very far from maintaining true equality of rate and force, even within a short given period. They have times of acceleration and of decrease, which it is hard to account for. Within three or four minutes, even, their speed and strength will vary, although the insect be kept in unbroken quiet. And if different individuals be compared, it does not appear that the vigour of each insect, or the probable length of its life, can be calculated by the force of the respiratory movements.

One of many similar observations may illustrate these statements.

Experiment i. May 19, 1848.—I watched a dragon-fly (*Libellula depressa*) which had been caught the day before. Just before its capture it was pursuing its prey, and flying very swiftly, in all directions, in the sunshine. The temperature was moderate, and there was a slight breeze stirring. The respiratory movements corresponded with the activity of the insect, and were very quick and vigorous. I placed it under a glass, and noted its state on the following morning. The insect had been free from any source of disturbance; the room was but faintly lighted; the temperature of the air was 52° Fahr.; circumstances which would have led me to anticipate that it would be in a very sluggish condition. I found it completely quiet, and making no respiratory movements which were visible on my first noting it. A touch aroused it somewhat, and it began to respire visibly, and to move voluntarily for a time or two; but it soon seemed like a thing half torpid, and ceased to move. I continued to observe it:—at one time it made plain respiratory movements; then, the force of these diminished, and they appeared to cease; and this train of circumstances was repeated several times, although great care was taken to maintain unchanged all the external conditions in which the insect was placed.

I afterwards proceeded to try the effects of mechanical irritation of the insect. On touching it with a feather between the points where the wings are attached, the wings moved rapidly several times;—I think in an involuntary manner, for I have produced precisely such movements, by similar means, in the decapitated dragon-fly. At this time the respiratory movements were lively and powerful; but they remained so for only a few seconds. I counted eleven in a quarter of a minute, but only three or four very feeble ones in the immediately following half-minute; then they became a little accelerated, but were very soon again impaired, and could not be perceived any more for some time.

An hour later I again watched the insect, which had been kept covered with a glass basin. Sometimes the respiratory movements were unseen; sometimes they were very slow and feeble; sometimes they were quick and strong; though, all the while, the insect did not stir.

The influence of excitement and mental emotion, mentioned in this observation, was again noted in many of the following. In all cases, voluntary efforts and agitation, of whatever kind, provoke the respiratory movements when they are too faint to be observed, and accelerate and strengthen them when they are already evident. But that variations of the movements are not wholly due to those of the mental state, will appear from the experiments that show similar variations in decapitated insects.

In like manner, the variations in the respiratory movements, according to the temperature of the insect, the accelerations with the rise, and the retardations with the fall of temperature, may be observed, in some measure, in those that are decapitated.

Influence of Decapitation on the Respiratory Movements.

The effects of decapitation, involving the removal of the supra- and sub-œsophageal ganglia (the analogues of the brain and medulla oblongata) of the insect, vary accordingly as the head is removed by a sharp instrument, or is suddenly crushed. In the latter case, the influence of 'shock' is added to that of the removal of the ganglia. In the following observations the head was severed with as little violence as possible:—

Exp. ii. Sept. 1, 1845.—I took a recently caught dragon-fly, and twice counted its respiratory movements; they were at the rate of sixty-four in a minute. I then separated the head of the insect with a sharp knife. There ensued convulsive movements of the body, and gaspings of the head: the respirations were fifty in the minute; but after the lapse of four minutes, they were reduced to thirty-eight, and after four more minutes to thirty-five; the respiration was performed more feebly than just after the head's removal; and in ten minutes it was reduced to thirty-one times in the minute. During all this time the insect was still, unless it was touched; then movements of the wings ensued. Fæces were spontaneously passed.

I now left the insect, and on returning in an hour, found the respirations in number as before, and quite as powerful. It lay in a tranquillity in which no motions but those of its breathing could be seen, and to one unobservant of the removal of the head, it would have seemed as if sleeping. Fæces were occasionally passed. Four hours later the breathing was feebler, its number being the same: the gaspings of the separated head, when it was touched, had less vigour; on my drawing a feather lightly over the surface, the legs retracted; slight movements of the wings also could be thus occasioned.

Fourteen hours elapsed before I renewed my observations. The insect was now respiring at the rate of twenty-six in the minute. Reflex movements could be yet excited in the body, but the head was quiet in spite of stimulus. In six hours more, the respiratory movements were still twenty-six, and regular. Eighteen hours later, they were still discernible, though very faint.

Exp. iii. Oct. 2.—I counted the respirations of a dragon-fly when it was tranquil, and found them 108 in the minute. I removed the head, and found them forty. Ten minutes afterwards they were fourteen: they were quite equable, and continued at this rate for some time longer.

Exp. iv.—The respiratory movements of another insect were sixty. On the head being removed they were twenty-five, and most regular. Ten minutes passed, and they were fifteen. Half an hour later they were seventeen, and feebler. In another instance they were reduced, on decapitation, from sixty-six to twenty-nine.

The foregoing experiments show that the effect of decapitation is always to diminish the frequency of the respiratory movements of the insect. They are confirmed by those which follow; but these prove other facts also; and first, that the irregularities in the mode and rate of breathing, which are noted in the natural respiration of the insect, are equally observed after decapitation, and are therefore not to be assigned to the will, or any mental state.

Exp. v.—On a day when the temperature was 59° Fahr., I counted the respirations of a cricket, in several minutes, with intervals of a quarter of a minute between each two. They were 84, 106, 79, 64, 59, 90. On the head being removed, they were 19, 27, 28, 20, 19, 14; and were less vigorous.

Exp. vi.—In a temperature of 52° Fahr., the respirations of another were counted like those of the last, and were 90, 70, 73, 72, 50, 62, and unequal in force as well as in rate. The head was removed, and they became 29, 20, 14, 11, 11, 14, and were considerably reduced in force. In another insect they were reduced from 56, 37, 46, 40, 37, 36, to 20, 14, 12, 11, 9, 6.

Exp. vii.—I counted the respirations of a dragon-fly eleven times, in as many minutes, with intervals of a quarter of a minute between each two. They were 100, 101, 98, 100, 110, 106, 108, 108, 108, 108, 106. I then removed the head, and numbered the respirations in the same manner again, and found them 55, 61, 70, 70, 66, 70, 68, 67, 70, 70, 62.

The evidence of all the foregoing observations, proving the diminution, in both rate and force, of the respiratory movements when the head is removed, may serve to illustrate the admitted correspondence of the supra-œsophageal and sub-œsophageal ganglia of the insect, with the brain and medulla oblongata of the vertebrate animal. But that these movements should be only diminished, and should not cease, as they do when, in a vertebrate animal, the medulla oblongata is destroyed,—this may be regarded as indicating that there is, in the insect, a multiplication and diffusion of the nervous centres for the respiratory movements, corresponding with the plan of multiplicity in the respiratory organs. The same conclusion may be derived from experiments, which will be related, of the effects of dividing the body into segments.

Influence of Shock *on the Respiratory Movements.*

Exp. viii. May 11, 1848.—Wishing to ascertain the effects of shock upon the respiratory movements, I took a dragon-fly (*Libellula depressa*) which had been caught the day before, and was breathing pretty vigorously, between forty and forty-five times in the minute, and crushed its head completely and suddenly. There followed perfect stillness; the respiratory movements were quite indiscernible, until between three and four minutes had passed; then, they could be just seen; but they soon ceased again. After about four minutes more had elapsed, the respiratory movements were marked with rather more power, and persisted; but they were as yet very faint and unequal. In two hours, though they continued, they were still without vigour; and so they remained twenty-four hours later.

Exp. ix. May 13, 1848. Temperature 64°.—I took a dragon-fly, which was breathing at the rate of thirty-six, and crushed the head and upper part of the thorax. The respiratory movements ceased, but in less than a minute were resumed; they were, however, very feeble and so remained. In about eight minutes from the infliction of the shock, the movements were at the rate of thirty-nine, but their power was exceedingly diminished, and they required minute watching to observe them properly.

Exp. x.—I crushed the head and thorax of a dragon-fly; the respiratory movements ceased for two minutes. At the same time, another, while in full activity, was struck a sharp blow which inflicted a like injury. The respiratory movements stopped instantly, were suspended for many minutes, and never recovered any degree of power.

Exp. xi. May 22.—To another dragon-fly whose respiratory movements were similarly feeble, after recovering from the shock of crushing the head, I applied stimulus by holding it over water of the temperature of 185° Fahr., and the strength and frequency of the movements were greatly increased.

Exp. xii.—In a dragon-fly, which had been decapitated and was breathing distinctly at the rate of sixty-three, I crushed the last two abdominal segments. For a minute and a half the respiratory movements were but dimly visible; then they revived gradually. This shows the retrograde influence of shock.

These experiments, while they confirm those which illustrate the influence of mere decapitation, prove also that the influence of shock, *i. e.* of sudden violent destruction of a part, is essentially the same in the insect as in the vertebrate animal. They show that the separation and evident distinctness of nervous centres do not so dissociate the parts, with which they are severally connected, as to place any of them beyond the influence of the injuries inflicted on the rest. And yet, while these effects of shock may prove the mutual relations of the several ganglia, the following experiments on the division of the body into segments may show how each ganglion is the centre for the respiratory movements of its own segments.

Respiration in the separate segments of the Insect.

Exp. xiii. Sept. 7, 1845.—I decapitated a dragon-fly, which was breathing vigorously at the rate of fifty-five in a minute. In five minutes the respiratory movements were forty-two, and considerably less forcible than before the decapitation. Eight minutes later they were thirty-nine. I now divided the abdomen of the insect into three segments, with care to produce as little shock as possible. At first I could discern respiratory contractions in that division alone which was attached to the thorax, and I waited as long as seven minutes before I could distinguish them in the remaining segments; and now, in all, they were very weak in comparison with those in the thorax, and still more so in comparison with what had been noted in the perfect insect. Had this been owing to shock, they would probably have been restored; but twenty minutes later they were still indistinct, and at times imperceptible. The superior division continued to respire by far the most strongly. On placing the middle segment in my hand, the movements became, in about a minute, very distinct, and increased to the rate of 100 in the minute; but, on replacing it on the table, they became again obscure, and soon imperceptible. Three hours after the operation no respiratory movements were observable, nor could any be excited, though the legs of the insect could be excited to retraction.

Exp. xiv. Oct. 14, 1845.—I took a small dragon-fly, which was breathing powerfully at the rate of 116 per minute, the movements being possibly quickened by emotion. On removing the head, they were diminished, in one minute, to fifty-eight (just half the former frequency), and were much weaker. I divided the abdomen into two equal parts. The respiratory actions ceased awhile; but the effects of the shock beginning to subside, they became apparent. In ten minutes they were more feeble but plain, and could be observed in both divisions. I could not succeed in counting those in the upper half; but fifteen minutes later they were fifty in the lower half. Two hours and a half after the segmentation, the respiratory movements were still visible.

Exp. xv. Oct. 14.—I counted the respiratory movements of a dragon-fly, of the same kind and size as the last. They were only sixty; and the difference is curious; for the two insects were caught together, and examined in the same manner, and in the same hour. On removing the head, the respirations were reduced to forty, and became much less distinct. The abdomen was bisected, and the respiratory movements ceased in it. In ten minutes they were renewed, but most faintly and slowly, and in the lower half alone. On breathing on the upper half, it contracted languidly for a few moments and then became motionless again.

Exp. xvi. June 29, 1846.—At a quarter before 2 P.M. I ascertained the respirations of a dragon-fly, while it was calm and at rest, to be thirty-two. Then I removed the head, and involuntary movements of the body, wings, and legs were noticed, which in a little while ceased, but were renewed, at pleasure, by excitation. In five minutes the respirations were reduced to twenty-eight, and were less powerful. The abdomen was now divided into three equal parts, two complete segments being in each. The respiratory movements were well-marked in both the upper divisions, most strongly in the uppermost; in the lowest, or caudal division, they were not observable. In the uppermost division of the abdomen, the respirations were at the rate of thirty-two; in the middle division, at the rate of forty-eight in the minute; in the caudal division, they were not excited even by breathing on it.

At half-past two the respiratory movements in the parts of the abdomen were feebler; in the thoracic division, they were still twenty-eight; in the middle division, fifty-two. At half-past three, the only respiratory movements discernible were those of the upper division; but they were much weaker and only twenty-four. Powerful respiratory contractions were excited by very lightly drawing a sharp point down the dark mesial line of the abdomen; and an incurvation of the body, and a peculiar movement of the legs towards the part irritated, were observed. At a quarter past seven the respiratory movements had ceased, but were renewed, though faintly, by breathing on the upper divisions of the abdomen. It was not easy to say why the caudal division did not respire, for its excito-motory power was not extinguished; it could be excited to motion, and fæces were expelled by it.

Exp. xvii.—The subject of this experiment was a lively dragon-fly, which breathed at the rate of thirty-five. The head was removed, and the respirations, which did not pause perceptibly, were forty-two. In five minutes, the respirations being at the same frequency, the abdomen was divided into three nearly equal parts. The caudal division breathed powerfully; the thoracic feebly; the middle one not at all. I placed this motionless part in my hand; in ten minutes it revived, and breathed at the rate of forty-eight. It was laid on the table, and in ten minutes it ceased to respire. In the other segments the breathing went on languidly, and soon ceased entirely.

The foregoing experiments, on the effects of dividing the abdomen of the insect, are confirmed in their evidence of the several ganglia being the centres, each in its own segment, of the respiratory, as well as of other involuntary movements, by some that show the influence of chloroform and ether.

Having ascertained that the complete immersion of insects in the vapour of chloroform or sulphuric ether quickly suspends the power of voluntary, and then of the respiratory and other involuntary movements, I wished to learn the influence of these liquids applied in a more limited manner.

Exp. xviii.—In a dragon-fly (*Libellula vulgata*), which was respiring vigorously, I completely suspended the respiratory movements by moistening the under surface of the abdomen with a camel's-hair brush dipped in chloroform. The insect remained quite lively; and though so much of its respiratory movement was thus checked, yet no difference could be observed in the strength or activity of its voluntary actions.

In another, I suspended the abdominal respiratory movements by a similar application of chloroform, but it was so little affected in its general motor power that it took flight on being released from the hand.

In another, I suspended the respiratory motion of the two upper segments of the abdomen, by applying chloroform to them alone. The other segments remained in action, till, by similarly moistening them, segment after segment, I gradually stayed the action of them all.

These effects were not due to the tracheæ being filled with liquid. None like them were produced by immersing dragon-flies in water; for in this liquid the respiratory movements continue vigorous, and are accelerated when the water is of high temperature. On the other hand, the same suspension of respiratory motion ensues when only the vapour of chloroform is locally applied. I put about five drops of chloroform into a glass bottle, and tied a piece of white leather tightly over it; and then, through a small hole in the leather, I passed only the abdomen of the insect into the bottle. When, in a minute, I withdrew it, the respiratory movements had altogether ceased; but the insect remained conscious, and could be readily excited, presenting then the curious spectacle of the legs and wings quickly moving, while not a breathing movement could be seen. In a few minutes very slight respiratory actions became visible; and they were gradually restored to nearly their former rate and power. I then repeated the experiment, with the same result.

It might be certainly anticipated, from these experiments, that destruction of all the ganglia would abolish all respiratory movements; but it seemed right to determine the point by experiment.

Exp. xix.—I decapitated a dragon-fly (*Libellula vulgata*); there were still vigorous respiratory movements, thirty in the minute; and the most lively reflex actions were produced by mechanical irritation, especially by irritating the last segment of the abdomen. I cut off this segment, and then introduced a long pin, which I moved up and down cautiously, with the view of destroying the ganglia, taking care not to injure the abdominal walls. The respiratory movements ceased entirely, and were not renewed; it became impossible, also, to occasion reflex actions by irritating the abdomen.

This experiment I have twice repeated with the same results.

Influence of Temperature and Galvanism on the Respiratory Movements.

The respirations of the dragon-fly present the same variations in vigour, as do the other movements, and the whole apparent life, according to the temperature of the medium in which it is placed. A kind of sleep overcomes the insect in the cold nights of autumn, after which it is found numb and motionless upon hedges, appearing as if dead, but easily revived by the sun's heat, or by artificial warmth. At these times its breathing is probably often suspended for a long period; and were it not that (as the following among many experiments show) the movements can be renewed after long suspension, the insect would be much more short-lived, and would perish at any considerable fall of temperature.

Exp. xx.—I put a dragon-fly (*Libellula depressa*), which was breathing at the rate of about forty-five in the minute, into water of the temperature of 100° F. The respiratory movements increased to 110. On removing the insect, and exposing it to the air, they were reduced to fifty. On replacing it in water of the same heat, the respiration again became greatly accelerated, but very quickly ceased, and was not renewed for a long time.

Another dragon-fly, breathing at the rate of thirty-five, was put into water at 90°, and its respiratory movements were raised instantly to seventy.

Exp. xxi.—I put a dragon-fly into a freezing mixture at 33°, and kept it in for thirty seconds. When taken out it was not respiring, but in less than a minute began to respire at the rate of thirty-five in the minute. I again immersed it for nearly three minutes. The respiratory movements this time ceased much longer and were more gradually renewed. The general activity of the insect was also more deeply affected. At first it seemed nearly insensible, and its voluntary movements were few and sluggish; but by degrees it became active. When it was breathing faintly and irregularly, in consequence of its exposure to the cold, I put it into water of the temperature of 98°. The respirations rose to 108, but they were still feeble, and the insect showed considerable excitement.

Exp. xxii.—The temperature of the air being 70° F., I put a dragon-fly, which was breathing vigorously, into water at 34°. After two minutes I removed it, and all movement had entirely ceased; but in a few minutes the respiratory movements were renewed, and it quickly recovered. I immersed it again with the same result. Afterwards, having let it remain for half an hour, I galvanized it while it was perfectly motionless and insensible, and produced both respiratory motions, and movements of the legs and wings*.

The dragon-fly is very susceptible of galvanism ; but if the wires be applied to the outer surface of the insect they produce little effect, because it is so bad a conductor. In all the experiments, therefore, from which I have drawn any conclusions, I first removed the head of the insect, and then, placing its under surface uppermost, passed a pin through the centre of the most anterior part of the thorax, and another through the last segment of the abdomen, and then fixed the insect to a table or flat piece of wood. I thus obtained a ready conduction for the galvanic current, when the heads of the pins were touched with the wires of the battery, and had the insect in a convenient position for observing its movements.

In regard to the influence of galvanism on the respiratory motions (to which alone, as in the former observations, I shall here refer), its almost constant effect is to accelerate them. I have, by its influence, increased the respirations from thirty to 150 in the minute, but a less increase (*e. g.* from thirty to fifty) was more usual. Sometimes, while galvanising in the method described above, I have observed an almost persistent respiratory contraction, so long as the current was continued; and on breaking the current, the respiratory relaxations and contractions alternated as before.

The effects of galvanism on the respiratory movements are well seen when the insect has been influenced by chloroform. For example, I exposed a recently caught dragon-fly to chloroform vapour. Soon, sensibility and all motions ceased, including the respiratory. I tried many plates of a trough excited first by water, and then by acid, but equally without effect; but having waited till the respiratory movements were resumed, I again employed the same power, and excited the contractions as often as I chose, making them much more powerful than they were when not thus stimulated.

The following chief conclusions may be drawn from the series of observations that have been related:—

1. The respiratory movements of dragon-flies (and probably of other insects) are naturally subject to considerable and frequent variations in force and rate, the causes of many of these variations being as yet unknown.

2. The respirations are always quickened by exercise, emotion, rise of temperature, galvanism, and mechanical irritation; and the last three agents quicken them in the decapitated, as well as in the perfect insect.

3. The respiratory movements of each segment of the trunk are, in some measure,

* In these experiments the changes of temperature were produced by submersion in water; but many others showed that submersion in water of the same temperature as the air produced no similar effects, and that those here described were equally produced in air of similarly varied temperature.

independent of the rest: they may be performed in separate segments, provided their nervous ganglia and cords are entire, and not paralysed by such influence as that of chloroform or ether.

4. The removal of the head, including the supra- and sub-œsophageal ganglia, does not, like the removal of the medulla oblongata of the vertebrate animal, put a stop to the respiratory movements, but diminishes their frequency and force, and deprives them of all influence of the will and of mental emotions.

5. The shock inflicted in the sudden destruction of the head, or of the terminal part of the abdomen, generally stops all the respiratory movements for a time, and greatly enfeebles them during the remainder of the insect's life.

6. The general tendency of the observations is to corroborate the opinion of the self-sufficiency of the several ganglia for the movements of their appropriate segments, and, thus far, of their essential independence; at the same time, their mutual relation and influence are proved by the co-ordinate similar movements of the segments, and by the diffused influence of shocks.

VI. *On the Structure of certain Limestone Nodules enclosed in seams of Bituminous Coal, with a Description of some Trigonocarpons contained in them.* *By* Joseph Dalton Hooker, *M.D., and* Edward William Binney, *Esq.*

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THE specimens of plants which we are about to describe were found imbedded in nodules of limestone, enclosed in a thin seam of bituminous coal not above 6 inches thick, in the lower part of the Lancashire coal-field. Their relative position is best understood from the following section (in a descending order).

1. Black shales containing *Avicula papyracea, Goniatites Listeri, Orthoceras attenuatum* and other Mollusca, apparently of marine origin.

2. Bituminous coal enclosing a horizontal layer of limestone nodules containing fossil vegetable remains.

3. Fire-clay full of *Stigmaria ficoides.*

The roof of the seam is also full of fossil shells, and those in the shales lie in immediate contact with the bituminous coal.

The nodules of limestone occur at short irregular distances, and their size varies from that of a walnut to lumps weighing half a hundred weight. The smaller nodules are spherical, the larger are vertically compressed, being oval, compressed oval, cylindrical or flattened cylinders. The presence of the small nodules may be readily detected by the weight of the coal containing them, whilst the larger ones cause the coal to bulge out, both in the roof and floor of the mine. The surfaces of the nodules are extremely hard, but frequently present faint traces of lamination, or more rarely of concentric foliation.

An examination of the surface offers very little indication of the fossil contents of these nodules, except that iron pyrites is more abundant in those containing *Halonia, Lepidodendron* and *Stigmaria,* causing in some instances a partial decomposition of the fossils.

The origin of these nodules may probably be ascribed to the presence of mineral matter, held in solution in water and precipitated upon, or aggregated around certain centres, in the mass of vegetable matter now for the most part turned into coal. The effect of this has been to preserve certain portions of the mass from becoming bituminous and to produce their calcification. We however offer this explanation with considerable diffidence, being aware that the whole subject of the formation of nodules of one mineral in a matrix of another, is one that involves many considerations, and shall therefore confine ourselves to remarking, that the appearances are of these nodules being sealed masses of fossil vegetable remains, and as such are probably fair samples of the vegetation that has produced the surrounding coal. The immediate cause of the calcification was no doubt due to the abundance of fossil shells in the shales immediately overlying the coal and nodules.

The remains of fossil plants hitherto met with comprise the following genera, which are given in the order of relative abundance in which they occurred, viz. *Calamodendron, Halonia, Sigillaria, Lepidodendron, Stigmaria, Trigonocarpon, Anabathra, Lycopodites, Lepidostrobus, Medullosa,* and others that are indeterminable. An analysis of a piece of fossil-wood (*Calamodendron*) taken out of the centre of a nodule, has been made for us by Mr. Hermann.

Carbonate of lime	76·66
Carbonate of magnesia . .	12·87
Sesquioxide of iron	4·95
Sulphate of iron	0·73
Carbonaceous matter . . .	4·95

From the above observations, it would appear, that the fossils in question are possessed of a double interest; the geologist recognises in them an association of vegetables that certainly prevailed throughout the epoch of the Coal formation, and in all probability contributed largely, if not almost exclusively, to the formation of that mineral; whilst the botanist detects in them characters of the greatest value as throwing light upon the affinities of the Flora of the period.

A section of any of these nodules shows a confused mass of decayed and apparently decaying vegetable remains; they present no appearance of these remains having been brought together by any mechanical agency; they appear to be associated together just as they fell from the plants that produced them, and to be the rotting remains of a redundant and luxuriant vegetation. Fruits of *Trigonocarpon*, entire or more or less decayed, occur abundantly, in the masses, along with the stems and roots of ferns and other cryptogamic plants and isolated masses of wood of unknown affinity. *Lepidostrobi* of various sizes, and apparently in all states of growth, are intermingled with these; but we have found no traces of coniferous wood, nor of the fronds of ferns; the absence of the latter may readily be accounted for from the fact of the nodules never cleaving so as to expose flat surfaces of any of the vegetables, and it is difficult to conceive the delicate fronds of ferns so preserved, that their structure should be recognized on a transverse section of them in a fossil state. The absence of coniferous wood is not so easily accounted for; and coupled with it we may remark, that we have not hitherto found any tissue at all resembling that which occurs occasionally abundantly in bituminous coal, and is known as mineral charcoal and mother-coal.

In none of the extensive series of sections that we have made and examined is there any appearance of that longitudinal arrangement of the mineral matter traversed by parallel canals of amber-coloured deposit that is so conspicuous in many good bituminous coals, and which has been considered by some eminent microscopical observers to be positive evidence of such coal being compressed vegetable tissue: on the contrary, the cellular and vascular tissues of our specimens, wherever they have decayed, present a homogeneous black or brown mass; and where no such decomposition has supervened, the vegetable tissues are so preserved that their real nature is evident. We are not, however, prepared to lay any particular stress upon this point, because, even if it be allowed that these nodules present a fair sample of the vegetable constituents of the coal surrounding them, it does not follow that the same assemblage of species has formed other coals; we may however remark, that with regard to some bituminous coals at any rate, we are inclined to regard the appearance of fibrous tissues as due to a molecular arrangement of the particles of that mineral, which no doubt had its origin in vegetable matter, but in which every trace of structure has been destroyed previous to, or during its mineralization.

The absence of *Calamites* (one of the most typical and universal of coal plants) is another curious fact connected with these fossils; the explanation is however very simple, for it has long been known to one of us, that some species of this genus represent the casts of the hollow or cellular axis of *Sigillaria* and *Calamodendron*, and perhaps of many other genera, as *Sternbergia* does of *Dadoxylon*; this is a subject however to which we shall recur at another time, when, having completed the analysis of the specimens of *Calamodendron* contained in these nodules, we shall hope to lay the results before the Royal Society. In the mean time we shall proceed to describe the structure of *Trigonocarpon*, the most interesting of the genera which we have named.

The usual form in which *Trigonocarpon* occurs is well known, and has been repeatedly figured. That this, however, was that of an incomplete organ has long been considered probable, and almost confirmed by the discovery of such specimens as those of *T. ovatum*, figured in Lindley and Hutton's 'Fossil Flora' (tab. 142 a), and in the 'Records of the Geological Survey of the United Kingdom' (vol. i. p. 430). There have also been found in the coal shales compressed *Trigonocarpons* surrounded by a disc, as if lying in the concavity of a scale, and suggesting the possibility of these fruits having been detached from a cone similar to that of a pine. In none of the specimens preserved in the limestone nodules are any such appearances presented; and it may be assumed that the appearance in question is due to the compression of the fleshy coat of the *Trigonocarpon*. The presence of this integument and of various others was stated in a notice printed in the 'Proceedings of the Royal Society' for March 30, 1854, and was one of the first results of our examination of these fossils: the more detailed analysis and figures were, as was then stated, reserved for an after communication.

Plate IV. fig. 1 represents a very beautiful specimen of *Trigonocarpon*, exposed by breaking a nodule of limestone. It is crossed by a fissure, dividing it into portions A. and B., and to understand its structure it is necessary to refer to fig. 2, which represents the same fossil, with the portion A. removed; fig. 3 represents the under surface of A; figs. 4 and 5 are of thin slices taken off fig. 2; and finally, figs. 11 to 17 represent highly magnified figures of the minute anatomy of the organs represented above, but taken from a large selection of specimens, some of which are represented at figs. 6 to 10.

The outermost integument (Plate IV. fig. 1*a*) is entirely cellular; it encloses the whole seed, except at the perforated apex; it presents neither cuticle nor epidermis, having apparently undergone partial decomposition; and in many specimens its tissue is entirely confounded with that of the surrounding vegetable remains, so that its limits cannot be defined. At fig. 12*b* it is seen almost in contact with a fragment of *Anabathra* (fig. 12*a*). This integument is composed of large utricles, that appear hexagonal when cut across (*b* of figs. 11 to 14); the individual cells do not retain any traces of having been nucleated, nor do they present any markings on their walls; they become smaller and closer in approaching the next integument, into which this outer one seems to pass insensibly, without any interruption of continuity.

The second integument (Plate IV. fig. 1*b*) consists of a much denser tissue than the former, and forms the body of the fruit; it is frequently preserved in a fossil state, but more often the cast of its cavity alone remains: it varies from one line to a quarter of an inch in thickness, and at the rounder end of the fruit it presents an annular ridge, surrounding a cuspidate point. This ridge, it may be assumed, surrounds the base of the seed; on a vertical section it presents the appearance of shoulders on each side (Plate IV. fig. 7 & 9*a*), the intervening space being probably the surface of attachment. At the opposite or narrow end of the fruit these integuments are prolonged as a conical cylindrical or trigonous beak, traversed by a narrow canal leading to the cavity of the second coat. The termination of this beak is always decomposed, but its base appears in some cases to be surrounded by an annular ridge, seen in a longitudinal section at Plate IV. figs. 5, 9 & 10*b*: it is remarkable, however, that though this ridge is evident on the slice fig. 5, which was taken from specimen fig. 1 at a considerable distance from the axis of the fruit, no traces of it are seen at figs. 1, 2 & 4, which are from nearer the axis of the same specimen.

The structure of this part of the fruit is curious; it appears to consist of parenchyma, the cells of which radiate upwards and outwards from the inner walls of the integument: the outer layer of cells (figs. 13 & 14*c*) is much transversely elongated; in passing inwards they become shorter, irregular, tortuous and confused (fig. 13*d*); towards the inner wall (figs. 13 & 15*e*) they are very small and short, and suddenly become longer (figs. 13 & 15*f*); they form a lining of long slender tubes to the whole cavity of the fruit. Amongst these last-mentioned cells some may be found marked with annular or spiral bands. All the cellular tissue of this integument is almost filled with dark golden-brown or blackish contents, and it is the presence of these contents that defines this integument from that surrounding it. The real nature of these cell-contents can only be conjectured; they may be the coloured inner walls of the cells, or a deposit of chlorophyle or resin in a peculiar condition: the appearance is exactly that presented by the cellular tissue of *Salisburia* and *Phyllocladus* fruits, where the rich brown colour is probably due to a deposit of resin.

Within these integuments is a large oval cavity, full of carbonate of lime and magnesia, of a yellowish white colour, and very compact. When the fossil is fractured, this mineral presents a cast of the surface of the cavity (as represented at Plate IV. fig. 1), and, when sliced, exposes a delicate membrane or sometimes two concentric ones, as represented in most of the specimens figured. These membranes have always more or less collapsed, and apparently are broken up into layers (Plate IV. figs. 7 & 8). These appearances of a double or treble membrane are probably due to the breaking up of one, or to the decomposition of the walls of the cavity; for all are uniform in structure, and shreds of cellular tissue and scalariform vessels are often found uniting them, and they are further identical in structure with the walls of the cavity. A very highly magnified view of a portion, taken from the section figured fig. 7*e*, is given at fig. 15, where the arrow indicates the position of the vascular bundles, which are more highly magnified at fig. 16. At fig. 15 the section of this membrane is seen bent at an angle on the right-hand side, and traversing the transparent carbonate of lime; to the left of it is a broad fissure in the mineral, and to the extreme left of the circle at *f* are the tissues of the inner wall of the second integument of the fruit. At fig. 17 is seen another section of the same membrane, formed of cylindrical utricles, with no traces of vascular tissue.

We have been thus particular in describing these structures, because we find them to be uniform throughout the very numerous suites of sliced specimens which we have examined: it is scarcely necessary to add, that though they exhaust our materials, they leave much to be desired in our knowledge of the fossil to which they belong. Nothing, however, has occurred during our study of them, to warrant our expecting to find any further structure in the cavity of the seeds taken from the limestone nodules.

Although no positive proof of the real nature and affinities of *Trigonocarpon* can be offered until the discovery of embryos or spores within the cavity of the fruit, there are so many important points now shown to exist in their structure, as to warrant more exact comparisons than have ever yet been instituted. We have already mentioned *Salisburia* as the nearest existing analogue known to us, and shall accordingly proceed to discuss it.

Salisburia is a drupe-bearing coniferous tree, a native of China and Japan, long cultivated in Europe, but only producing fruit in the middle and southern regions of this continent; these are oblong, about the size and colour of a damson plum, and are produced on the terminal branches of the tree, from which they are easily detached when ripe. The drupe consists of three integuments, which are the metamorphosed coverings of a naked ovule. The external integument is thick and fleshy as in *Trigonocarpon*, is covered with a delicate cuticle (Plate V. fig. 6), and is formed of membranous utricles; the outer layers of these cells are empty, but the inner, which gradually become longer, are filled with a viscid resinous fluid. Towards the inner surface of this integument the cells are much elongated, and become mixed with scalariform and annulate and subspiral vessels and long empty cells (fig. 8 *a* and fig. 9); within this is a thin crustaceous integument, formed of densely packed, vertically elongated sclerogen cells (figs. 10, 11), and this again is lined by a delicate coat of annulated cells similar to those outside it (fig. 15): all these integuments are perforated in a young state for the impregnation of the ovule, and this is the only explanation or analogue which we can offer of the canal leading down to the cavity of the *Trigonocarpon.* An extremely delicate membrane (fig. 16) surrounds the albumen of *Salisburia*, and the latter is formed of a densely packed mass of cells (fig. 17) enclosing minute starch granules* (fig. 18).

The absence of any crustaceous integument exactly similar to *Salisburia* is to be remarked in *Trigonocarpon*; but these organs are so extensively modified in the allies of *Salisburia,* that the suppression of one in *Trigonocarpon,* or its representation by the middle coat (which certainly appears to have been much indurated), is a consideration of comparatively little moment, whilst the resemblance between the structures which we have described in the two genera is very remarkable.

The supposed alliance of *Trigonocarpon* with *Coniferæ* does not, however, rest on the above comparison alone, but to a certain extent upon collateral evidence: thus, the presence of wood, closely resembling what is supposed to be typical of *Coniferæ,* is abundant in the carboniferous formation; while the absence of cones and of foliage similar to that of those *Coniferæ* of the present day, whose seeds are similarly arranged, renders it probable that the drupe-bearing division of the Order, which is now chiefly prevalent in the Southern hemisphere, predominated in the carboniferous æra. The remarkable fact too, of the resemblance of *Noggerathia* leaves (a carboniferous genus) to that of *Salisburia,* was long ago indicated by LINDLEY and HUTTON; and though adduced by BRONGNIART in favour of *Trigonocarpon* (which is sometimes found associated with these leaves) being allied to *Cycadeæ,* accords better with the assumption of both being coniferous.

The association of *Trigonocarpon,* in the nodules we have examined, unfortunately offers no clue to their affinities, as we find neither cycadeous nor coniferous wood along with them; while of plants belonging to or allied to Filices and Lycopodiaceæ we find abundant remains in close proximity with the *Trigonocarpons,* but in the latter there are no traces of the tissues so prevalent in these plants. It is further to be remarked, in connection with these plants, that *Salisburia* fruit not only presents no trace of coniferous tissues, but abounds in scalariform annular and subspiral vessels, which are supposed to be very rare in the order to which it belongs.

* These analyses are added in the hope that they may aid others in the investigation of this interesting subject, should any one be so fortunate as to detect any contents within the cavity of *Trigonocarpon.* We have also at fig. 12 figured some loose sclerogen cells that are found occasionally on the inner wall of the crustaceous integument, and at figs. 13 and 14 some modifications of the cellular tissue in the inner coat.

The abundance of these *Trigonocarpons* in the nodules, and the fact of their lying at all angles and in all positions, suggests the probability of their having fallen from a height into a soft or spongy mass of decaying vegetable matter; and it may be noticed, that the very similar fruits of *Podocarpus ferruginea*, a New Zealand drupe-bearing Conifer, are in like manner shed in profusion from the lofty trees that produce them, and become imbedded in the swamps out of which the trees grow; and that the latter are often covered with Ferns and Lycopodia in great profusion, the decay of all which produces a spongy bog, in consistence not unlike that of the mass in which the *Trigonocarpons* are imbedded. Such comparisons cannot, however, be carried far; for whereas these New Zealand bogs, and the clay upon which they almost invariably rest, are everywhere traversed by woody roots of coniferous trees, we find in the substance of the limestone nodules and in the underclay of the coal no trace of these, but in the latter *Sigillaria* roots abundantly (viz. *Stigmaria ficoides*). In our present state of knowledge (or rather ignorance) of the physiognomy, as well as of the botanical characters of the vegetation of the coal epoch, all references of detached organs are extremely rash, and in the present case we cannot venture beyond alluding to the facts, that the flower and fruit of *Sigillaria* are totally unknown, and that these and *Trigonocarpon* fruit and *Noggerathia* leaves are very abundant throughout the coal formation. There is another curious point, to which also we can only incidentally allude, which is, that *Salisburia* has several embryos in each seed, which in germinating become as many young plants: these often coalesce at a very early period, and the result is a compound tree, with one main axis, but as many primary roots as there were embryos: though offering no explanation of the phenomenon, it may be mentioned as a curious circumstance, that the base of every *Sigillaria* trunk is marked by a cruciform ridge, separating the four primary divisions of the root, of which ridge no explanation has ever been offered.

With regard to the evidence of *Coniferæ* having existed during the carboniferous period, we are far from considering that afforded by the wood abounding in discs as conclusive, however much these resemble the discs of *Araucaria*: it is now well known that very similar discs abound in the wood of many Natural Orders that have no alliance with *Coniferæ*; but it is not hitherto known that there is a coniferous tree, in which the discs are not present in all parts of the wood, but are totally absent from one-half of each annual wood deposit. This tree, probably a drifted one, was discovered on the shores of Wellington Channel by Sir Edward Belcher, and it renders it not impossible that coniferous wood may be found in which these discs are totally absent. It is now, however, universally admitted by those botanists who have made both the anatomical structure and affinities of plants their study, that the structure of the axis of Exogens affords no guide to their affinity; *Coniferæ* have been supposed to form the best marked exception to this rule, and there is no doubt that they do so; but the coniferous woods of the coal epoch present so many remarkable deviations from those existing at the present time, that it becomes dangerous to speculate upon them.

On Plate V. are four highly magnified views of the tissues in the limestone matrix surrounding the *Trigonocarpons*: these appear to belong to very different groups of vegetables, and being a very small proportion of the variety that does exist, lead to the conclusion that the flora of the period was a varied and profuse one. We have few remarks to offer upon them. Fig. 1 represents a tissue resembling that of *Dadoxylon.* Fig. 2 probably belongs to a plant allied to the Ferns. Fig. 3, apparently an exogenous wood, crossed by medullary rays. Fig. 4 is also a Fern.

VII. *On the Theory of Definite Integrals.* *By* W. H. L. Russell, *Esq., B.A.* *Communicated by* A. Cayley, *Esq., F.R.S.*

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I PROPOSE in the following paper to investigate some new methods for summing various kinds of series, including almost all of the more important which are met with in analysis, by means of definite integrals, and to apply the same to the evaluation of a large number of definite integrals. In a paper which appeared in the Cambridge and Dublin Mathematical Journal for May 1854, I applied certain of these series to the integration of linear differential equations by means of definite integrals. Now Professor Boole has shown, in an admirable memoir which appeared in the Philosophical Transactions for the year 1844, that the methods which he has invented for the integration of linear differential equations in finite terms, lead to the summation of numerous series of an exactly similar nature, whence it follows that the combination of his methods of summation with mine, will lead to the evaluation of a large number of definite integrals, as will be shown in this paper. It is hence evident that the discovery of other modes of summing these series by means of definite integrals must in all cases lead to the evaluation of new groups of definite integrals, as will also be shown in the following pages. I then point out that these investigations are equivalent to finding all the more important definite integrals whose values can be obtained in finite terms by the solution of linear differential equations with variable coefficients. Again, there are certain algebraical equations which can be solved at once by Lagrange's series, and by common algebraical processes; the summation of the former by means of definite integrals affords us a new class of results, which I next consider. A continental mathematician, M. Smaasen, has given, in a recent volume of Crelle's Journal, certain methods of combining series together which give us the means of reducing various multiple integrals to single ones. The series hitherto considered are what have been denominated "factorial series"; but, lastly, I proceed to show that analogous processes extend to series of a very complicated nature and of an entirely different form, and for that purpose sum by means of definite integrals certain series whose values are obtained in finite terms in the 'Exercices des Mathématiques' by means of the Residual Calculus. The total result will be the evaluation of an enormous number of definite integrals on an entirely new type, and the application of definite integrals to the summation of many intricate series.

Let us first consider the series whose general term is

$$\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{\beta(\beta+1)\ldots(\beta+n-1)}\cdot\frac{\alpha'(\alpha'+1)\ldots(\alpha'+n-1)}{\beta'(\beta'+1)\ldots.(\beta'+n-1)}\cdot\frac{x^n}{1.2.3\ldots n}.$$

Its sum will be found to be

$$\frac{\Gamma\beta}{\Gamma\alpha\Gamma(\beta-\alpha)}\cdot\frac{\Gamma\beta'}{\Gamma\alpha'\Gamma(\beta'-\alpha')}\cdot\cdot\int_0^1\int_0^1\ldots v^{\alpha-1}z^{\alpha'-1}\ldots(1-v)^{\beta-\alpha-1}(1-z)^{\beta'-\alpha'-1}\ldots\varepsilon^{xvz\ldots}dvdz.$$

Next, if we consider the series, whose general term is

$$\frac{1}{\beta(\beta+1)\ldots.(\beta+n-1)\ \beta'(\beta'+1)\ldots(\beta'+n-1)}\cdot\frac{x^n}{1.2.3\ldots n},$$

we find for the sum

$$\frac{\Gamma\beta.\varepsilon}{2\pi}\cdot\frac{\Gamma\beta'.\varepsilon}{2\pi}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dz\ dz'\ldots\frac{\varepsilon^{i(z+z'\ldots)}}{(1+iz)^{\beta}(1+iz')^{\beta'}\cdots}\,\varepsilon^{\frac{x}{(1+iz)(1+iz')\ldots.}}.$$

We may easily reduce this to a possible form by putting $z=\tan\theta$, $z'=\tan\theta'$, &c. If the series to be summed is of the nature of both the kinds of series we have been discussing, we must combine the two methods of summation together.

Now consider the following differential equation:

$$u+\phi(\mathrm{D})\varepsilon^{r\omega}u=0,\ \text{where}\ \varepsilon^{\omega}=x.$$

This equation can always be satisfied when the factors in the denominator of $\phi(\mathrm{D})$ are real and unequal by a series of the form

$$u=1+\frac{\alpha\beta\gamma..}{\alpha'\beta'\gamma'..}x+\frac{\alpha(\alpha+1)\beta(\beta+1)\gamma(\gamma+1)\ldots}{\alpha'(\alpha'+1)\beta'(\beta'+1)\gamma'(\gamma'+1)..}\frac{x^2}{1.2}+\text{\&c.}$$

We shall suppose that the number of the quantities α, β, γ &c. is always less than the number of the quantities α', β', γ' &c., and, for the present, that the magnitude of α, β, γ &c. is always less than that of α', β' &c., each to each. Then the sum of this series by means of definite integrals can always be found by the preceding theorems. Now Professor BOOLE has given, in the memoir I have before mentioned, the conditions which are necessary in order that the equation $u+\phi(\mathrm{D})\varepsilon^{r\omega}u=0$ may be integrable in finite terms, which are therefore the conditions that the sum of the above series, and consequently the value of any definite integral equivalent to it, may be found in finite terms. I shall now give some instances of the evaluation of definite integrals by the application of these principles. Let us consider the symbolical equation

$$u-\frac{\mu^2\varepsilon^{2\theta}u}{(\mathrm{D}-1)(\mathrm{D}-4)}=0,\ \text{where}\ \varepsilon^{\theta}=x,$$

and assume for its solution

$$v-\frac{\mu^2\varepsilon^{2\theta}v}{(\mathrm{D}-1)(\mathrm{D}-2)}=0,\ \text{so that}\ u=(\mathrm{D}-2)v,$$

whence

$$v=\mathrm{C}_1x\varepsilon^{\mu x}+\mathrm{C}_2x\epsilon^{-\mu x}.$$

Hence

$$u=\mathrm{C}_1(\mu x^2-x)\varepsilon^{\mu x}+\mathrm{C}_2(\mu x^2+x)\varepsilon^{-\mu x};$$

and we find from this the series

$$x^4\left\{1+\frac{\mu^2}{\frac{5}{2}.1}\cdot\frac{x^2}{2^2}+\frac{\mu^4}{\frac{5}{2}.\frac{7}{2}.1.2}\cdot\frac{x^4}{2^4}+\&c.\right\}=\frac{3}{2\mu^3}\{(\mu x^2-x)\varepsilon^{\mu x}+(\mu x^2+x)\varepsilon^{-\mu x}\} \quad . \quad . \quad (I.)$$

Whence we find, putting μ for $\frac{\mu^2x^2}{2^2}$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\sqrt{\cos\theta}\,\varepsilon^{\mu\cos^2\theta}\cos\left(\mu\sin\theta\cos\theta+\frac{5\theta}{2}-\tan\theta\right)$$

$$=\frac{\sqrt{\pi}}{2\mu^2\varepsilon}\left\{2\mu\varepsilon^{2\sqrt{\mu}}-\sqrt{\mu}\,\varepsilon^{2\sqrt{\mu}}+2\mu\varepsilon^{-2\sqrt{\mu}}+\sqrt{\mu}\,\varepsilon^{-2\sqrt{\mu}}\right\}.$$

Next consider the symbolical equation

$$(D-1)(D-3)(D-5)u-\mu^3\varepsilon^{3\omega}u=0, \text{ where } \varepsilon^{\omega}=x;$$

and assume as the transformed equation

$$(D-1)(D-2)(D-3)v-\mu^3\varepsilon^{3\omega}v=0.$$

Then $$u=(D-2)v,$$

and $$v=C_1x\varepsilon^{\mu x}+C_2x\varepsilon^{\mu\alpha x}+C_3x\varepsilon^{\mu\beta x};$$

where 1, α, β are the three cube roots of unity.

Hence $$u=C_1(\mu x^2-x)\varepsilon^{\mu x}+C_2(\alpha\mu x^2-x)\varepsilon^{\alpha\mu x}+C_3(\beta\mu x^2-x)\varepsilon^{\beta\mu x}.$$

We must determine C_1, C_2, C_3 according to the series we have to sum.

If $$C_1=\frac{8}{3\mu^4},\quad C_2=-\frac{4(1+\sqrt{-3})}{3\mu^4},\quad C_3=-\frac{4(1-\sqrt{-3})}{3\mu^4},$$

we find

$$\left.\begin{aligned}&x^5\left\{1+\frac{1}{\frac{5}{3}.\frac{7}{3}.1}\cdot\frac{\mu^3x^3}{3^3}+\frac{1}{\frac{5}{3}.\frac{8}{3}.\frac{7}{3}.\frac{10}{3}.1\cdot2}\,\frac{\mu^6x^6}{3^6}+\&c.\right\}\\&=\frac{8}{3\mu^4}(\mu x^2-x)\varepsilon^{\mu x}+\frac{8}{3\mu^4}(2\mu x^2+x)\varepsilon^{-\frac{\mu x}{2}}\cos\frac{\sqrt{3}}{2}\mu x\\&-\frac{8\sqrt{3}}{3\mu^4}x\varepsilon^{-\frac{\mu x}{2}}\sin\frac{\sqrt{3}}{2}\mu x\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

Whence $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta d\phi\varepsilon^{\mu\cos\theta\cos\phi\cos(\theta+\phi)}\cos^{-\frac{1}{3}}\theta\cos^{\frac{1}{3}}\phi$$

$$\cos\left\{\mu\cos\theta\cos\phi\sin(\theta+\phi)+\frac{5\theta}{3}+\frac{7\phi}{3}-(\tan\theta+\tan\phi)\right\}$$

$$=\frac{2\pi}{3\sqrt{3}\varepsilon^2\sqrt[3]{\mu^4}}\left\{(3\sqrt[3]{\mu}-1)\varepsilon^{3\sqrt[3]{\mu}}+(6\sqrt[3]{\mu}+1)\varepsilon^{-\frac{3\sqrt[3]{\mu}}{2}}\cos\frac{3\sqrt{3}\sqrt[3]{\mu}}{2}\right.$$

$$\left.-\sqrt{3}\varepsilon^{-\frac{3\sqrt[3]{\mu}}{2}}\sin\frac{3\sqrt{3}\sqrt[3]{\mu}}{2}\right\}.$$

Also

$$\left.\begin{aligned} & x^6\left\{1+\frac{1}{\frac{7}{3}\cdot\frac{8}{3}\cdot 1}\,\frac{\mu^3x^3}{3^3}+\frac{1}{\frac{7}{3}\cdot\frac{10}{3}\cdot\frac{8}{3}\cdot\frac{11}{3}\cdot 1\cdot 2}\,\frac{\mu^6x^6}{3^6}+\&c.\right\} \\ & =\frac{40}{3\mu^5}(\mu x^2-2x)\varepsilon^{\mu x}-\frac{40x}{3\mu^5}(\mu x-2)\varepsilon^{-\frac{\mu x}{2}}\cos\frac{\sqrt{3}}{2}\mu x \\ & \quad +\frac{40\sqrt{3}x}{3\mu^5}(\mu x+2)\varepsilon^{-\frac{\mu x}{2}}\sin\frac{\sqrt{3}}{2}\mu x \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

Whence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta d\phi\,\varepsilon^{\mu\cos\theta\cos\phi\cos(\theta+\phi)}\cos^{\frac{1}{3}}\theta\cos^{\frac{2}{3}}\phi$$

$$\cos\left\{\mu\cos\theta\cos\phi\sin(\theta+\phi)+\frac{7\theta}{3}+\frac{8\phi}{3}-(\tan\theta+\tan\phi)\right\}$$

$$=\frac{2\pi}{3\sqrt{3}\varepsilon^2\sqrt[3]{\mu^5}}\left\{(3\sqrt[3]{\mu}-2)\varepsilon^{3\sqrt[3]{\mu}}-(3\sqrt[3]{\mu}-2)\varepsilon^{-\frac{3\sqrt[3]{\mu}}{2}}\cos\frac{3\sqrt{3}\sqrt[3]{\mu}}{2}\right.$$

$$\left.+\sqrt{3}(3\sqrt[3]{\mu}+2)\varepsilon^{-\frac{3\sqrt[3]{\mu}}{2}}\sin\frac{3\sqrt{3}\sqrt[3]{\mu}}{2}\right\}.$$

Again, let the symbolical equation be

$$(D-1)(D-2)(D-5)u-\mu^2(D-3)\varepsilon^{2\omega}u=0,$$

and let the transformed equation be

$$(D-1)(D-2)v-\mu^2\varepsilon^{2\omega}v=(D-1)(D-2)V,$$

whence $u=(D-3)v, \qquad 0=(D-3)V.$

Hence we find $V=Cx^3,$

and $v=C_1x+C_2x\varepsilon^{\mu x}+C_3x\varepsilon^{-\mu x},$

whence $u=-2C_1x+C_2(\mu x^2-2x)\varepsilon^{\mu x}+C_3(-\mu x^2-2x)\varepsilon^{-\mu x}$;

we determine C_1, C_2, C_3 according to the series we have to sum. Hence we find

$$x^5\left\{1+\frac{2}{\frac{5}{2}.3.1}\,\frac{\mu^2x^2}{2^2}+\frac{2.3}{\frac{5}{2}.\frac{7}{2}.3.4.1.2}\,\frac{\mu^4x^4}{2^4}+\&c.\right\}=\frac{24x}{\mu^4}+\frac{6}{\mu^4}(\mu x^2-2x)\varepsilon^{\mu x}-\frac{6}{\mu^4}(\mu x^2+2x)\varepsilon^{-\mu x}. \text{ (IV.)}$$

Hence we have

$$\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta dv\; v(1-v)^{-\frac{1}{2}}\cos\theta\,\varepsilon^{\mu v\cos^2\theta}\cos(\mu v\sin\theta\cos\theta+3\theta-\tan\theta)$$

$$=\frac{2\pi}{\mu^2\varepsilon}+\frac{\pi}{\mu^2\varepsilon}(\sqrt{\mu}-1)\varepsilon^{2\sqrt{\mu}}-\frac{\pi}{\mu^2\varepsilon}(\sqrt{\mu}+1)\varepsilon^{-2\sqrt{\mu}}.$$

By a similar method we find

$$x^4\left\{1+\frac{2}{3.\frac{7}{2}.1}\,\frac{\mu^2x^2}{2^2}+\frac{2.3}{3.4.\frac{7}{2}.\frac{9}{2}.1.2}\,\frac{\mu^4x^4}{2^4}+\&c.\right\}=\frac{120x^2}{\mu^4}+\frac{30}{\mu^5}(\mu x^2-3x)\varepsilon^{\mu x}+\frac{30}{\mu^5}(\mu x^2+3x)\varepsilon^{-\mu x}, \text{(V.)}$$

whence we have

$$\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta dv\, v(1-v)^{\frac{1}{2}}\cos\theta\,\varepsilon^{\mu v\cos^2\theta}\cos(\mu v\sin\theta\cos\theta+3\theta-\tan\theta)$$

$$=\frac{2\pi}{\mu^2\varepsilon}+\frac{\pi}{4\mu^{\frac{5}{2}}\varepsilon}(2\sqrt{\mu}-3)\varepsilon^{2\sqrt{\mu}}+\frac{\pi}{4\mu^{\frac{5}{2}}\varepsilon}(2\sqrt{\mu}+3)\varepsilon^{-2\sqrt{\mu}}.$$

It is to be particularly remarked, that we may in many cases simplify the final results, which we obtain by means of these summations, by the use of the theorem

$$\Gamma\frac{1}{n}\,\Gamma\frac{2}{n}\,\Gamma\frac{3}{n}\cdots\Gamma\frac{n-1}{n}=(2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}}.$$

Again, let $(D-1)(D-3)(D-5)u-\mu(D-2)(D-4)\varepsilon^\theta u=0,$

and assume as the transformed equation

$$(D-1)v-\mu\varepsilon^\theta v=0.$$

Then $u=(D-2)(D-4)v$

$$0=(D-2)(D-4)V,$$

whence $V=Ax^2+Bx^4,$

and $v=C_1(\mu x^2-x)\varepsilon^{\mu x}+C_2x^3+C_3x,$

whence $u=C_1(\mu^2x^3-3\mu x^2+3x)\varepsilon^{\mu x}+C_2x^3+C_3x,$

where the constants must be determined by comparison of this expression with the series to be summed. Thus we have

$$x^5\left\{1+\frac{2.4}{3.5}\mu x+\frac{2.3.4.5}{3.4.5.6}\,\frac{\mu^2x^2}{1.2}+\text{\&c.}\right\}=\frac{8}{\mu^4}\varepsilon^{\mu x}(\mu^2x^3-3\mu x^2+3x)-\frac{24x}{\mu^4}+\frac{4}{\mu^2}x^3. \quad . \quad . \quad \text{(VI.)}$$

Hence $\displaystyle\int_0^1\int_0^1 vz^3\varepsilon^{\mu vz}dvdz=\frac{\varepsilon^\mu}{\mu^4}(\mu^2-3\mu+3)-\frac{3}{\mu^4}+\frac{1}{2\mu^2}.$

Moreover we shall find

$$x^6\left\{1+\frac{2.5}{4.6}\mu x+\frac{2.3.5.6}{4.5.6.7}\cdot\frac{\mu^2x^2}{1.2}+\text{\&c.}\right\}=\frac{30}{\mu^5}(\mu^2x^3-4\mu x^2+4x)+\frac{10}{\mu^2}\left(x^4+\frac{3x^3}{\mu}\right)-\frac{120}{\mu^5}x, \quad . \quad \text{(VII.)}$$

whence $\displaystyle\int_0^1\int_0^1 vz^4(1-v)\varepsilon^{\mu vz}dvdz=\frac{\varepsilon^\mu}{\mu^5}(\mu-2)^2+\frac{1}{3\mu^2}\left(1+\frac{3}{\mu}\right)-\frac{4}{\mu^5}.$

We shall also find

$$x^4\left\{1+\frac{2}{4}\mu x+\frac{2.3}{4.5}\cdot\frac{\mu^2x^2}{1.2}+\frac{2.3.4}{4.5.6}\cdot\frac{\mu^3x^3}{1.2.3}+\text{\&c.}\right\}=\frac{6}{\mu^3}(\mu x^2-2x)\varepsilon^{\mu x}+\frac{6}{\mu^3}(\mu x^2+2x). \quad \text{(VIII.)}$$

Hence $\displaystyle\int_0^1 v(1-v)\varepsilon^{\mu v}dv=\frac{1}{\mu^3}(\mu-2)\varepsilon^\mu+\frac{1}{\mu^3}(\mu+2).$

These three last integrals can be obtained by ordinary integration. I have introduced them here partly for the sake of system, and partly because we shall require the series which they represent on other occasions.

We may extend this process, by performing operations with respect to the quantity (μ). Thus we may operate on any of the integrals we have obtained by such a symbol as $F\left(\frac{d}{d\mu}\right)$, where F is any rational function; and if it is an entire function, we have merely differentiations to perform. If it is a rational fraction, and the factors of the denominator are real and unequal, we may decompose it into simple rational fractions, each of which may, in its turn, be transformed into a simple integral. If we apply this operation to any of the results we have obtained, we immediately have a definite integral $\int\int ..P\varepsilon^{\mu Q}F(Q)\,dv...\,d\theta...$ expressed in a series of single integrals, where the integrations are performed with respect to (μ), and (μ) may be taken between any limits. But (μ) must in no case pass through zero, as the definite integrals, on which we operate with respect to (μ), cannot be found for that value of μ by the processes we have been investigating. There are many other operations of a similar nature, which it is easy to imagine.

I am now come to the second part of this memoir, the investigation of those new methods of summation, and of the definite integrals corresponding to them, to which I have before alluded. Let us consider the series

$$1+\frac{x}{\beta}+\frac{x^2}{\beta(\beta+1).1.2}+\frac{x^3}{\beta(\beta+1)(\beta+2).1.2.3}+\&c.,$$

where (β) is an integer. The following integral is known:

$$\int_0^\pi d\theta\,\varepsilon^{a\cos\theta}\cos(a\sin\theta)\cos n\theta=\frac{\pi}{2}\cdot\frac{a^n}{1.2.3...n};$$

$$\therefore\quad \frac{1}{\Gamma\beta}=\frac{1}{\pi a^{\beta-1}}\int_{-\pi}^{\pi}d\theta\,\varepsilon^{a\cos\theta}\cos(a\sin\theta)\,\varepsilon^{(\beta-1)i\theta}.$$

Hence we find for the sum of the above series,

$$\frac{\Gamma\beta}{\pi a^{\beta-1}}\int_{-\pi}^{\pi}d\theta\,\varepsilon^{a\cos\theta}\cos(a\sin\theta)\,\varepsilon^{(\beta-1)i\theta}\,\varepsilon^{\frac{x\varepsilon^{i\theta}}{a}}.$$

Next let us consider the same series when (β) is a fraction. We have

$$\frac{\Gamma(\beta-1)\,\Gamma(n+1)}{\Gamma(\beta+n)}=\int_0^1 dv\,v^n(1-v)^{\beta-2};$$

$$\therefore\quad \frac{\Gamma\beta}{\Gamma(\beta+n)}=\frac{\beta-1}{\pi a^n}\int_0^1\int_{-\pi}^{\pi}d\theta dv\,v^n(1-v)^{\beta-2}\varepsilon^{a\cos\theta}\cos(a\sin\theta)\,\varepsilon^{ni\theta},$$

except for $n=0$, when

$$\frac{2\Gamma\beta}{\Gamma\beta}=\frac{\beta-1}{\pi}\int_0^1\int_{-\pi}^{\pi}d\theta dv(1-v)^{\beta-2}\varepsilon^{a\cos\theta}\cos(a\sin\theta);$$

and we find for the sum of the series,

$$\frac{\beta-1}{\pi}\int_0^1\int_{-\pi}^{\pi}(1-v)^{\beta-2}\varepsilon^{a\cos\theta}\cos(a\sin\theta)\,\varepsilon^{\frac{vx\varepsilon^{i\theta}}{a}}d\theta dv-1.$$

The following are instances of the application of this method obtained by using series I., III., IV.:—

$$\int_0^1\int_{-\pi}^{\pi} d\theta dz(1-z)^{\frac{1}{2}}\varepsilon^{\mu(\alpha+z)\cos\theta}\cos(a\mu\sin\theta)\cos(\mu z\sin\theta)$$

$$=\frac{2\pi}{3}+\frac{\pi}{8\mu^3\alpha^{\frac{3}{2}}}\left\{(2\mu\sqrt{\alpha}-1)\varepsilon^{2\mu\sqrt{\alpha}}+(2\mu\sqrt{\alpha}+1)\varepsilon^{-2\mu\sqrt{\alpha}}\right\}$$

$$\int_0^1\int_0^1\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varepsilon^{\alpha\cos\theta+\beta\cos\phi+\mu vz\cos(\theta+\phi)}(1-v)^{\frac{1}{3}}(1-z)^{\frac{2}{3}}$$

$$\cos(\alpha\sin\theta)\cos(\beta\sin\phi)\cos\left(\mu vz\sin(\theta+\phi)\right)d\theta d\phi dvdz$$

$$=\frac{27\pi^2}{20}+\frac{2\pi^2}{81(\mu\alpha\beta)^{\frac{5}{3}}}(3\sqrt[3]{\mu\alpha\beta}-2)\varepsilon^{3\sqrt[3]{\mu\alpha\beta}}-\frac{2\pi^2}{81(\mu\alpha\beta)^{\frac{5}{3}}}\varepsilon^{\frac{-3\sqrt[3]{\mu\alpha\beta}}{2}}$$

$$\left\{(3\sqrt[3]{\mu\alpha\beta}-2)\cos\frac{3\sqrt{3}}{2}\sqrt[3]{\mu\alpha\beta}-\sqrt{3}(3\sqrt[3]{\mu\alpha\beta}+2)\sin\frac{3\sqrt{3}}{2}\sqrt[3]{\mu\alpha\beta}\right\}$$

$$\int_0^1\int_{-\pi}^{\pi} d\theta\, dv\, v(1-v)^{-\frac{1}{2}}\varepsilon^{(\alpha+\mu v)\cos\theta}\cos(2\theta+\mu v\sin\theta)\cos(\alpha\sin\theta)$$

$$=\frac{\pi}{\mu^2}+\frac{\pi}{2\mu^2}(\sqrt{\alpha\mu}-1)\varepsilon^{2\sqrt{\alpha\mu}}-\frac{\pi}{2\mu^2}(\sqrt{\alpha\mu}+1)\varepsilon^{-2\sqrt{\alpha\mu}}.$$

Again, we know that

$$\int_0^{\frac{\pi}{2}} d\theta\cos^\beta\theta\cos n\theta=\frac{\pi\Gamma(\beta+1)}{2^{\beta+1}\Gamma\left(\frac{\beta+n}{2}+1\right)\Gamma\left(\frac{\beta-n}{2}+1\right)},$$

from which we may deduce the following:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{a+b-2}\theta\,\varepsilon^{(a-b)i\theta}d\theta=\frac{\pi\Gamma(a+b-1)}{2^{a+b-2}\Gamma a\Gamma b}.$$

Now consider the series

$$1+\frac{\alpha}{\beta}x+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\cdot\frac{x^2}{1.2}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\cdot\frac{x^3}{1.2.3}+\&c.,$$

where (α) is greater than β. Then by the above formula

$$\frac{\Gamma(\alpha+n)}{\Gamma(\beta+n)}=\frac{2^{\alpha+n-1}}{\pi}\Gamma(\alpha-\beta+1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\cos^{\alpha+n-1}\theta\,\varepsilon^{(2\beta-\alpha+n-1)i\theta};$$

and we find for the sum of the series,

$$\frac{2^{\alpha-1}}{\pi}\cdot\frac{\Gamma\beta\Gamma(\alpha-\beta+1)}{\Gamma\alpha}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\cos^{\alpha-1}\theta\,\varepsilon^{(2\beta-\alpha-1)i\theta}\,\varepsilon^{2\cos\theta\,\varepsilon^{i\theta}x}.$$

In like manner we can find the sum of the series

$$1+\frac{\alpha}{\beta}\cdot\frac{\alpha'}{\beta'}x+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\cdot\frac{\alpha'(\alpha'+1)}{\beta'(\beta'+1)}\cdot\frac{x^2}{1.2}+\&c.,$$

where α is greater than β, α' than β'.

The use of this integral will give an important extension of the method I have employed for expressing the integrals of differential equations by means of definite integrals. For in order to the success of that method, it is necessary, as is shown in my paper in the Cambridge Mathematical Journal before alluded to, that the magnitude of the factorials (if any) in the numerator of each term of the series to be summed, should be less than that of the corresponding factorials in the denominator; whereas this integral enables us to sum series in which the reverse is the case. I shall now apply the series, whose sum we have just found, to the evaluation of definite integrals, using series VI. and VII. Hence

$$\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta dv\ v(1-v)^2\cos^3\theta\ \varepsilon^{2\mu v\cos^2\theta}\cos(2\mu v\sin\theta\cos\theta+\theta)=\frac{\pi}{4\mu^4}(\mu^2-3\mu+3)\varepsilon^{\mu}-\frac{3\pi}{4\mu^4}+\frac{\pi}{8\mu^2}$$

$$\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta dv\ v(1-v)^3\cos^4\theta\ \varepsilon^{2\mu v\cos^2\theta}\cos(2\mu v\sin\theta\cos\theta+2\theta)$$

$$=\frac{3\pi}{8\mu^5}(\mu-2)^2\varepsilon^{\mu}+\frac{\pi}{8\mu^3}(\mu+3)-\frac{3\pi}{2\mu^5}.$$

By a process similar to those used above, we find

$$1+\frac{2}{\frac{3}{2}.3.1}\ \frac{\mu^2x^2}{2^2}+\frac{2.3}{\frac{3}{2}.\frac{5}{2}.3.4.1.2}\ \frac{\mu^4x^4}{2^4}+\&c.$$

$$=-\frac{8}{\mu^4x^4}+\frac{2}{\mu^4x^4}(\mu^2x^2-2\mu x+2)\varepsilon^{\mu x}+\frac{2}{\mu^4x^4}(\mu^2x^2+2\mu x+2)\varepsilon^{-\mu x}.$$

Hence $\displaystyle\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta d\phi\ \varepsilon^{\alpha\cos\phi+2\mu\cos\theta\cos(\theta+\phi)}\cos\theta\cos(\alpha\sin\phi)\cos 2\big(\phi+\mu\cos\theta\sin(\theta+\phi)\big)$

$$=-\frac{\pi}{2\mu^2}+\frac{\pi}{4\mu^2}(2\mu\alpha-2\sqrt{\mu\alpha}+1)\ \varepsilon^{2\sqrt{\mu\alpha}}+\frac{\pi}{4\mu^2}(2\mu\alpha+2\sqrt{\mu\alpha}+1)\ \varepsilon^{-2\sqrt{\mu\alpha}}.$$

The following formulæ are found in CRELLE's Journal:—

$$\int_0^{\frac{\pi}{2}}\cos^{a-2}\theta\cot^b\theta\cos a\theta d\theta=\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}\cdot\frac{\pi}{2\cos\frac{b\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}}\cos^{a-2}\theta\cot^b\theta\sin a\theta d\theta=\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}\cdot\frac{\pi}{2\sin\frac{b\pi}{2}};$$

$$\therefore\ \int_0^{\frac{\pi}{2}}\cos^{a-2}\theta\cot^b\theta\ \varepsilon^{ai\theta}d\theta=\frac{\Gamma(a+b-1)}{\Gamma a\Gamma b}\cdot\frac{\pi}{\sin b\pi}\varepsilon^{i(1-b)\frac{\pi}{2}},$$

whence we find $\displaystyle\frac{\Gamma(a+b-1)}{\Gamma a}=\frac{\Gamma b\sin b\pi}{\pi}\varepsilon^{-i(1-b)\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos^{a-2}\theta\cot^b\theta\ \varepsilon^{ai\theta}d\theta.$

In this formula we suppose (b) to be less than unity.

Now put $b=\frac{1}{2}$, then

$$\frac{\Gamma\left(a-\frac{1}{2}\right)}{\Gamma a}=\frac{1}{\sqrt{\pi}}\varepsilon^{-\frac{i\pi}{4}}\int_0^{\frac{\pi}{2}}\cos^{a-2}\theta\cot^{\frac{1}{2}}\theta\,\varepsilon^{\theta i a};$$

and putting $\alpha=n+\frac{5}{2}$, we have

$$\frac{\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)}=\frac{1}{\sqrt{\pi}}\varepsilon^{-\frac{i\pi}{4}}\int_0^{\frac{\pi}{2}}\cos^{n+\frac{1}{2}}\theta\cot^{\frac{1}{2}}\theta\,\varepsilon^{\theta i(n+\frac{5}{2})},$$

whence we find, from series IV.,

$$1+\frac{2}{\frac{5}{2}.3}\mu+\frac{2.3}{\frac{5}{2}.\frac{7}{2}.3.4}\cdot\frac{\mu^2}{1.2}+\&c.$$

$$=\frac{3}{2\pi}\varepsilon^{-\frac{i\pi}{4}}\int_0^{\frac{\pi}{2}}\int_{-\pi}^{\pi}\cos^{\frac{1}{2}}\theta\cot^{\frac{1}{2}}\theta\,\varepsilon^{\frac{5\theta i}{2}}\varepsilon^{\cos\phi}\cos\sin\phi\,\varepsilon^{2i\phi}d\phi d\theta\,\varepsilon^{\mu\cos\theta}\,\varepsilon^{i(\theta+\phi)},$$

$$\therefore\ \left(\text{since }\frac{2\pi}{3}\varepsilon^{\frac{i\pi}{4}}=\frac{\pi\sqrt{2}}{3}+\frac{\pi i\sqrt{2}}{3}\right)\text{ we have}$$

$$\int_0^{\frac{\pi}{2}}\int_{-\pi}^{\pi}\cos^{\frac{1}{2}}\theta\cot^{\frac{1}{2}}\theta\,\varepsilon^{\cos\phi}\cos(\sin\phi)\varepsilon^{\mu\cos\theta\cos(\theta+\phi)}d\phi d\theta\cos\left\{\mu\cos\theta\sin(\theta+\phi)+\frac{5\theta}{2}+2\phi\right\}$$

$$=\frac{\pi}{\mu^2\sqrt{2}}+\frac{\pi}{2\sqrt{2}\mu^2}(\sqrt{\mu}-1)\varepsilon^{2\sqrt{\mu}}-\frac{\pi}{2\sqrt{2}\mu^2}(\sqrt{\mu}+1)\varepsilon^{-2\sqrt{\mu}}.$$

Let us again consider the series

$$1+\frac{\alpha}{\beta}x+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\cdot\frac{x^2}{1.2}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\cdot\frac{x^3}{1.2.3}+\&c.$$

Then making use of the integrals

$$\Gamma(\alpha+n)=\int_0^{\infty}\varepsilon^{-z}z^{\alpha+n-1}\,dz,\text{ and }\Gamma(\alpha+n)=h^{\alpha+n}\varepsilon^{-\frac{\pi i}{2}(\alpha+n)}\int_0^{\infty}\varepsilon^{hiz}z^{\alpha+n-1}dz,$$

where (h) is a constant quantity, we find as the sum of this series,

$$\frac{\Gamma\beta}{\Gamma\alpha}\cdot\frac{\varepsilon}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}dvdz\,z^{\alpha-1}\varepsilon^{-z}\frac{\varepsilon^{iv}}{(1+iv)^{\beta}}\varepsilon^{\frac{xz}{1+iv}},$$

and

$$\frac{\Gamma\beta}{\Gamma\alpha}\cdot\frac{\varepsilon}{2\pi}h^{\alpha}\varepsilon^{-\frac{\pi i\alpha}{2}}\int_{-0}^{\infty}\int_{-\infty}^{\infty}dvdz\,z^{\alpha-1}\varepsilon^{hiz}\frac{\varepsilon^{iv}}{(1+iv)^{\beta}}\varepsilon^{-\frac{ihzx}{1+iv}};$$

also when β is an integer, we may find the following expressions as the sum of the same series:—

$$\frac{\Gamma\beta}{\Gamma\alpha}\cdot\frac{1}{\pi c^{\beta-1}}\int_0^{\infty}\int_{-\pi}^{\pi}d\theta dz\,z^{\alpha-1}\varepsilon^{-z}\cos(c\sin\theta)\,\varepsilon^{c\cos\theta}.\varepsilon^{(\beta-1)i\theta}.\varepsilon^{\frac{xz\varepsilon^{i\theta}}{c}},$$

and also

$$\frac{\Gamma\beta}{\Gamma\alpha}\cdot\frac{1}{\pi c^{\beta-1}}\cdot h^{\alpha}\varepsilon^{-\frac{\pi i\alpha}{2}}\int_0^{\infty}\int_{-\pi}^{\pi}d\theta dz\,z^{\alpha-1}\varepsilon^{hiz}\cos(c\sin\theta)\varepsilon^{c\cos\theta}.\varepsilon^{(\beta-1)i\theta}.\varepsilon^{-\frac{ihzx\varepsilon^{i\theta}}{c}}.$$

From hence we obtain, using the first of the two integrals and the series

$$1+\frac{2}{4}\mu+\frac{2.3}{4.5}\cdot\frac{\mu^2}{1.2}+\frac{2.3.4}{4.5.6}\cdot\frac{\mu^3}{1.2.3}+\&c.=\frac{6}{\mu^3}(\mu-2)\varepsilon^\mu+\frac{6}{\mu^3}(\mu+2)$$

$$\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta dz\varepsilon^{\mu z\cos^2\theta-z}z\cos^2\theta\cos(\mu z\sin\theta\cos\theta+4\theta-\tan\theta)=\frac{2\pi}{\mu^3\varepsilon}(\mu-2)\varepsilon^\mu+\frac{2\pi}{\mu^3\varepsilon}(\mu+2),$$

and also
$$\int_0^\infty\int_{-\pi}^{\pi}d\theta dz\varepsilon^{c\cos\theta+\frac{\mu z\cos\theta}{c}-z}z\cos(c\sin\theta)\cos\left(\frac{\mu z\sin\theta}{c}+3\theta\right)$$
$$=\frac{\pi c^3}{\mu^3}(\mu-2)\varepsilon^\mu+\frac{\pi c^3}{\mu^3}(\mu+2).$$

The second integral will require in its applications, that we equate possible and impossible parts, in other respects the results will be analogous to those we have just obtained.

There are one or two other methods of summation which I shall briefly notice.

We see at once that

$$1+\frac{\mu}{2}+\frac{1}{1.2}\cdot\frac{\mu^2}{3}+\frac{1}{1.2.3}\cdot\frac{\mu^3}{4}+\&c.=\frac{\varepsilon^\mu-1}{\mu}.$$

Now if (r) be any integer, $\frac{1}{r}=\frac{2(-1)^{r-1}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\log_\varepsilon\cos\theta\varepsilon^{2ri\theta}.$

Hence
$$1+\frac{\mu}{2}+\frac{1}{1.2}\cdot\frac{\mu^2}{3}+\frac{1}{1.2.3}\cdot\frac{\mu^3}{4}+\&c.=\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\log_\varepsilon\cos\theta\,\varepsilon^{2i\theta}.\varepsilon^{-\mu\varepsilon^{2i\theta}}.$$

Whence
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\log_\varepsilon\cos\theta\,\varepsilon^{-\mu\cos2\theta}\cos(\mu\sin2\theta-2\theta)=\frac{\pi}{2}\cdot\frac{\varepsilon^\mu-1}{\mu}.$$

The integral
$$\int_0^\pi\theta\sin\theta\cos^{2r}\theta=\frac{\pi}{2r+1}$$

can be employed in the same way.

Again,
$$\int_0^{\frac{\pi}{2}}\cos^n\theta\cos n\theta\,d\theta=\frac{\pi}{2}\cdot\frac{1}{2^n},$$

whence
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\cos^n\theta\,\varepsilon^{ni\theta}=\frac{\pi}{2^n}.$$

Hence using the series
$$1+\frac{\mu^2}{\frac{1}{2}\cdot1}\cdot\frac{1}{2^2}+\frac{\mu^4}{\frac{1}{2}\cdot\frac{3}{2}\cdot1\cdot2}\cdot\frac{1}{2^4}+\&c.=\frac{\varepsilon^\mu+\varepsilon^{-\mu}}{2},$$

we find

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta d\phi\cos^{-\frac{3}{2}}\phi\,\varepsilon^{\mu^2\cos^2\theta\cos\phi\cos(2\theta-\phi)}\cos\left(\mu^2\sin(2\theta-\phi)\cos^2\theta\cos\phi+\tan\phi-\frac{\phi}{2}\right)$$
$$=\frac{\sqrt{\pi}}{\varepsilon}(\varepsilon^\mu+\varepsilon^{-\mu}).$$

There are some other definite integrals which we may use in the summation of factorial series, as

$$\int_0^{\frac{\pi}{2}} d\theta \cos^n\theta \cos n\theta \cos 2r\theta = \frac{\pi}{4} \cdot \frac{n(n-1)\dots(n-r+1)}{1.2.3\dots r} \frac{1}{2^n},$$

$$\int_0^{\pi} \frac{d\theta \sin^{2n}\theta}{(1-2a\cos\theta+a^2)} = \frac{(2n-1)(2n-3)\dots3.1}{2n(2n-2)\dots4.2} \cdot \frac{\pi}{2},$$

$$\int_0^1 \frac{dx\, x^{\alpha-1}(1-x)^{\beta-1}}{(x+a)^{\alpha+\beta}} = \frac{1}{a^\beta(1+a)^\alpha} \cdot \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)},$$

$$\int_{-\infty}^{\infty} \frac{dx}{(a+ix)^m(b-ix)^n} = 2\pi(a+b)^{1-m-n} \cdot \frac{1.2.3\dots m+n-2}{1.2.3\dots m-1.1.2.3\dots n-1},$$

$$\int_0^{\infty} \frac{x^{m-\frac{1}{2}}dx}{\{(x+a)(x+b)\}^i} = \frac{\Gamma\frac{1}{2}\Gamma\left(n-\frac{1}{2}\right)}{\Gamma n} \frac{1}{(\sqrt{a}+\sqrt{b})^{2n-1}},$$

and probably some besides.

I shall now offer a few observations on the nature of the integrals we have been discussing. The preceding investigations appear to be equivalent to a solution of the following problem:—"To find the definite integrals, whose values can be determined in finite terms by the solution of linear differential equations with variable coefficients." It should seem that the definite integrals, which we have considered in this paper, are the most general ones of any importance, whose values can be found in this way, for the following reasons:—If we expand any definite integral, which is a solution of a differential equation, and its equivalent in terms of the principal variable, and equate like powers of that variable, we obtain a series of definite integrals of a simpler kind, each equal to a fraction whose numerator and denominator consist of factorials, and can therefore be expressed by the products of Eulerian integrals, or to the sum of such fractions. Now I have employed all the more important definite integrals of this class, which are yet known, in the summation of the series which satisfy the differential equation

$$(ax^n+bx^{n-r})\frac{d^ny}{dx^n}+(a'x^{n-1}+b'x^{n-r-1})\frac{d^{n-1}y}{dx^{n-1}}+\&c.=0;$$

and as the properties of the Eulerian integrals have been much studied, and the integrals whose values are dependent on them consequently well known, it is probable that the definite integrals, which we have considered in this paper, embrace all the more important ones whose values can be determined in finite terms by the solution of the above equation. Were we to employ equations of a more general form, we should find that the successive terms of the series which express their solutions, would be given by equations of finite differences, in which the members equated to zero would each consist of more than two terms. Consequently we should be unable in the general case to sum the resulting series by means of definite integrals; and in those cases in which we might find this possible, the integration of the differential equations in finite terms would be practicable in very few cases. The following method of determining a well-known definite integral is here added, to show the connexion between previous investigations relative to definite integrals, and those given in the present memoir.

We know that $$1-r^2+\frac{r^4}{1.2}-\frac{r^6}{1.2.3}+\dots=\varepsilon^{-r^2},$$

or $$1-\frac{(2r)^2}{1.2}\cdot\frac{1}{2}+\frac{(2r)^4}{1.2.3.4}\cdot\frac{1.3}{2^2}-\frac{(2r)^6}{1.2.3.4.5.6}\cdot\frac{1.3.5}{2^3}+\&c.=\varepsilon^{-r^2}.$$

Hence remembering that $$\int_0^\infty dz\, z^{2n}\varepsilon^{-z^2}=\frac{1.3..2n-1}{2^n}\cdot\frac{\sqrt{\pi}}{2},$$

we find $$\int_0^\infty \varepsilon^{-z^2}\cos 2rz=\frac{\sqrt{\pi}}{2}\varepsilon^{-r^2}.$$

I shall now enter on some investigations connected with LAGRANGE's theorem.

Let $1-y+\alpha y^r=0$ be an algebraical equation. Then LAGRANGE's theorem gives us the following series:—

$$y^m=1+m\alpha+\frac{m(m+2r-1)}{1.2}\alpha^2+\&c.+\frac{m(m+nr-1)\,(m+nr-2)\dots\dots(m+n(r-1)+1)}{1.2.3\dots n}\alpha^n+\&c.$$

If we apply the usual test of convergency to this series, we find that $(r-1)\alpha$ must be less than unity.

Then we see that

$$\frac{1}{m}\cdot\frac{dy^m}{d\alpha}=1+(m+2r-1)\alpha+\frac{(m+3r-1)\,(m+3r-2)}{1.2}\alpha^2+\&c.$$
$$+\frac{(m+nr-1)\,(m+nr-2)\dots(m+n(r-1)+1)}{1.2.3\dots(n-1)}\alpha^{n-1}+\&c.$$

Now $$(m+nr-1)\,(m+nr-2)\dots(m+n(r-1)+1)=\frac{\Gamma(m+nr)}{\Gamma(m+n(r-1)+1)};$$

wherefore, since $$\frac{\Gamma(a+b-1)}{\Gamma a\Gamma b}=\frac{2^{a+b-2}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{a+b-2}\theta\,\varepsilon^{(a-b)i\theta}d\theta,$$

we have $$(m+nr-1)\dots(m+n(r-1)+1)$$
$$=\frac{2^{m+nr-1}\Gamma(n)}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{m+nr-1}\theta\,\varepsilon^{(m+n(r-2)+1)i\theta}d\theta$$
$$=\frac{2^{m+nr-1}}{\pi}\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta dz\,\cos^{m+nr-1}\theta\,\varepsilon^{(m+n(r-2)+1)i\theta}.z^{n-1}\varepsilon^{-z}.$$

Hence we have $$1+(m+2r-1)\alpha+\frac{(m+3r-1)\,(m+3r-2)}{1.2}\alpha^2+\&c.$$

$$=\frac{2^{m+r-1}}{\pi}\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta dz\,\cos^{m+r-1}\theta\,\varepsilon^{2^r\alpha z\cos^r\theta\cos(r-2)\theta-z}$$
$$\cos(2^r\alpha z\cos^r\theta\sin(r-2)\theta+(m+r-1)\theta);$$

$$\therefore \quad \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta dz \cos^{m+r-1}\theta \, \varepsilon^{2^r \alpha z \cos^r\theta \cos(r-2)\theta - z}$$

$$\cos(2^r\alpha z \cos^r\theta \sin(r-2)\theta + (m+r-1)\theta) = \frac{\pi}{2^{m+r-1}m} \cdot \frac{d.y^m}{da}.$$

Let $r=2$, then we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos^{m+1}\theta \cos(m+1)\theta}{1-c\cos^2\theta} = \frac{2\pi}{m} \cdot \frac{d}{dc}\left\{\frac{1-\sqrt{1-c}}{c}\right\}^m,$$

where (c) is of course less than unity; an integral given by ABEL.

When $2^r\alpha$ is less than unity we can always integrate with respect to (z), but may obtain a single integral more simply by proceeding as follows:—

We have $$\frac{(m+nr-1)\ (m+nr-2)\ldots(m+n(r-1)+1)}{1.2.3\ldots n-1}$$

$$= \frac{2^{m+nr-1}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cos^{m+nr-1}\theta \, \varepsilon^{(m+n(r-2)+1)i\theta};$$

consequently we find by summing a geometrical progression,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cos^{m+r-1}\theta \left\{\frac{\cos(m+r-1)\theta - 2^r a \cos^r\theta \cos(m+1)\theta}{1-2^{r+1}a\cos^r\theta\cos(r-2)\theta + 2^{2r}a^2\cos^{2r}\theta}\right\} = \frac{\pi}{2^{m+r-1}m} \frac{dy^m}{da}.$$

When $r=2$ this result coincides with that last obtained. We may obtain a very general result by applying FOURIER's theorem to the series of LAGRANGE and LAPLACE as follows:—

If $u=f(y)$, and $y=z+x\phi(y)$,

we have $$u=f(z)+\{\phi(z)f'(z)\}x+\frac{d}{dz}\{\phi^2zf'z\}\frac{x^2}{1.2}+\&c.;$$

$$\therefore \quad \frac{du}{dx}=\phi(z)f'(z)+\frac{d}{dz}\{\phi^2(z)f'z\}x+\frac{d^2}{dz^2}\{\phi^3(z)f'(z)\}\frac{x^2}{1.2}+\&c.;$$

Now we generally have $$F(z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\alpha(z-z')Fz'\frac{d\alpha.dz'}{2\pi},$$

whence $$\phi^n(z)f'z=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon^{i\alpha(z-z')}\phi^n(z')f'(z')\frac{d\alpha.dz'}{2\pi}$$

and $$\frac{d^{n-1}}{dz^{n-1}}\phi^n(z)f'(z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon^{i\alpha(z-z')}(i\alpha)^{n-1}\phi^n(z')f'(z')\frac{d\alpha.dz'}{2\pi}.$$

Hence substituting in the above series, we find

$$\frac{du}{dx}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon^{i\alpha(z-z')}\phi(z')f'(z')\varepsilon^{i\alpha\phi(z')x}\frac{d\alpha dz'}{2\pi}.$$

Consequently we find the following definite integral:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\alpha dz \phi(z')f'(z')\cos\alpha\big(z-z'+x\phi(z')\big)=2\pi\frac{du}{dx}.$$

Again, from LAPLACE's theorem, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\alpha dz' \cos\alpha(z-z'+x\phi_1\phi_2 z')\phi_1\phi_2 z' f'\phi_2 z' = 2\pi\frac{du}{dx},$$

where $u=f(y),\ y=\phi_2(z+x\phi_1 y)$.

These theorems of course suppose the series from whence they were derived to be convergent.

As examples we may take the following.

Let $y=1+xy^3$,

then
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\alpha dz' \cos\alpha(1-z+xz^3)z^3$$

$$=2\pi\frac{d}{dx}\left\{\sqrt[3]{\left(\frac{1}{2x}+\sqrt{\left(\frac{1}{4x^2}-\frac{1}{27x^3}\right)}\right)}+\sqrt[3]{\left(\frac{1}{2x}-\sqrt{\left(\frac{1}{4x^2}-\frac{1}{27x^3}\right)}\right)}\right\}.$$

Also let $y=1+x\varepsilon^y$,

then
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\alpha(1-z'+x\varepsilon^{z'})\varepsilon^z d\alpha dz=\frac{2\pi\varepsilon^y}{1-x\varepsilon^y},$$

which we may modify thus; by eliminating (x)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\alpha\left(\frac{(y-1)\varepsilon^z-(z-1)\varepsilon^y}{\varepsilon^y}\right)\varepsilon^z d\alpha dz=\frac{2\pi\varepsilon^y}{2-y}.$$

Analogous methods apply to series involving BERNOUILLI's numbers; thus we have

$$\frac{x}{\varepsilon^x-1}=1-\frac{x}{2}+\frac{B_1}{1.2}x^2-\frac{B_3}{1.2.3.4}\cdot x^4+\&c.$$

$$\frac{B_{2n-1}}{\Gamma(2n+1)}=\frac{1}{2^{2n-1}\pi^{2n}}\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\&c.\right)$$

$$=\frac{1}{2^{2n-1}\pi^{2n}\Gamma 2n}\int_0^1\frac{\left(\log_\varepsilon\frac{1}{z}\right)^{2n-1}dz}{1-z};$$

$$\therefore\quad \frac{x}{\varepsilon^x-1}+\frac{x}{2}-1=\frac{x}{\pi}\int_0^1\frac{dz\sin\left(\frac{x}{2\pi}\log_\varepsilon\frac{1}{z}\right)}{1-z}.$$

Hence we have
$$\int_0^1\frac{\sin(\alpha\log_\varepsilon z)\,dz}{z-1}=\frac{\pi}{2}\cdot\frac{\varepsilon^{2\alpha\pi}+1}{\varepsilon^{2\alpha\pi}-1}-\frac{1}{2\alpha}.$$

In this formula (α) must lie between 0 and 1, as it is necessary for the convergence of the above series that x should be less than 2π.

I now enter upon the consideration of the processes I have before mentioned for reducing multiple integrals to single ones. We easily see the truth of the following equation:—

$$1+\frac{\mu}{\frac{1}{2}.1^2.2^2}+\frac{\mu^2}{\frac{1}{2}.\frac{3}{2}.1^2.2^2.2^4}+\frac{\mu^3}{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.1^2.2^2.3^2.2^6}+\&c.$$

$$=2+\frac{\mu}{1.2.1}+\frac{\mu^2}{1.2.3.4.1.2}+\frac{\mu^3}{1.2.3.4.5.6.1.2.3}+\&c.-1.$$

Hence we have $$\frac{\Gamma\frac{1}{2}\cdot\varepsilon}{2\pi}\cdot\frac{\Gamma.1\,\varepsilon}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\varepsilon^{i(z+z')}dzdz'}{(1+iz)^{\frac{1}{2}}(1+iz)}\,\varepsilon^{\frac{\mu}{(1+iz)(1+iz').2^2}}$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}d\theta\varepsilon^{\alpha\cos\theta}\cos(\alpha\sin\theta)\,\varepsilon^{\frac{\varepsilon^{2i\theta}}{\alpha^2}\cdot\mu}-1.$$

Hence $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varepsilon^{\frac{\mu}{4}\cos\theta\cos\phi\cos(\theta+\phi)}\cos^{-\frac{3}{2}}\theta\cos^{-1}\phi.d\theta d\phi,$$

$$\cos\left(\frac{\mu}{4}\cos\theta\cos\phi\sin(\theta+\phi)+\frac{\theta}{2}+\phi-(\tan\theta+\tan\phi)\right)$$

$$=\frac{4\sqrt{\pi}}{\varepsilon^2}\int_{-\pi}^{\pi}d\theta\varepsilon^{\alpha\cos\theta}\cos\alpha\sin\theta.\varepsilon^{\frac{\mu\cos 2\theta}{\alpha^2}}\cos\frac{\mu\sin 2\theta}{\alpha^2}-\frac{4\pi^{\frac{3}{2}}}{\varepsilon^2}.\quad.\quad.\quad.\quad.\quad.\quad(A.)$$

But we may effect these reductions systematically by means of the following proposition due to M. SMAASEN:—

If $$a_0+a_1x+a_2x^2+a_3x^3+\&c.=\phi_1(x),$$

and $$b_0+b_1x+b_2x^2+b_3x^3+\&c.=\phi_2(x),$$

then $$a_0b_0+a_1b_1x+a_2b_2x^2+\&c.$$

$$=\frac{1}{2\pi}\int_0^{\pi}d\theta\{(\phi_1(x\varepsilon^{i\theta})+\phi_1(x\varepsilon^{-i\theta}))(\phi_2(\varepsilon^{i\theta})+\phi_2(\varepsilon^{-i\theta}))\}.$$

M. SMAASEN has also proved in the same paper, that if the sums of the three series

$$a_0+a_1x+a_2x^2+a_3x^3+\&c.$$
$$b_0+b_1x+b_2x^2+b_3x^3+\&c.$$
$$c_0+c_1x+c_2x^2+c_3x^3+\&c.$$

are known, we may determine the sum of the series

$$a_0b_0c_0+a_1b_1c_1x+a_2b_2c_2x^2+\&c.$$

by means of a double integral, but we shall not want this in what follows.

Now $$1+\frac{x}{1.2}+\frac{x^2}{1.2.3.4}+\frac{x^3}{1.2.3.4.5.6}+\&c.=\frac{\varepsilon^{\sqrt{x}}+\varepsilon^{-\sqrt{x}}}{2}$$

$$1+\mu x+\frac{\mu^2x^2}{1.2}+\frac{\mu^3x^3}{1.2.3}+\&c.=\varepsilon^{\mu x};$$

consequently $$1+\frac{\mu x}{\frac{1}{2}\cdot1^2\cdot2^2}+\frac{\mu^2x^2}{\frac{1}{2}\cdot\frac{3}{2}\cdot1^2\cdot2^2\cdot2^4}+\&c.$$

$$=\frac{1}{2\pi}\int_0^{\pi}d\theta\left\{\frac{\varepsilon^{\sqrt{x\varepsilon^{i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{-i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{-i\theta}}}}{2}\right\}\{\varepsilon^{\mu\varepsilon^{i\theta}}+\varepsilon^{\mu\varepsilon^{-i\theta}}\}.$$

Now $$\varepsilon^{\sqrt{x\varepsilon^{i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{-i\theta}}}+\varepsilon^{-\sqrt{x\varepsilon^{-i\theta}}}$$

$$=2\varepsilon^{\sqrt{x}\cos\frac{\theta}{2}}\cos\left(\sqrt{x}\sin\frac{\theta}{2}\right)+2\varepsilon^{-\sqrt{x}\cos\frac{\theta}{2}}\cos\left(\sqrt{x}\sin\frac{\theta}{2}\right);$$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varepsilon^{\frac{\mu}{4}\cos\theta\cos\phi\cos\theta+\phi}\cos^{-\frac{3}{2}}\theta\cos^{-1}\phi\, d\theta d\phi,$$

$$\cos\left(\frac{\mu}{4}\cos\theta\cos\phi\sin(\theta+\phi)+\frac{\theta}{2}+\phi-(\tan\theta+\tan\phi)\right)$$

$$=\frac{4\sqrt{\pi}}{\varepsilon^2}\int_0^\pi d\theta\left\{(\varepsilon^{\cos\frac{\theta}{2}}+\varepsilon^{-\cos\frac{\theta}{2}})\cos\sin\frac{\theta}{2}\right\}\{\varepsilon^{\mu\cos\theta}\cos(\mu\sin\theta)\} \quad . \quad . \quad . \quad (B.)$$

Hence we find, by comparing (A.) with (B.),

$$\int_0^\pi d\theta\,\varepsilon^{\mu\cos\theta}\cos(\mu\sin\theta)\left\{2\varepsilon^{\frac{\cos 2\theta}{\mu}}\cos\frac{\sin 2\theta}{\mu}-(\varepsilon^{\cos\frac{\theta}{2}}+\varepsilon^{-\cos\frac{\theta}{2}})\cos\sin\frac{\theta}{2}\right\}=\pi.$$

We have already proved that

$$1+\frac{2}{4}x+\frac{2.3}{4.5}\cdot\frac{x^2}{1.2}+\frac{2.3.4}{4.5.6}\cdot\frac{x^3}{1.2.3}+\&c.$$

$$=\frac{6}{x^3}(x+2)+\frac{6}{x^3}(x-2)\varepsilon^x.$$

Hence
$$1+\frac{3}{5}\cdot\frac{x}{2}+\frac{3.4}{5.6}\cdot\frac{x^2}{2.3}+\&c.$$

$$=\frac{12}{x^4}(x+2)+\frac{12}{x^4}(x-2)\varepsilon^x-\frac{2}{x},$$

and
$$1+\mu x+\frac{\mu^2x^2}{1.2}+\frac{\mu^3x^3}{1.2.3}+\&c.=\varepsilon^{\mu x}.$$

Consequently the theorem of M. SMAASEN will give us the sum of the series

$$1+\frac{3}{2}\cdot\frac{\mu x}{5.1}+\frac{3.4}{2.3}\cdot\frac{\mu^2x^2}{5.6.1.2}+\frac{3.4.5}{2.3.4}\cdot\frac{\mu^3x^3}{5.6.7.1.2.3}+\&c.$$

by means of a single integral, and we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta d\phi\,\varepsilon^{2\mu\cos\theta\cos\phi\cos(\theta-\phi)}\cos^2\theta\cos^3\phi\cos\{2\mu\cos\theta\cos\phi\sin(\theta-\phi)+\tan\phi-5\phi\}$$

$$=\frac{\pi}{6\varepsilon}\int_0^\pi d\theta\{6(\cos 3\theta+2\cos 4\theta)+6\varepsilon^{\cos\theta}\cos(3\theta-\sin\theta)$$

$$-12\varepsilon^{\cos\theta}\cos(4\theta-\sin\theta)-\cos\theta\}\varepsilon^{\mu\cos\theta}\cos(\mu\sin\theta).$$

The fundamental idea of the preceding calculations, as will be readily seen, is as follows: to reduce every term of the series proposed to be summed by means of definite integrals to the form of the general term of the series whose sum is given by the common exponential theorem, and then to find the sum of the whole quantity contained under the signs of integration by means of that theorem. The factorials in the numerator of each term may be taken in any order we please relative to those of the denominator, provided that the same relative order is observed in every term throughout the whole series; moreover, we may use different integrals to express the same factorials, so that we can deduce the value of many definite integrals from one series.

I shall now give an example of the summation of a factorial series of a somewhat different nature.

Consider the series—

$$1+\frac{x}{a^2+2^2}+\frac{x^2}{(a^2+2^2)(a^2+4^2)}+\frac{x^3}{(a^2+2^2)(a^2+4^2)\dots(a^2+2^2n^2)}+\text{\&c.},$$

we know that $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varepsilon^{a\theta}(\cos\theta)^n=\frac{1.2.4\dots 2n}{(a^2+2^2)(a^2+4^2)\dots(a^2+2^2n^2)}\cdot\frac{\varepsilon^{\frac{a\pi}{2}}-\varepsilon^{-\frac{a\pi}{2}}}{a}.$$

Hence by substitution the above series becomes

$$\frac{a}{\varepsilon^{\frac{a\pi}{2}}-\varepsilon^{-\frac{a\pi}{2}}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\varepsilon^{a\theta}\left\{1+\frac{x\cos^2\theta}{1.2}+\frac{x^2\cos^4\theta}{1.2.3.4}+\text{\&c.}\right\}$$

$$=\frac{a}{2\left(\varepsilon^{\frac{a\pi}{2}}-\varepsilon^{-\frac{a\pi}{2}}\right)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\varepsilon^{a\theta}\{\varepsilon^{\sqrt{x}\cos\theta}+\varepsilon^{-\sqrt{x}\cos\theta}\}.$$

There are other series of an analogous nature which may be summed in a similar manner: the object of introducing the above summation in this paper, is to point out the use of the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varepsilon^{a\theta}(\cos\theta)^n$, when impossible factors occur in the denominators of the successive terms of a factorial series.

In the 'Exercices de Mathématiques,' Cauchy has proved that if z be a quantity of the form $\varrho(\cos\varphi+i\sin\varphi)$, and $z\varphi(z)$ continually approach zero as ϱ indefinitely increases whatever be φ, then the residue of $\varphi(z)$ is equal to zero, the limits of ϱ being 0 and (∞), and those of φ, π and $-\pi$. From this theorem he deduces the sums of certain series, which I shall presently consider; but must first give certain results which will be useful in the sequel.

Since $$\int_0^\infty \varepsilon^{-ax^2}\cos 2x dx=\frac{\sqrt{\pi}}{2\sqrt{a}}\varepsilon^{-\frac{1}{a}}$$

$$\therefore \quad \varepsilon^{-\frac{1}{a}}=\frac{\sqrt{a}}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\varepsilon^{-\frac{ax^2}{4}}\cos x dx.$$

Again, since $$\int_{-\infty}^{\infty}\varepsilon^{x-ax^2}=\frac{\sqrt{\pi}}{\sqrt{a}}\,\varepsilon^{\frac{1}{4a}},$$

we find $$\varepsilon^{\frac{1}{a}}=\frac{\sqrt{a}}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\varepsilon^{x-\frac{ax^2}{4}}dx,$$

whence we have $$\varepsilon^{\frac{1}{a}}-\varepsilon^{-\frac{1}{a}}=\frac{\sqrt{a}}{2\sqrt{\pi}}\int_{-\infty}^{\infty}dx(\varepsilon^{x}-\cos x)\varepsilon^{-\frac{ax^2}{4}}.$$

The first series we propose to consider is the following:—

$$\frac{1}{x^2-\frac{1}{x^2}}\tan\frac{\pi x^2}{2}+\frac{\frac{1}{3}}{\frac{x^2}{9}-\frac{9}{x^2}}\tan\frac{\pi x^2}{6}+\frac{\frac{1}{5}}{\frac{x^2}{25}-\frac{25}{x^2}}\tan\frac{\pi x^2}{10}+\&c.=\frac{\pi}{16}\left\{\left(\frac{\varepsilon^{\frac{\pi x}{2}}-\varepsilon^{-\frac{\pi x}{2}}}{\varepsilon^{\frac{\pi x}{2}}+\varepsilon^{-\frac{\pi x}{2}}}\right)^2-\tan^2\frac{\pi x}{2}\right\}.$$

Put $\pi x^2=\varrho$, then this series with its sign changed may be resolved into the three following:—

$$\frac{x^2}{2(1+x^2)}\tan\frac{\varrho}{2}+\frac{x^2}{2(9+x^2)}\cdot\frac{1}{3}\tan\frac{\varrho}{3.2}+\frac{x^2}{2(25+x^2)}\frac{1}{5}\tan\frac{\varrho}{5.2}+\&c.$$

$$\frac{x}{4(1-x)}\tan\frac{\varrho}{2}+\frac{x}{4(3-x)}\frac{1}{3}\tan\frac{\varrho}{3.2}+\frac{x}{4(5-x)}\cdot\frac{1}{5}\tan\frac{\varrho}{5.2}+\&c.$$

$$-\frac{x}{4(1+x)}\tan\frac{\varrho}{2}-\frac{x}{4(3+x)}\cdot\frac{1}{3}\tan\frac{\varrho}{3.2}-\frac{x}{4(5+x)}\cdot\frac{1}{5}\cdot\tan\frac{\varrho}{5.2}+\&c.$$

The general terms of these series are respectively,

$$\frac{x^2}{2\{(2n+1)^2+x^2\}}\cdot\frac{1}{2n+1}\tan\frac{\varrho}{2(2n+1)}$$

$$\frac{x}{4\{(2n+1)-x\}}\cdot\frac{1}{2n+1}\tan\frac{\varrho}{2(2n+1)}$$

$$-\frac{x}{4\{(2n+1)-x\}}\cdot\frac{1}{2n+1}\tan\frac{\varrho}{2(2n+1)}.$$

These terms become after transformation, since

$$\int_0^\infty\frac{dz(\varepsilon^{\alpha z}-\varepsilon^{-\alpha z})}{\varepsilon^{\pi z}-\varepsilon^{-\pi z}}=\frac{1}{2}\tan\frac{\alpha}{2},$$

$$\frac{x}{2}\int_0^\infty du\,\varepsilon^{-(2n+1)u}\sin xu\int_0^\infty\frac{dz}{(\varepsilon^{\pi z}-\varepsilon^{-\pi z})\sqrt{\pi\varrho z}}\int_{-\infty}^\infty dv(\varepsilon^v-\cos v)\varepsilon^{-\frac{(2n+1)v^2}{4\rho z}}\int_0^1\frac{ds}{\sqrt{\pi}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}.s^{2n}$$

$$+\frac{x}{4}\int_0^\infty du\,\varepsilon^{-(2n+1-x)u}\int_0^\infty\frac{dz}{(\varepsilon^{\pi z}-\varepsilon^{-\pi z})\sqrt{\pi\rho z}}\int_{-\infty}^\infty dv(\varepsilon^v-\cos v)\varepsilon^{-\frac{(2n+1)v^2}{4\rho z}}\int_0^1\frac{ds}{\sqrt{\pi}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}.s^{2n}$$

$$-\frac{x}{4}\int_0^\infty du\,\varepsilon^{-(2n+1+x)u}\int_0^\infty\frac{dz}{(\varepsilon^{\pi z}-\varepsilon^{-\pi z})\sqrt{\pi\varrho z}}\int_{-\infty}^\infty dv(\varepsilon^v-\cos v)\varepsilon^{-\frac{(2n+1)v^2}{4\rho z}}\int_0^1\frac{ds}{\sqrt{\pi}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}.s^{2n}.$$

Each of the series is consequently reduced to a geometrical progression; wherefore, summing the three progressions and taking the aggregate, we have

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_0^1\frac{(\varepsilon^v-\cos v)\varepsilon^{-u-\frac{v^2}{4\pi x^2 z}}\log_\varepsilon^{-\frac{1}{2}}\frac{1}{s}(2\sin xu+\varepsilon^{xu}-\varepsilon^{-xu})dsdvdudz}{(\varepsilon^{\pi z}-\varepsilon^{-\pi z})\sqrt{z}(1-\varepsilon^{-2u-\frac{v^2}{2\pi x^2 z}}s^2)}$$

$$=\frac{\pi^2}{4}\left\{\tan^2\frac{\pi x}{2}-\left(\frac{\varepsilon^{\frac{\pi x}{2}}-\varepsilon^{-\frac{\pi x}{2}}}{\varepsilon^{\frac{\pi x}{2}}+\varepsilon^{-\frac{\pi x}{2}}}\right)^2\right\}.$$

Next, let us consider the series

$$\frac{1}{x^2-\frac{1}{x^2}}\cot\pi x^2+\frac{\frac{1}{2}}{\frac{x^2}{4}-\frac{4}{x^2}}\cot\frac{\pi x^2}{2}+\frac{\frac{1}{3}}{\frac{x^2}{9}-\frac{9}{x^2}}\cot\frac{\pi x^3}{3}+\&c.$$

$$=\frac{\pi}{8}\left\{\cot^2\pi x-\left(\frac{\varepsilon^{\pi x}+\varepsilon^{-\pi x}}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}}\right)^2\right\}.$$

Let each term of this series be transformed by means of the integral

$$\int_0^\infty \frac{z^{\alpha-1}dz}{1-z}=\pi\cot\alpha\pi,$$

and we have

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_0^1 \frac{\varepsilon^{-u+v-\frac{v^2}{4x^2\log_\varepsilon z}}\log_\varepsilon{}^{-\frac{1}{2}}\frac{1}{s}(2\sin xu+\varepsilon^{xu}-\varepsilon^{-xu})\,dsdvdudz}{(z-z^2)\log_\varepsilon^{\frac{1}{2}}z(1-\varepsilon^{-u-\frac{v^2}{4x^2\log_\varepsilon z}}s)}$$

$$=\pi^{\frac{5}{2}}\left\{\left(\frac{\varepsilon^{\pi x}+\varepsilon^{-\pi x}}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}}\right)^2-\cot^2\pi x\right\}.$$

Again,

$$\frac{1}{x^2-\frac{1}{x^2}}\sec\frac{\pi x^2}{2}-\frac{\frac{1}{3}}{\frac{x^2}{9}-\frac{9}{x^2}}\sec\frac{\pi x^2}{6}+\frac{\frac{1}{5}}{\frac{x^2}{25}-\frac{25}{x^2}}\sec\frac{\pi x^2}{10}-\&c.$$

$$=\frac{\pi}{16}\left\{\left(\frac{2}{\varepsilon^{\frac{\pi x}{2}}+\varepsilon^{-\frac{\pi x}{2}}}\right)^2-\sec^2\frac{\pi x}{2}\right\}.$$

Here we reduce each term by means of the integral

$$\int_0^\infty \frac{dz(\varepsilon^{\alpha z}+\varepsilon^{-\alpha z})}{\varepsilon^{\pi z}+\varepsilon^{-\pi z}}=\frac{1}{2}\sec\frac{\alpha}{2};$$

and we have

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_0^1 \frac{(\varepsilon^v+\cos v)\,\varepsilon^{-u-\frac{v^2}{4\pi x^2 z}}\log_\varepsilon{}^{-\frac{1}{2}}\frac{1}{s}(2\sin xu+\varepsilon^{xu}-\varepsilon^{-xu})\,dsdvdudz}{(\varepsilon^{\pi z}+\varepsilon^{-\pi z})\sqrt{z}(1+\varepsilon^{-2u-\frac{v^2}{2\pi x^2 z}}s^2)}$$

$$=\frac{\pi^2}{4}\left\{\sec^2\frac{\pi x}{2}-\left(\frac{2}{\varepsilon^{\frac{\pi x}{2}}+\varepsilon^{-\frac{\pi x}{2}}}\right)^2\right\}.$$

Also, since

$$\frac{1}{x^2-\frac{1}{x^2}}\operatorname{cosec}\pi x^2-\frac{\frac{1}{2}}{\frac{x^2}{4}-\frac{4}{x^2}}\operatorname{cosec}\frac{\pi x^2}{2}+\frac{\frac{1}{3}}{\frac{x^2}{9}-\frac{9}{x^2}}\operatorname{cosec}\frac{\pi x^2}{3}-\&c.$$

$$=\frac{\pi}{8}\left\{\left(\frac{2}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}}\right)^2-\operatorname{cosec}^2\pi x\right\},$$

we have, transforming each term of the series by means of the integral

$$\int_0^\infty \frac{z^{\alpha-1}dz}{1+z}=\pi\operatorname{cosec}\alpha\pi,$$

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^1 \frac{\varepsilon^{-u+v-\frac{v^2}{4x^2\log_\varepsilon z}} \log_\varepsilon^{-\frac{1}{2}} \frac{1}{s} (2\sin xu + \varepsilon^{xu} - \varepsilon^{-xu})\, ds dv du dz}{(z+z^2)\log_\varepsilon^{\frac{1}{2}} z (1+\varepsilon^{-u-\frac{v^2}{4x^2\log_\varepsilon z}} s)}$$

$$=\pi^{\frac{5}{2}}\left\{\operatorname{cosec}^2 \pi x - \left(\frac{2}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}}\right)^2\right\}.$$

Let us next consider the series

$$\frac{\sin\theta}{1+\alpha^2}+\frac{2\sin 2\theta}{2^2+\alpha^2}+\frac{3\sin 3\theta}{3^2+\alpha^2}+\&c.=\frac{\pi}{2}\cdot\frac{\varepsilon^{\alpha(\pi-\theta)}-\varepsilon^{-\alpha(\pi-\theta)}}{\varepsilon^{\alpha\pi}-\varepsilon^{-\alpha\pi}}.$$

The general term of this series is $\frac{n \sin n\theta}{n^2+\alpha^2}$;

also, we have
$$\frac{n}{n^2+\alpha^2}=\int_0^\infty \varepsilon^{-nz}\sin\alpha z dz,$$

and
$$\varepsilon^{-z}\sin\theta+\varepsilon^{-2z}\sin 2\theta+\varepsilon^{-3z}\sin 3\theta+\&c.=\frac{\varepsilon^{-z}\sin\theta}{1-2\varepsilon^{-z}\cos\theta+\varepsilon^{-2z}}.$$

Hence
$$\int_0^\infty \frac{\varepsilon^{-z}\sin\alpha z dz}{1-2\varepsilon^{-z}\cos\theta+\varepsilon^{-2z}}=\frac{\pi}{2\sin\theta}\cdot\frac{\varepsilon^{\alpha(\pi-\theta)}-\varepsilon^{-\alpha(\pi-\theta)}}{\varepsilon^{\alpha\pi}-\varepsilon^{-\alpha\pi}}.$$

In like manner from the series

$$\frac{\cos\theta}{1+\alpha^2}+\frac{\cos 2\theta}{2^2+\alpha^2}+\frac{\cos 3\theta}{3^2+\alpha^2}+\&c.=\frac{\pi}{2\alpha}\cdot\frac{\varepsilon^{\alpha(\pi-\theta)}+\varepsilon^{-\alpha(\pi-\theta)}}{\varepsilon^{\alpha\pi}-\varepsilon^{-\alpha\pi}}-\frac{1}{2\alpha^2}$$

$$\int_0^\infty \cos\alpha z\frac{(\varepsilon^{-z}\cos\theta-\varepsilon^{-2z})\,dz}{1-2\varepsilon^{-z}\cos\theta+\varepsilon^{-2z}}=\frac{\pi}{2}\cdot\frac{\varepsilon^{\alpha(\pi-\theta)}+\varepsilon^{-\alpha(\pi-\theta)}}{\varepsilon^{\alpha\pi}-\varepsilon^{-\alpha\pi}}-\frac{1}{2\alpha}.$$

Let us next consider the series

$$\frac{1}{\varepsilon^{\pi}-\varepsilon^{-\pi}}-\frac{2}{\varepsilon^{2\pi}-\varepsilon^{-2\pi}}+\frac{3}{\varepsilon^{3\pi}-\varepsilon^{-3\pi}}-\&c.=\frac{1}{4\pi}.$$

We know that
$$\int_0^\infty \frac{\sin\mu z . dz}{\varepsilon^{2\pi z}-1}=\frac{1}{4}\cdot\frac{\varepsilon^{\mu}+1}{\varepsilon^{\mu}-1}-\frac{1}{2\mu},$$

$$\therefore \int_0^\infty \frac{\sin 2n\pi z dz}{\varepsilon^{2\pi z}-1}=\frac{1}{2}\,\frac{1}{1-\varepsilon^{-2n\pi}}-\frac{1}{4}-\frac{1}{4n\pi},$$

$$\therefore \frac{1}{\varepsilon^{n\pi}-\varepsilon^{-n\pi}}=2\varepsilon^{-n\pi}\int_0^\infty \frac{\sin 2n\pi z . dz}{\varepsilon^{2\pi z}-1}+\frac{\varepsilon^{-n\pi}}{2}+\frac{\varepsilon^{-n\pi}}{2n\pi}.$$

Now $x\sin\theta-x^2\sin 2\theta+x^3\sin 3\theta-\&c.,$

$$=\frac{x\sin\theta}{1+2x\cos\theta+x^2}.$$

From whence we have $x\sin\theta-2x^2\sin 2\theta+3x^3\sin 3\theta-\&c.$

$$=\frac{x\sin\theta(1-x^2)}{(1+2x\cos\theta+x^2)^2}.$$

It is hence evident that

$$\frac{1}{4\pi}=2\varepsilon^{\pi}(\varepsilon^{2\pi}-1)\int_0^\infty \frac{dz\sin 2\pi z}{(\varepsilon^{2\pi}+2\varepsilon^{\pi}\cos 2\pi z+1)^2(\varepsilon^{2\pi z}-1)}+\frac{1}{2}\,\frac{\varepsilon^{\pi}}{(\varepsilon^{\pi}+1)^2}+\frac{1}{2\pi}\cdot\frac{1}{\varepsilon^{\pi}+1};$$

$$\therefore \int_0^\infty \frac{dz . \sin 2\pi z}{(\varepsilon^{2\pi}+2\varepsilon^{\pi}\cos 2\pi z+1)^2(\varepsilon^{2\pi z}-1)}=\frac{1}{4\varepsilon^{\pi}(\varepsilon^{\pi}+1)^2(\varepsilon^{\pi}-1)}\left\{\frac{\varepsilon^{2\pi}-1}{2\pi}-\varepsilon^{\pi}\right\}.$$

Again, we have $\frac{1}{\varepsilon^{\pi}-\varepsilon^{-\pi}}-\frac{2^{4m+1}}{\varepsilon^{2\pi}-\varepsilon^{-2\pi}}+\frac{3^{4m+1}}{\varepsilon^{3\pi}-\varepsilon^{-3\pi}}-\&c.=0.$

We must transform the element n^{4m+1} thus:

$$n^{4m+1}=\frac{1}{\Gamma(4m+1)}\int_0^1 x^{\frac{1}{n}-1}\left(\log_{\varepsilon}\frac{1}{x}\right)^{4m}dx=\frac{1}{\Gamma(4m+1)}\int_0^1 \varepsilon^{\frac{\log_{\varepsilon}x}{n}}\frac{1}{x}\left(\log_{\varepsilon}\frac{1}{x}\right)^{4m}dx.$$

Again, $$\varepsilon^{\frac{\log_{\varepsilon}x}{n}}=\frac{\sqrt{n}}{2\sqrt{\pi\log_{\varepsilon}x}}\int_{-\infty}^{\infty}\varepsilon^{z-\frac{nz^2}{4\log_{\varepsilon}x}}dz.$$

Lastly, $$\sqrt{n}=\frac{2n}{\sqrt{\pi}}\int_0^{\infty}\varepsilon^{-nv^2}dv.$$

Hence, combining these integrals together, and substituting for $\frac{1}{\varepsilon^{n\pi}-\varepsilon^{-n\pi}}$, as before, we are able to transform the above series into one which can be summed by the ordinary rules. The resulting definite integral will of course be equal to zero.

Cauchy has applied the methods of the residual calculus to the determination of the sum of the series whose general term is

$$(-1)^{n-1}\frac{\varepsilon^{n\alpha}-\varepsilon^{-n\alpha}}{\varepsilon^{n\pi}-\varepsilon^{-n\pi}}\cdot\frac{n\cos n\alpha}{n^4+c^4}$$

in finite terms. We may transform the element $\frac{1}{n^4+c^4}$ thus:

$$\frac{1}{n^4+c^4}=\frac{1}{c^2}\int_0^{\infty}\varepsilon^{-n^2z^2}\sin c^2z dz.$$

Again, $$\varepsilon^{-n^2z^2}=\frac{2}{\sqrt{\pi z}}\int_0^{\infty}\varepsilon^{-\frac{v^2}{z^2}}\cos 2nz.$$

Wherefore, combining these integrals, and transforming the other elements as before, we may find its sum by means of definite integrals. We may resolve $\frac{n}{n^4+c^4}$ into its partial fractions, and then find the sum of the series, which would be simpler.

The transformation of $\varepsilon^{-n^2z^2}$ which I have used above, is due to Professor Kummer, who has applied it in the seventeenth volume of Crelle's Journal, in a paper to which I am indebted for many ideas relative to the connexion of definite integrals with series, to the expression of the series

$$1+q+q^4+q^9+\&c.,$$

and others of a similar nature by means of a definite integral. The integral $\int_0^{\infty}\frac{\sin\mu z.dz}{\varepsilon^{2\pi z}-1}$ was first applied to the summation of series, whose terms involve elements of the form $\frac{1}{\alpha-\beta x^n}$, by Poisson in his Memoir on the Distribution of Electricity in two electrized spheres, which mutually act upon each other. He proves that the calculation of the electrical arrangement depends upon the value of the definite integral,

$$\int_0^\infty \frac{\sin cz\,.\,dz}{(\varepsilon^{2\pi z}-1)\,(a+b\sin^2 cz)}.$$

I mention this on account of its analogy with the definite integral

$$\int_0^\infty \frac{dz \sin 2\pi z}{(\varepsilon^{2\pi}+2\varepsilon^{\pi}\cos 2\pi z+1)^2\,(\varepsilon^{2\pi z}-1)},$$

whose value is found above. The principles contained in this paper will enable us at once to find the sums of the series

$$1+x+\frac{x^{\frac{1}{2}}}{1.2}+\frac{x^{\frac{1}{3}}}{1.2.3}+\frac{x^{\frac{1}{4}}}{1.2.3.4}+\&c.$$

$$1+\frac{\tan\theta}{1}+\frac{\tan 2\theta}{1.2}+\frac{\tan 3\theta}{1.2.3}+\&c.$$

$$1+\frac{\sec\theta}{1}+\frac{\sec\frac{\theta}{2}}{1.2}+\frac{\sec\frac{\theta}{3}}{1.2.3}+\&c.$$

and of many others which can be imagined, by means of definite integrals. The definite integral of Poisson given above occurs in the solution of a functional equation; and it is probable that series similar to those I have been discussing in this paper, may be useful in enabling us to express the solutions of other functional equations by definite integrals.

PHILOSOPHICAL TRANSACTIONS.

VIII. *On Circumstances modifying the Action of Chemical Affinity.* *By* J. H. Gladstone, *Ph.D., F.R.S.*

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It is among the facts in chemical science which admit of no dispute, that a substance frequently shows a greater tendency to combine with one body than with another. This has usually received the appellation "elective attraction," or "elective affinity." It is also perhaps universally allowed that the manifestations of this elective affinity are greatly influenced by the insolubility, or the volatility of the original substances, or of the resulting compounds. The degree of temperature, the respective masses of the different substances, the presence of other bodies, and many circumstances beside these, are supposed to modify the result.

The attempt has frequently been made to construct tables showing the relative strength of affinity of different substances for some particular body, and Guyton de Morveau even endeavoured to give a numerical expression to them. In treating of this subject the elaborate disquisition of Bergman, 'De Attractionibus Electivis,' must be referred to; in which he illustrates at once the chemical fact, and the meaning of the term, by supposing A to be a substance united to *c*, and that on the addition of *b*, the *c* is excluded, and the union of the latter substance with A is brought about; in which case he says *b* has a stronger elective attraction for A than *c* has*. He, in common with most chemists both of his own and of later times, takes it for granted that if *b* decomposes A*c*, it does so completely. The Swedish chemist gives the results of nearly 2000 reactions in one table, the first column of which exhibits the following substances arranged according to their affinity for sulphuric acid, commencing with what he conceived the most powerful:—baryta, potash, soda, lime, magnesia, am-

* The whole passage runs thus:—"Sit A materies, quam aliæ heterogeneæ *a, b, c,* &c. adpetunt: ponatur ulterius A τῳ *c* unitum ad saturationem (quod per A*c* in sequentibus indicamus), addito *b*, ejusdem ambire unionem cum exclusione του *c*, A dicitur fortius adtrahere *b*, quam *c*, vel etiam *b* gaudere attractione electiva fortiori, quam *c*: tandem A*b*, addito *a*, priora vincula laxet, *b* respuat, et ejus loco *a* eligat, hinc intelligitur *a* τῳ *b* vi attractiva præpollere, et ratione efficaciæ seriem quamdam constituere *a*, *b*, et *c*."

monia, zinc, manganese, iron, lead, tin, cobalt, copper, nickel, bismuth, arsenic, mercury, antimony, silver, gold, platinum, alumina, sesquioxide of iron, water, phlogiston. The suitability of some of the methods employed for arriving at these results has never, as far as I know, been questioned; for instance, that zinc has a stronger affinity for sulphuric acid than manganese, or iron, or lead has, because it will separate any one of these metals from its solution in the said acid. Other methods however are more open to objection, such, for example, as that which led BERGMAN to place baryta at the head of the series, because it took sulphuric acid from every other base. To such deductions as this, drawn from precipitation, it may be objected, that the tendency of the two bodies to combine has arisen more or less from the insolubility of the compound. BERTHOLLET adopted this view; and in his 'Recherches sur les Lois de l'Affinité,' he endeavoured to prove "que les affinités électives n'agissent pas comme des forces absolues par lesquelles une substance seroit déplacée par une autre dans une combinaison; mais que, dans toutes les compositions et les décompositions qui sont dues à l'affinité élective, il se fait un partage de l'objet de la combinaison entre les substances dont l'action est opposée, et que les proportions de ce partage sont déterminées non seulement par l'énergie de l'affinité de ces substances, mais aussi par la quantité avec laquelle elles agissent, de sorte que la quantité peut suppléer à la force de l'affinité pour produire un même degré de saturation*."

These two conflicting views were much discussed at the time when they were propounded; the attention subsequently paid to the laws of stoichiometry has removed much of the difficulty in which the subject was then involved; GAY-LUSSAC has pointed out the erroneous idea of cohesion that obscured the reasoning of BERTHOLLET; and yet the amount of truth contained in either of these opposite opinions remains still an open question.

It is now some years since I first began to reason, and occasionally to experiment upon this subject. Since that time MALAGUTI has published a paper bearing upon it, which will be referred to subsequently; BUNSEN and DEBUS have experimented, and independently arrived at a very remarkable law; and WILLIAMSON has on more than one occasion vindicated the views of BERTHOLLET.

BUNSEN† exploded together carbonic oxide, hydrogen or cyanogen with oxygen, and, after varying his experiments greatly, deduced the following conclusions:—

1. When two or more bodies B, B′, ... are presented in excess to the body A, under the circumstances most favourable to their union, the body A takes from each of them B, B′, ... quantities which always stand to one another in a simple relation; so that for 1, 2, 3, 4 .... atoms of the one compound, there are formed 1, 2, 3, 4 .... atoms of the

* BERGMAN even had some perception of the influence of quantity, as when he says,—"Jam ulterius restat explorandum, num omne *d* sufficiente *c* possit unione pristina extrudi. Probe hoc in genere notetur oportet, decomponentis *c* duplo, triplo, immo interdum sextuplo majore opus esse quantitate, quam quæ τῷ A libero saturando sufficit." He failed, however, to see any particular significancy in this fact.

† Ann. Ch. Pharm. lxxxv. 137.

other. 2. If in this manner there is formed one atom of the compound A+B, and one atom of the compound A+B′, the mass of the body B may be increased in the presence of that of B′ up to a certain point, without any change in that atomic proportion: but if a certain limit be passed, the relation of the atoms, which was that of 1 : 1, changes suddenly and becomes 1 : 2, 1 : 3, 2 : 3, and so on. 3. When a body A, acting on an excess of the compound BC, reduces it, so that AB is formed, and C is set at liberty; then if C can, in its turn, reduce the newly-formed compound, the final result is that the reduced part of B+C is in a simple atomic proportion to the unreduced part. 4. The second observation applies also to these reductions. Debus* examined the phenomena presented in the precipitation of a mixture of the hydrates of lime and baryta by carbonic acid, and of the hydrochlorates of these earths by carbonate of soda, and arrived at results analogous to those given above.

In each of these cases, however, the first products of the chemical combination have been removed at once from the field of action. In Bunsen's experiment the carbonic acid and water will not react on one another; in that of Debus the carbonates separate immediately in an insoluble condition. It is evidently quite another case when the products themselves of the chemical action remain free to react. A mixture of two salts in solution, which do not produce a precipitate, affords a case where this requisite is fulfilled. Let AB and CD be such salts. According to the one view, when mixed they will either remain without mutual action, or, should the affinities so preponderate, they will become simply AC and BD, the excess of either original salt remaining inactive. According to the other view, A will divide itself in certain proportions between B and D, while C will do the same in the inverse ratio, the said proportions being determined not solely by the differences of energy in the affinities, but also by the differences of the quantities of the bodies. Again, supposing the latter view to be correct, another question will arise—Does the amount of AD or CB produced increase in a gradual manner with the relative increase of AB, or do sudden transitions take place under these circumstances, such as Bunsen and Debus observed in their experiments?

It was to the elucidation of these questions that I applied myself. In the majority of instances it is impossible to ascertain what has taken place when a mixture of the kind alluded to has been made; but the physical properties of salts will sometimes give an indication. Colour seemed to offer the best means of solving the problem; yet even here a difficulty arose from the fact that many bases, such as nickel, give the same coloured solution when combined with different acids, and *vice versâ*. Sesquioxide of iron, however, appeared to promise good results, since many of its salts are intensely coloured, while others are almost colourless. The formation of the sulphocyanide was first submitted to a full investigation; after which other ferric salts were more cursorily examined; and after them a number of other binary compounds in order to extend the range of observation.

* Ann. Ch. Pharm. lxxxv. 103; lxxxvi. 156; lxxxvii. 238.

Ferric Sulphocyanide.

If a soluble sulphocyanide be mixed with a ferric salt, a red solution results indicating the formation of the ferric sulphocyanide. Suppose three equivalents of the sulphocyanide be mixed with one equivalent of the metallic salt, we have the exact proportions theoretically necessary for the production of $Fe_2, 3S_2 Cy$. The first question to be solved is,—In such a case does the whole of the iron and of the sulphocyanogen combine as ferric sulphocyanide, or does it not? If BERGMAN's view be correct, the decomposition will be in accordance with the following simple formula (Rd standing for any salt radicle, and M for any metal)—

$$Fe_2 Rd_3 + 3M, S_2 Cy = Fe_2, 3S_2 Cy + 3M Rd ;$$

and it will not matter what metal is represented by M, or what salt radicle by Rd, provided only that a double decomposition *does* take place. Beside which, the addition of a larger quantity of either one of the original compounds will not increase the colour, for there is but one sesquisulphocyanide of iron, and the whole of the iron, or of the sulphocyanogen (as the case may be), has been already saturated. If, however, BERTHOLLET's view be correct, the decomposition will not be so complete as to form merely $Fe_2, 3S_2 Cy$ and 3M Rd, but in addition to these two salts there will be certain portions of the two original salts still remaining as such in the solution. This will become manifest by an amount of colour being obtained which is not equal to what would have been produced had the whole of the iron entered into combination with the sulphocyanogen: and the requirements of the theory will lead us moreover to expect that the amount of ferric sulphocyanide (and consequently the depth of colour) will depend in a great measure on the nature of M or Rd, and will be increased by each addition of either the soluble sulphocyanide or the ferric salt.

The following were the preliminaries for the complete determination of this question:—

Solutions of the ferric chloride, nitrate, sulphate, acetate, and citrate were prepared, each containing an amount of iron equivalent to 0·162 grm. of sesquioxide in every 1000 grain measures of water. The salts were made principally by dissolving pure hydrated ferric oxide in the pure acid; but it was found very difficult to obtain them of a definite constitution. Yet those employed, if not absolutely coinciding with the expression $Fe_2 Rd_3$, came very close to it; and fortunately the general result will be found not to depend upon great precision in the perfectly atomic composition of the iron salts. A solution of pure sulphocyanide of potassium was prepared containing 2·376 grms. of the salt in every 1000 gr. measures, that is to say, twelve equivalents in the same quantity of solution as contained one equivalent of the ferric salts. Solutions of other potash salts, containing the same amount of potassium in 1000 gr. measures, were prepared; as also solutions of known strength of sulphocyanide of barium and of mercury, and of hydrosulphocyanic acid.

In order to compare the depth of colour produced on the admixture of these solutions, it was necessary to have vessels of colourless glass of a uniform character. Ordinary precipitating glasses, holding about five ounces, were found peculiarly fitted for the purpose: being blown, and not moulded, they are very translucent; they are easily obtained devoid of colour; and, although not strictly uniform in size, it was easy to pick out a sufficient number which would furnish every requisite for the experiment. This was tested by dividing a coloured solution into two equal parts, and pouring one-half into one glass, and the other half into another; if the two solutions appeared then of a perfect equality of tint, nothing more could be desired. When two or more coloured solutions were to be compared, the best method was found to be to place the glasses containing them on a stand before the window, across the pane of which was stretched a piece of tissue-paper about 3 inches in depth. The experiments were almost always performed when the sun was shining, but not on the window itself, as that was found to be disadvantageous. By observing the coloured solutions against the evenly illuminated tissue-paper, most exact results could be obtained, especially after a little practice. If the object was to observe the amount of dilution necessary to reduce one coloured solution to the same tint as another, distilled water was added, and thoroughly mixed with it, until equal bulks of the two solutions appeared alike. The amount of water added was of course easily measured in a graduated vessel. My own observation was always checked by that of my assistant, and if we differed I generally adopted his view, since having no idea of what result was to be expected, his judgment was the more impartial. I may also state in this place, that it was found unnecessary to let a freshly mixed solution containing a sulphocyanide stand any length of time, for it assumed instantaneously its proper amount of colour: two mixtures, similarly prepared, were always found to be of precisely the same shade; and everything conspired to give me great and increasing confidence in the validity of testimony drawn from the colour of a solution.

The first object to be determined evidently was, whether on mixing three equivalents of sulphocyanide of potassium with one equivalent of the ferric salt, say the chloride, the full depth of colour possible from the combination of all the sulphocyanogen with all the iron was actually obtained. That this was not the case was seen at once, for on the addition to such a mixture of either more sulphocyanide of potassium, or more chloride of iron, the colour was increased. This showed also the influence of mass, which will be exhibited quantitatively in due course; but before doing so it is necessary to advert to another part of the inquiry, viz.—

The dependence of the amount of the coloured salt on the nature of the other substances present in the solution, but which are not immediately concerned in its formation.—In order to investigate this point, 25 gr. meas. of each of the ferric salts, which it will be remembered contained exactly the same amount of iron, were mixed with 6·25 gr. meas. of the sulphocyanide of potassium solution, that is one equiv. of the former to three of the latter. The five mixtures were equally diluted. At a glance it was evident that a widely different amount of red sulphocyanide of iron had been formed. The solution containing the ferric citrate was still green; that containing the acetate was red, but by no means deep in colour, and of a yellowish tint; while those containing the sulphate, nitrate, or chloride were of an intense red. The following are the relative amounts of dilution required to bring these four last-mentioned solutions to the same tint:—

3 equivs. sulphocyan. potassium + 1 equiv. ferric nitrate diluted to 100 parts.
3 equivs. sulphocyan. potassium + 1 equiv. ferric chloride diluted to 89·4 parts.
3 equivs. sulphocyan. potassium + 1 equiv. ferric sulphate diluted to 65·2 parts.
3 equivs. sulphocyan. potassium + 1 equiv. ferric acetate diluted to 20 parts.

The numbers 100, 89·4, 65·2, and 20 therefore represent the relative amounts of the ferric sulphocyanide contained in these several mixtures*.

In reference to the mixture of ferric citrate with sulphocyanide of potassium, the question presents itself,—Does absolutely no interchange take place between them, or does a partial though very minute formation of ferric sulphocyanide occur in accordance both with the law of Berthollet and the analogy of the other cases? The latter conclusion will appear probable from the following observations. Although the yellowish-green tint of the citrate still remains after the addition of three equivalents of the sulphocyanide, six equivalents almost remove it, and a larger quantity renders the solution colourless. No red colour ever appears in a very dilute solution, but this destruction of the green appears to point to the presence of a sufficient amount of the complementary colour to neutralize it; and if sulphocyanide of potassium be added in large excess to a strong solution of citrate of iron, an unmistakeable red ensues.

Similar experiments were tried in which the same ferric salt was employed, but different sulphocyanides. Two portions of nitrate of iron, each representing one equivalent, were mixed, the one with six equivalents of sulphocyanide of barium, the other with a corresponding amount of the potassium salt. A deep red resulted in both instances, but 1000 gr. meas. of the solution containing the potassium compound required the dilution of that containing the barium salt to only 880 gr. meas. to bring it to an equality of colour. The solution of sulphocyanide of mercury produced a scarcely perceptible reddening when added to the ferric nitrate.

These experiments suffice to show, that on mixing together solutions of soluble sulphocyanides and of ferric salts, the amount of sesquisulphocyanide of iron formed depends in a great measure on the nature of the substances previously combined with the sulphocyanogen and with the metallic oxide. The question naturally arises,—Does the converse of this hold good? If a solution of sesquisulphocyanide of iron be mixed with some other salt, not capable of forming a precipitate with it, will that also cause the distribution of the elements into *four* salts, manifesting itself by a diminution of the colour? and if so, will that vary according to the nature of the other salt?

* See Note A.

In order to test this, pure sesquisulphocyanide of iron was required; but I found the greatest difficulty in preparing such a salt; and even when obtained in tolerable purity, it was always liable to spontaneous decomposition, giving rise to protoxide of iron and a yellow powder (hydropersulphocyanic acid or pseudo-sulphocyanogen). The nearest approach to a pure salt was made by decomposing sulphocyanide of barium by an equivalent quantity of ferric sulphate, but this was not absolutely free from either protoxide of iron or sulphuric acid. Six equal portions of this were taken: one was kept as a standard; to the other five were added respectively equal portions of the nitrate, hydrochlorate, sulphate, acetate, and citrate of potash. In each case the colour was reduced. Column I. in the annexed Table shows the amount to which the standard had to be diluted before it was brought down to an equality in colour with the different mixtures. Column II. represents a similar experiment, in which a red mixture of one equivalent of sesquichloride iron, and twelve equivalents of sulphocyanide of potassium, was employed instead of the actual ferric sulphocyanide. The two following columns give the results of two experiments selected from a number of very early ones, the testimony of all of which was similar. In III. a mixture of ferric sulphate and sulphocyanide of potassium was employed; in IV. the same, with a large excess of sulphuric acid.

	I.	II.	III.	IV.
Volume of original solution + salt added ...	70 m.	64 m.	34 m.	60 m.
Mixture containing the nitrate	80	77	41	62
Mixture containing the chloride	90	90	56	68
Mixture containing the sulphate	150	160	124	85
Mixture containing the acetate...............	270	220	164	120
Mixture containing the citrate	trace of red	green		

These experiments might be varied *ad infinitum*, all proving the influence on the resulting colour of the *nature* of a substance mixed with the ferric sulphocyanide. Other organic acids, such as the oxalic and the tartaric, were found to reduce the red very rapidly*. As might be expected also, the diversity of effect is not confined to differences in the *acid* present. The addition of a protosalt of iron has a very great effect in reducing the colour of the ferric sulphocyanide: baryta salts act powerfully; lime salts have so much influence that the experiments here detailed would have been rendered abortive if spring water had been used for dilution, and even the quality of the distilled water employed could be detected by its effect on some of the succeeding experiments. A solution of chloride of mercury (as was observed long ago) very speedily removes the colour.

The colour of a mixture not dependent on the manner in which the constituents were originally arranged.—On more closely examining the above results, it will be seen that whether ferric citrate be mixed with sulphocyanide of potassium, or ferric sulphocyanide with citrate of potash, the resulting liquids contain but the merest trace of

* Pelouze has observed the different effects of different acids, Ann. Chim. et Phys. xliv. 216.

the red salt; it appears also that the deepest colour is obtained on mixing either ferric nitrate with sulphocyanide of potassium, or ferric sulphocyanide with nitrate of potash; whilst the mixtures containing compounds of acetic, sulphuric, and hydrochloric acids are intermediate in colour in regular order. Mercury also seems to exert the most powerful affinity for sulphocyanogen in whatever way they are brought together. This suggests the conclusion that the amount of sulphocyanide of iron in a mixture of salts does not depend on the *manner* in which the different substances were at first combined. The experiment above described was incapable of affording a quantitative demonstration of this, as a perfectly pure and definite sulphocyanide of iron was not obtained; but the following arrangement was made to put it to a rigid test. Two solutions were made; the first by mixing together 50 grain measures of ferric nitrate, and 12·5 gr. meas. of sulphate of potash; the second by mixing 50 gr. meas. of ferric sulphate, and 12·5 gr. meas. of nitrate of potash; so that each contained one equiv. of ferric oxide, three equivs. of potash, three equivs. of nitric acid, and three equivs. of sulphuric acid. To each was added 6·25 gr. meas., that is, 1·5 equiv. of sulphocyanide of potassium. The colours resulting in the two cases were so nearly identical, that the first diluted to 1770 gr. meas. just equalled the second diluted to 1820 gr. meas. Another 6·25 gr. meas. of sulphocyanide of potassium were added to each. The two solutions appeared now identical in colour; they certainly did not differ by 1 degree in 80. The amount of sulphocyanide of potassium in each was then doubled: the resulting colours could not be distinguished from each other. It appears, therefore, that it makes no difference whether there be mixed in solution

$$Fe_2O_3, 3NO_5+3(KO, SO_3)+3KS_2Cy \quad \text{or} \quad Fe_2O_3, 3SO_3+3(KO, NO_5)+3KS_2Cy.$$

The influence of the mass of one of the substances that produce the coloured salt.—The influence of mass has yet to be considered quantitatively. For this purpose two mixtures were prepared, each containing 25 gr. meas. of the ferric nitrate, and 6·25 gr. meas. of the sulphocyanide of potassium solution. They were both diluted so as to occupy 880 gr. meas. The one was kept as a standard of comparison; with the other additional portions of sulphocyanide of potassium were mixed; and as that increased the colour, it was diluted till brought to an equality with the standard solution.

Original solution.	Additional sulphocyanide of potassium.	Dilution required*.	Original solution.	Additional sulphocyanide of potassium.	Dilution required*.
880 gr. m.	6·25 gr. m.	390 gr. m.	880 gr. m.	125 gr. m.	2680 gr. m.
880 gr. m.	13·75 gr. m.	680 gr. m.	880 gr. m.	200 gr. m.	3310 gr. m.
880 gr. m.	20 gr. m.	880 gr. m.	880 gr. m.	275 gr. m.	3690 gr. m.
880 gr. m.	27·5 gr. m.	1070 gr. m.	880 gr. m.	387·5 gr. m.	4200 gr. m.
880 gr. m.	33·75 gr. m.	1250 gr. m.	880 gr. m.	500 gr. m.	4510 gr. m.
880 gr. m.	52·5 gr. m.	1780 gr. m.	880 gr. m.	612·5 gr. m.	4720 gr. m.
880 gr. m.	90 gr. m.	2300 gr. m.	880 gr. m.	775 gr. m.	4990 gr. m.

Beyond this the experiment could not be carried, although evidently no termination of the action of additional sulphocyanide had been arrived at.

* This includes the water in which the sulphocyanide of potassium was dissolved.

But the original 880 gr. meas. expressed the amount of ferric sulphocyanide produced by the mixture of one equiv. of ferric nitrate and three equivs. of sulphocyanide of potassium. Every addition also of 6·25 gr. meas. of sulphocyanide of potassium is an addition of three equivalents. Hence the first line of the preceding Table signifies that, considering the amount of the aforesaid ferric sulphocyanide as 880 (or for greater convenience 88), the amount formed when the same *one* equivalent of ferric nitrate is mixed with *six* equivalents of sulphocyanide of potassium is 880+390, that is 1270 (or 127)*. On this principle the foregoing experimental results may be thus tabulated :—

Ferric nitrate. Sulphocyan. of potassium.	Red salt produced.	Ferric nitrate. Sulphocyan. of potassium.	Red salt produced.
1 equiv.+ 3 equivs.	88	1 equiv.+ 63 equivs.	356
1 equiv.+ 6 equivs.	127	1 equiv.+ 99 equivs.	419
1 equiv.+ 9·6 equivs.	156	1 equiv.+135 equivs.	457
1 equiv.+12·6 equivs.	176	1 equiv.+189 equivs.	508
1 equiv.+16·2 equivs	195	1 equiv.+243 equivs.	539
1 equiv.+19·2 equivs.	213	1 equiv.+297 equivs.	560
1 equiv.+28·2 equivs.	266	1 equiv.+375 equivs.	587
1 equiv.+46·2 equivs.	318		

On comparing these numbers, it will be at once evident that each addition of the sulphocyanide produced relatively a smaller increase of colour. The numbers in fact give the long curve of Plate VII. fig. 1, where the ordinates express the proportionate amount of red salt, and the abscissæ the number of equivalents of the sulphocyanide.

The influence of mass was again tried by means of additional portions of ferric nitrate instead of additional sulphocyanide of potassium. The experiment was conducted in a precisely similar manner to the last. I have not thought it necessary to record the amounts of the salts and the water employed: the following are the results of the observations reduced as before :—

Sulphocyan. of potassium. Ferric nitrate.	Red salt produced.	Sulphocyan. of potassium. Ferric nitrate.	Red salt produced.
3 equivs.+1 equiv.	88	3 equivs.+ 5 equivs.	138
3 equivs.+2 equivs.	110·5	3 equivs.+ 6 equivs.	144
3 equivs.+3 equivs.	122	3 equivs.+10 equivs.	161
3 equivs.+4 equivs.	131	3 equivs.+14 equivs.	174

These numbers also give a curve. It is that designed in Plate VII. fig. 11.

It is not difficult to bring this experiment into uniformity with the preceding so as to form in fact a continuation of it, one equiv. of ferric nitrate being combined with less than three equivs. of sulphocyanide of potassium. For, expressing the number of equivalents of the iron salt by x, and the comparative amount of ferric sulphocyanide by y, the general formula of the terms in the above Table will be, K denoting the potassium and F the iron salt,—

$$3K+xF=y,$$

* See Note A.

which may evidently be reduced to a unity of F by dividing by x, thus—

$$F+\frac{3}{x}K=\frac{y}{x}.$$

On this principle the experiment may be thus tabulated:—

Ferric nitrate. Sulphocyan. of potassium.	Red salt produced.	Ferric nitrate. Sulphocyan. of potassium.	Red salt produced.
1 equiv. + 3 equivs.	88	1 equiv. + 0·6 equiv.	27·6
1 equiv. + 1·5 equiv.	55·25	1 equiv. + 0·5 equiv.	24
1 equiv. + 1 equiv.	40·66	1 equiv. + 0·3 equiv.	16·1
1 equiv. + 0·75 equiv.	32·75	1 equiv. + 0·21 equiv.	12·43

which numbers are represented by the broken line in Plate VIII. fig. 1.

It need scarcely be explained, that had the whole of the sulphocyanogen present in the above experiment united itself with the iron, the second term would have indicated 44 degrees instead of 55·25, the third 29·33, and so on; and the diagram would have presented a straight line joining 0×0 and 3×88; while no excess of sulphocyanide would cause the line to sink below 88.

In order to confirm, by a more direct experiment, the result just arrived at, two mixtures were made, each consisting of 100 gr. meas. of ferric nitrate, and 2 gr. meas. of sulphocyanide of potassium; that is, one equiv. of the former to 0·24 equiv. of the latter. The experiment was conducted precisely as in the previous cases, additional quantities of the sulphocyanide being mixed with one of the red solutions. The results form the subjoined Table. The comparative amount of red salt produced is given in two columns: in the first 88 is taken as the starting-point (which was the case in the actual experiment), in the second 88 is assumed as the expression of the amount produced when one equiv. of ferric salt is mixed with three of the sulphocyanide, and the numbers are calculated accordingly, so that this column should tally with those of the previous experiments.

Ferric nitrate. Sulphocyan. of potassium.	Red salt produced.		Ferric nitrate. Sulphocyan. of potassium.	Red salt produced.	
1 equiv. + 0·24 equiv.	88	19	1 equiv. + 3·57 equivs.	450	97·8
1 equiv. + 0·48 equiv.	138	29·7	1 equiv. + 4·44 equivs.	502	108·2
1 equiv. + 0·78 equiv.	183	39·4	1 equiv. + 5·58 equivs.	552	119
1 equiv. + 1·05 equiv.	226	48·7	1 equiv. + 7·08 equivs.	623	134·3
1 equiv. + 1·31 equiv.	279	60·2	1 equiv. + 8·64 equivs.	683	147·3
1 equiv. + 1·83 equiv.	320	69	1 equiv. + 10·29 equivs.	733	158·1
1 equiv. + 2·22 equivs.	355	76·5	1 equiv. + 11·79 equivs.	778	167·8
1 equiv. + 2·88 equivs.	400	86·3			

These numbers form the long curve projected in Plate VIII. fig. 1. The dotted line is that of the previous experiment (vide above), showing an almost perfect agreement. The broken line seems to follow the more continuous course. Where the sulphocyanide was not in excess, there appears a slight discrepancy between the two experiments.

The regularity of this increase of colour (as exhibited to the eye in the curves) is a proof that no law obtains under the circumstances of the experiment, similar to that observed and enunciated by BUNSEN. There is nowhere any sudden increase in the amount of ferric sulphocyanide formed. If the partition of the bases and acids in the mixture really take place at first in atomic proportions, it is evident that, being at full liberty to act and react, the salts arrange themselves according to their respective mass, without reference to their respective atomic weights.

The effect of mass on the formation of ferric sulphocyanide in a mixture of salts, where other substances replaced the nitric acid or the potash, was also tried.

Two solutions were prepared, each containing 50 gr. meas. of the ferric sulphate mixed with 12·5 gr. meas. of sulphocyanide of potassium, and the experiment was conducted as in the former instances. The weakness of the colour produced when a sulphate is present was the reason why the amount of the salts employed was doubled. The following are the results deduced from the observations:—

Ferric sulphate.	Sulphocyan. of potassium.	Red salt produced.	Ferric sulphate.	Sulphocyan. of potassium.	Red salt produced.
1 equiv.+	3 equivs.	88	1 equiv.+	45 equivs.	318
1 equiv.+	6 equivs.	128	1 equiv.+	57 equivs.	355
1 equiv.+	9 equivs.	153	1 equiv.+	69 equivs.	390
1 equiv.+	12 equivs.	177	1 equiv.+	81 equivs.	418
1 equiv.+	15 equivs.	198	1 equiv.+	93 equivs.	440
1 equiv.+	20 equivs.	223	1 equiv.+	105 equivs.	458
1 equiv.+	24 equivs.	241	1 equiv.+	123 equivs.	486
1 equiv.+	30 equivs.	263	1 equiv.+	147 equivs.	513
1 equiv.+	36 equivs.	288	1 equiv.+	195 equivs.	538

An experiment precisely analogous to the preceding was tried with the ferric chloride in the place of the sulphate. The results were:—

Ferric chloride.	Sulphocyan. of potassium.	Red salt produced.	Ferric chloride.	Sulphocyan. of potassium.	Red salt produced.
1 equiv.+	3 equivs.	88	1 equiv.+	65·4 equivs.	338
1 equiv.+	9 equivs.	148	1 equiv.+	83·4 equivs.	370
1 equiv.+	15 equivs.	190	1 equiv.+	107·4 equivs.	400
1 equiv.+	21 equivs.	216	1 equiv.+	131·4 equivs.	428
1 equiv.+	28·8 equivs.	246	1 equiv.+	155·4 equivs.	456
1 equiv.+	41·4 equivs.	286	1 equiv.+	191·4 equivs.	488
1 equiv.+	53·4 equivs.	312	1 equiv.+	239·4 equivs.	528

These two series of numbers give respectively the broken and the dotted lines in Plate VII. fig. 1.

A glance at these curves will show that, although the actual amount of ferric sulphocyanide produced from the same quantity of the sesquinitrate, chloride, or sulphate of iron varies greatly, yet the increase of colour on the addition of more sulphocyanide of potassium maintains a somewhat similar ratio in each case. In Plate VIII. fig. 11 the commencement of the three curves is shown on an enlarged scale, and their close approximation becomes still more evident. The variations that do exist arise, I am disposed to think, mainly from errors in the experiment; and this opinion is founded not only on the observations above detailed, but upon others of a shorter range, which it was not considered necessary to record, especially as the three given were the last of their respective kinds which I made, and on that account, I believe, worthy of the greater reliance. None of the others, I may remark, differed materially from them.

I have in vain endeavoured, by the aid of my friend, Mr. HENRY WATTS, to find an equation which will resolve the curves deduced from the above observations. They do not appear to belong to the second order.

For the purpose of seeing whether the same ratio was maintained where a much smaller proportion of red sulphocyanide was formed, the experiment was repeated with the ferric acetate. The following results were obtained:—

Ferric acetate. Sulphocyan. of potassium.	Red salt produced.	Ferric acetate. Sulphocyan. of potassium.	Red salt produced.
1 equiv.+1 equiv.	62·4	1 equiv.+11 equivs.	232
1 equiv.+3 equivs.	88	1 equiv.+13 equivs.	304
1 equiv.+5 equivs.	108	1 equiv.+15 equivs.	352
1 equiv.+7 equivs.	133	1 equiv.+19 equivs.	398
1 equiv.+9 equivs.	187		

Here we have not only an entirely different ratio, but the curve represented by these numbers (see the line composed of alternate lines and dots in Plate VIII. fig. 11) is of an irregular character. It is evident there is some interfering action; what that is will be seen when the ferric acetate itself is made the subject of experiment.

A similar experiment on the influence of mass was tried with hydrogen in the place of potassium; that is to say, nitrate of iron was mixed with successive portions of a solution of hydrosulphocyanic acid of known strength. The following are the results reduced to the same unit of comparison as in the preceding cases:—

Ferric nitrate. Hydrosulphocyanic acid.	Red salt produced.	Ferric nitrate. Hydrosulphocyanic acid.	Red salt produced.
1 equiv.+ 2 equivs.	66	1 equiv.+20 equivs.	288
1 equiv.+ 4 equivs.	108	1 equiv.+24 equivs.	315
1 equiv.+ 6 equivs.	142	1 equiv.+30 equivs.	353
1 equiv.+ 8 equivs.	168	1 equiv.+38 equivs.	400
1 equiv.+12 equivs.	217	1 equiv.+46 equivs.	440
1 equiv.+16 equivs.	257		

The curve represented by these numbers is the line broken by three dots in Plate VIII. fig. 11. It is as regular as the corresponding one from the ferric nitrate, the continuous line in the same Plate, but the curve is quite different, showing a more rapid ratio according to which the coloured salt is formed when hydrogen is substituted for potassium.

Method of determining the actual amount of the coloured salt in a given mixture.— From the experiments above recorded, it would seem probable that no amount of sulphocyanide of potassium added to a ferric salt will absolutely convert the whole of it into the sesquisulphocyanide of iron. Yet we can easily judge where this result will be very nearly attained. Thus 400 equivalents of the sulphocyanide, added to one of the ferric nitrate, must give a close approximation. A somewhat larger quantity will do the same with the sulphate. Indeed, it was found by experiment that 500 equivalents of the sulphocyanide added to each of the three principal ferric compounds, caused as nearly as possible the same intensity of colour. Such a mixture was made, assumed to be the proper tint for an equivalent of the ferric sulphocyanide, and employed as a standard. Mixtures were then made of three equivalents of sulphocyanide of potassium with one equivalent of the various ferric salts, each occupying 330 gr. meas. The standard red was then diluted till it was equal in colour to these several mixtures. The annexed Table gives the different amounts of dilution required.

1 eq. ferric nitrate+3 eq. sulphocyanide of potassium . .	1700 gr. m.
1 eq. ferric chloride+3 eq. sulphocyanide of potassium . .	1900 gr. m.
1 eq. ferric sulphate+3 eq. sulphocyanide of potassium . .	2650 gr. m.
1 eq. ferric acetate+3 eq. sulphocyanide of potassium . .	7000 gr. m. (about)

This affords us the elements requisite for a calculation of the actual amount of ferric sulphocyanide present in each of these mixtures; and had it been desirable, almost every observation given above might have been thus reckoned. The ratio between the volumes of the diluted standard ferric sulphocyanide, and that of any one of the other red mixtures (330 gr. meas.), gives the ratio between one equivalent and the fractional part existing in the said mixture. The four observations calculated on this principle give the following results:—

1 eq. ferric nitrate+3 eq. sulphocyan. of potas. give 0·1941 eq. ferric sulphocy.
1 eq. ferric chloride+3 eq. sulphocyan. of potas. give 0·1737 eq. ferric sulphocy.
1 eq. ferric sulphate+3 eq. sulphocyan. of potas. give 0·1245 eq. ferric sulphocy.
1 eq. ferric acetate+3 eq. sulphocyan. of potas. give 0·0471 eq. ferric sulphocy. (about)

If this mode of reckoning involve no fallacy, the proportion between these four numbers should be the same as that given near the commencement of this inquiry, where the colours produced on a different occasion by adding three equivalents of sulphocyanide of potassium to one equivalent of the ferric salts, were directly compared. That they do agree almost exactly will be seen from the following Table, where column I. gives the numbers of the former experiment, and column II. those of the last calculation reduced to the same unit of comparison.

	I.	II.
1 equiv. ferric nitrate + 3 equivs. sulphocyanide of potassium	100	100
1 equiv. ferric chloride + 3 equivs. sulphocyanide of potassium	89·4	89·5
1 equiv. ferric sulphate + 3 equivs. sulphocyanide of potassium	65·2	64·2
1 equiv. ferric acetate + 3 equivs. sulphocyanide of potassium	20	24·2?

This close agreement proves not only the correctness of the two independent experiments, but also the correctness of this method of reckoning the amount of the coloured salt in any given mixture.

Influence of the mass of a substance present in the solution, but which is not one of the constituents of the coloured salt.—It has already been remarked that the addition of a colourless salt will reduce the colour of a solution of ferric sulphocyanide. The influence of mass in this kind of action remains to be examined.

A mixture was made of ferric sulphate and sulphocyanide of potassium. The red solution that resulted contained of course sulphate of potash. Successive portions of a solution of this salt were added, and the amount of decomposition effected was determined by means similar to those employed in previous experiments.

Sulphate of potash added.	Water added to comparative solution.	Sulphate of potash added.	Water added to comparative solution.
5 measures=	22 measures.	30 measures=	92 measures.
10 measures=	38 measures.	40 measures=	115 measures.
15 measures=	52 measures.	60 measures=	155 measures.
20 measures=	67 measures.		

This action then proceeds in a gradually decreasing ratio. The above results give the curve projected in Plate VII. fig. 3, which, like those formerly examined, does not belong to the second order.

These very diversified experiments have put to a rigid test the truth of BERTHOLLET's view. Whatever were the circumstances under which the reactions were tried, they invariably showed that the results were dependent both upon the nature and upon the quantity of all the substances in solution.

I have purposely investigated, at considerable length, the reactions made evident by the colour of ferric sulphocyanide, both because I desired to prove the matter thoroughly, and because the scientific public will probably require (as indeed I did when I commenced the investigation) a greater amount of testimony, where it depends on the colour of a solution, than if it had depended on substances actually separated and weighed. I now proceed to the examination of reactions made evident by means of other coloured iron salts; but I shall not dwell at any length on these, unless there be some apparent anomaly to call for more particular attention.

Ferric Gallate.

A solution of gallic acid was made of known strength. Equal portions of it were added to equal portions of the different ferric salts.

1 equiv. ferric nitrate with 1 equiv. gallic acid gave 100 parts of black salt.
1 equiv. ferric chloride with 1 equiv. gallic acid gave 88 parts of black salt.
1 equiv. ferric sulphate with 1 equiv. gallic acid gave 70 parts of black salt.
1 equiv. ferric citrate with 1 equiv. gallic acid gave 10? parts of black salt.

The mixture containing the citrate could not be accurately compared on account of its greenish hue. That made from the acetate was of an intense blue.

When single equivalents of nitrate of iron and gallic acid were mixed, a solution resulted in which the gallic acid had to such an extent combined with the sesquioxide of iron, that the addition of several equivalents of either one of the constituent substances caused a scarcely perceptible increase of colour.

The ferric chloride was then tried.

Ferric chloride. Gallic acid.	Black salt produced.	Ferric chloride. Gallic acid.	Black salt produced.
1 equiv. + 1 equiv.	88	1 equiv. + 4 equivs.	128
1 equiv. + 2 equivs.	108	1 equiv. + 6 equivs.	133
1 equiv. + 3 equivs.	120		

These numbers are represented by the dotted curve in Plate IX. fig. 1.

The citrate afforded a better opportunity of obtaining a numerical result represent ing the influence of the mass of one of the constituents. The following are two observations made by means of it. A difficulty was experienced from the greenish hue of the mixture of single equivalents gradually changing to black as fresh acid was added; hence the observations given below do not commence at unity.

Ferric citrate. Gallic acid.	Black salt produced.	Ferric citrate. Gallic acid.	Black salt produced.
1 equiv. + 4 equivs.	88	1 equiv. + 12 equivs.	256
1 equiv. + 6 equivs.	133	1 equiv. + 16 equivs.	310
1 equiv. + 8 equivs.	176	1 equiv. + 18·6 equivs.	346
1 equiv. + 10 equivs.	220		

These numbers are represented by the broken line in Plate IX. fig. 1.

Ferric citrate. Gallic acid.	Black salt produced.	Ferric citrate. Gallic acid.	Black salt produced.
1 equiv. + 5 equivs.	155	1 equiv. + 13 equivs.	297
1 equiv. + 7 equivs.	200	1 equiv. + 17 equivs.	353
1 equiv. + 9 equivs.	237	1 equiv. + 25 equivs.	445
1 equiv. + 11 equivs.	270	1 equiv. + 33 equivs.	509

This is represented by the continuous line in the same Plate.

Experiments were also tried with gallate of potash in the place of gallic acid.

1 eq. ferric nitrate + 1 eq. gallate of potash gave 100 parts of black salt.
1 eq. ferric chloride + 1 eq. gallate of potash gave 97 parts of black salt.
1 eq. ferric sulphate + 1 eq. gallate of potash gave 68 parts of black salt.
1 eq. ferric citrate + 1 eq. gallate of potash gave a blue solution.
1 eq. ferric acetate + 1 eq. gallate of potash gave a precipitate.

A mixture of single equivalents of gallate of potash and ferric nitrate gave nearly, but apparently not quite the same depth of colour as when an equivalent of gallic acid was mixed with the same iron salt.

Ferric gallate was prepared by dissolving the hydrated sesquioxide of iron in gallic acid. It was divided into two equal parts, to one of which successive portions of hydrochloric acid were added, while the other was diluted after each addition till it had been reduced to the colour of the acid mixture.

Hydrochloric acid added.	Water added to comparative solution.	Hydrochloric acid added.	Water added to comparative solution.
1 measure =	3·4 measures.	3·25 measures =	10·1 measures.
2 measures =	6·6 measures.	4·5 measures =	12·0 measures.

It requires no further experiments to show that the ferric gallate bears the same testimony as the sulphocyanide.

Ferric Meconate.

Similar experiments were made with meconic acid and the iron salts. After mixing these substances it was found necessary to allow the solutions to stand a minute or two before observation, in order that the full colour might be developed.

1 eq. ferric nitrate + 1 eq. meconic acid ($3HO, C_{14} H_4 O_{14}$) gave 100 parts of red salt.
1 eq. ferric chloride + 1 eq. meconic acid ($3HO, C_{14} H_4 O_{14}$) gave 96 parts of red salt.
1 eq. ferric sulphate + 1 eq. meconic acid ($3HO, C_{14} H_4 O_{14}$) gave 72 parts of red salt.
1 eq. ferric citrate + 1 eq. meconic acid ($3HO, C_{14} H_4 O_{14}$) gave 42 parts of red salt.
1 eq. ferric acetate + 1 eq. meconic acid ($3HO, C_{14} H_4 O_{14}$) gave a red precipitate.

The influence of successive additions of meconic acid to a mixture of single equivalents of that substance and ferric nitrate was tried. The results were as follows:—

Ferric nitrate.	Meconic acid.	Red salt produced.	Ferric nitrate.	Meconic acid.	Red salt produced.
1 equiv. +	1 equiv.	88	1 equiv. +	4 equivs.	75
1 equiv. +	2 equivs.	80	1 equiv. +	6 equivs.	73
1 equiv. +	3 equivs.	76	1 equiv. +	8 equivs.	74

Here, instead of finding an increase of colour, as might have been expected by analogy, there is a distinct though small decrease. On examining the action more fully, and by repeated experiments, it was found that the maximum colour was obtained when the ferric nitrate and the meconic acid were mixed in single equivalents (or rather in the proportion of 12 atoms of the former to 11 of the latter); that the addition of more ferric nitrate to such a mixture did not notably increase the colour; that the addition to it of 0·25 equivalent of meconic acid made little change; and that a greater addition caused a decided diminution of the tint.

A mixture of one equivalent of meconic acid with one equivalent of sesquichloride of iron was examined in a similar manner. The addition of meconic acid was found in this case also to diminish the colour. The effect of successive additions of the ferric salt was more particularly tried.

Meconic acid.	Ferric chloride.	Red salt produced.	Meconic acid.	Ferric chloride.	Red salt produced.
1 equiv. +	1 equiv.	88	1 equiv. +	5 equivs.	99
1 equiv. +	1·2 equiv.	96	1 equiv. +	7 equivs.	93
1 equiv. +	1·8 equiv.	108	1 equiv. +	9 equivs.	99
1 equiv. +	2·6 equivs.	118	1 equiv. +	13 equivs.	119
1 equiv. +	3·8 equivs.	106			

These different determinations are exhibited by the continuous line in Plate IX. fig. 2. No regular curve can be described passing through these points; it is evident that there is some action which twice changes the order of the series.

When ferric sulphate and meconic acid are mixed, it requires about seven atoms of the former to five of the latter to produce the greatest intensity of colour. It then just about equals in tint the mixture of five atoms of the ferric nitrate with the same amount of meconic acid, and made up to the same volume.

No amount of ferric citrate added to the above-mentioned amount of meconic acid is capable of bringing the colour up to that of the former mixtures. This is partly, if not wholly, due to the greenish tint of the citrate neutralizing the red of the meconate.

There is then, in the case of the ferric meconate, some action interfering with the proper exhibition of the law of mass, which did not occur with the sulphocyanide or gallate. It seemed a natural supposition that this might arise from the power of meconic acid to form several compounds of different degrees of redness with ferric oxide. With a view to ascertain if meconate of iron itself entered into any such combination with meconic acid, the red salt was prepared by allowing hydrated sesquioxide to stand some hours with meconic acid. The experiment never gave the result anticipated, but this may have arisen from the more acid compound being always formed under such circumstances.

Thinking it would be desirable to repeat these experiments with meconate of potash instead of the acid, that salt was prepared by neutralizing meconic acid with the carbonate of the alkali. Three atoms of the base were found to combine with one of the acid. Single equivalents of this salt and of the various ferric salts were mixed.

1 eq. ferric nitrate $+3KO, C_{14}H_4O_{14}$ gave 100 parts of red salt.
1 eq. ferric chloride $+3KO, C_{14}H_4O_{14}$ gave 73 parts of red salt.
1 eq. ferric sulphate $+3KO, C_{14}H_4O_{14}$ gave 84 parts of red salt.
1 eq. ferric citrate $+3KO, C_{14}H_4O_{14}$ gave a trace of red salt.
1 eq. ferric acetate $+3KO, C_{14}H_4O_{14}$ gave a red precipitate.

These proportions differ considerably from those observed where meconic acid itself was employed.

The effect of successive additions of meconate of potash to the ferric nitrate was also tried.

Ferric nitrate.	Meconate of potash.	Red salt produced.	Ferric nitrate.	Meconate of potash.	Red salt produced.
1 equiv. +	0·33 equiv.	34	1 equiv. +	1 equiv.	88
1 equiv. +	0·5 equiv.	50	1 equiv. +	1·2 equiv.	84
1 equiv. +	0·8 equiv.	74	1 equiv. +	1·5 equiv.	69

Here the greatest intensity of colour evidently occurs when about single equivalents are mixed, the addition of a larger quantity of meconate of potash producing a rapid diminution of the colour.

The same was observed in respect to the ferric chloride. When single equivalents had been mixed, the addition of more ferric salt was not found to make any great difference in colour. This action, however, was examined quantitatively by means of the citrate.

Meconate of potash.	Ferric citrate.	Red salt produced.	Meconate of potash.	Ferric citrate.	Red salt produced.
1 equiv. +	0·8 equiv.	97	1 equiv. +	3·8 equivs.	102
1 equiv. +	1·2 equiv.	80	1 equiv. +	5 equivs.	125
1 equiv. +	1·8 equiv.	69	1 equiv. +	7 equivs.	154
1 equiv. +	2·6 equivs.	74	1 equiv. +	9·2 equivs.	165

These numbers are represented by the broken line in Plate IX. fig. 2.

Imagining that the rapid decrease of colour manifested when meconate of potash was added in excess to the ferric salt, might be due to the formation of some paler double compound, I added the potash salt to a solution of pure meconate. The colour was greatly diminished.

These results show satisfactorily enough that the amount of meconate of iron formed depends upon the nature of the various substances in solution, but their testimony in respect to the mass of these substances is obscured by the formation of these double compounds. Thinking to avoid this by always using the same amount of meconic acid and iron, and yet to exhibit the effect of mass, the following experiments were performed. Pure meconate of iron was treated with acetate of potash; it was quickly reduced to a pale yellow. Oxalate or phosphate of potash had the same effect. A saturated solution of sulphate of potash was tried, and gave these results:—

Sulphate of potash added.		Water added to comparative solution.
5 measures	=	12 measures.
12 measures	=	30 measures.

A strong solution of sulphate of soda was tried:—

Sulphate of soda added.		Water added to comparative solution.	Sulphate of soda added.		Water added to comparative solution.
5 measures	=	16 measures.	40 measures	=	58 measures.
10 measures	=	26 measures.	60 measures	=	80 measures.
20 measures	=	36 measures.			

Another solution of meconate of iron was similarly treated with dilute hydrochloric acid :—

Hydrochloric acid added.	Water added to comparative solution.	Hydrochloric acid added.	Water added to comparative solution.
2 measures =	3·6 measures.	7 measures =	15 measures.
3 measures =	5 measures.	9 measures =	20 measures.
5 measures =	10 measures.		

It is evident, that although a reduction of the amount of ferric meconate always takes place, there is some cause interfering with the regularity of the decrease of colour.

The amount of ferric meconate depends therefore upon the nature and upon the quantity of all the substances present at the same time in the solution; but the regularity of the action of mass, which was observed with the sulphocyanide and gallate, is not confirmed in this instance.

Ferric Pyromeconate.

Thinking that the irregularity in the influence of mass might be more or less connected with the tribasic character of meconic acid, it occurred to me that an examination of ferric pyromeconate would be desirable, since pyromeconic acid is monobasic, and yet strikes an intense red with the sesquioxide of iron, and in many other respects resembles the substance from which it is derived. Accordingly, some pyromeconic acid was prepared by submitting meconic acid to dry distillation :—

1 eq. ferric nitrate + 3 eq. pyromeconic acid gave 100 parts of red salt.
1 eq. ferric chloride + 3 eq. pyromeconic acid gave 86 parts of red salt.
1 eq. ferric sulphate + 3 eq. pyromeconic acid gave 39 parts of red salt.
1 eq. ferric citrate + 3 eq. pyromeconic acid gave 27 parts of red salt.
1 eq. ferric acetate + 3 eq. pyromeconic acid gave a red precipitate.

1 eq. ferric nitrate + 3 eq. pyromeconate of potash gave 100 parts of red salt.
1 eq. ferric chloride + 3 eq. pyromeconate of potash gave 74 parts of red salt.
1 eq. ferric sulphate + 3 eq. pyromeconate of potash gave 36 parts of red salt.
1 eq. ferric citrate + 3 eq. pyromeconate of potash gave 26 parts of red salt.
1 eq. ferric acetate + 3 eq. pyromeconate of potash gave a red precipitate.

On trying the influence of mass, it was found that the addition of pyromeconic acid rapidly *diminished* the colour of ferric pyromeconate; and that the colour was the deepest when the base was in large excess.

Thinking that the effect of a colourless salt upon the red pyromeconate might display more clearly the influence of mass, a mixture of ferric chloride and pyromeconic acid was experimented on with a solution of sulphate of potash :—

Sulphate of potash added.	Water added to comparative solution.	Sulphate of potash added.	Water added to comparative solution.
3·7 measures	= 12·5 measures.	20 measures	= 37·5 measures.
11·2 measures	= 27·5 measures.	30 measures	= 47·5 measures.

This is exhibited in Plate IX. fig. 3.

The monobasic pyromeconate appears therefore to be similar in its testimony to the tribasic meconate.

Ferric Acetate.

Similar experiments to those already described were made with the acetate of iron. As the red colour of this salt is very slight, as compared with that of the preceding compounds, it necessitated the employment of tolerably strong solutions.

Twelve equivalents of acetate of potash were added to one equivalent of each of the ferric salts, and gave the following proportions:—

1 eq. ferric nitrate + 12 eq. acetate of potash gave 100 parts of red salt.
1 eq. ferric chloride + 12 eq. acetate of potash gave 139 parts of red salt.
1 eq. ferric sulphate + 12 eq. acetate of potash gave 112 parts of red salt.
1 eq. ferric citrate + 12 eq. acetate of potash gave no red salt.

These proportions differ greatly from those which have been previously observed.

The effect of successive additions of acetate of potash was tried:—

Ferric nitrate.	Acetate of potash.	Red salt produced.	Ferric nitrate.	Acetate of potash.	Red salt produced.
1 equiv. +	3 equivs.	88	1 equiv. +	21 equivs.	87
1 equiv. +	6 equivs.	109	1 equiv. +	30 equivs.	64
1 equiv. +	9 equivs.	109	1 equiv. +	39 equivs.	61
1 equiv. +	12 equivs.	102	1 equiv. +	48 equivs.	52
1 equiv. +	15 equivs.	96	1 equiv. +	63 equivs.	46

Plate IX. fig. 4 exhibits these numbers; and here again, as in the case of the meconate, there is something producing a great irregularity of action.

To a mixture of one equivalent of ferric nitrate and three of acetate of potash, successive portions of the iron salt were added. They rendered the mixture much paler, reducing it at last almost to the colour of the nitrate itself.

In order to ascertain whether these changes of colour were due to the formation of double salts containing both iron and potash, three equal portions of the ferric acetate employed in the previous experiments were treated respectively with solutions of acetate of potash, acetic acid, and water. The potash salt caused a slight *increase* of colour, and the pure acid a great *decrease*, as compared with the effect of mere dilution. It is clear that there exist different combinations of acetic acid and sesquioxide of iron; indeed, it has been observed before by others, that a highly coloured solution of ferric acetate will spontaneously deposit red oxide and become almost colourless.

This irregularity of colour accounts for the extraordinary appearance when successive portions of sulphocyanide of potassium were added to the ferric acetate (*vide* Plate VIII. fig. 2); for, it must be remembered, the red colour observed in that experiment was due to the colour of the acetate as well as of the sulphocyanide.

The ferric acetate then confirms BERTHOLLET's view, but, like the meconate, its testimony in respect to the influence of mass is equivocal.

Ferric Ferrocyanide.

The ferric ferrocyanide, though insoluble in pure water, is soluble in the presence of oxalic acid, giving a deep blue. A mixture was made of a known amount of ferrocyanide of potassium with that acid, and it was added to the various ferric salts. The blue from the nitrate, chloride, or sulphate was very intense; the mixture containing the acetate was colourless the first minute, but gradually became blue; while that containing the citrate also deepened in tint on standing. After remaining about two hours, the coloured mixtures were in the following proportions:—

1 eq. ferric nitrate + 3 eq. ferrocyanide of potassium gave 100 parts of blue salt.
1 eq. ferric chloride + 3 eq. ferrocyanide of potassium gave 87 parts of blue salt.
1 eq. ferric sulphate + 3 eq. ferrocyanide of potassium gave 89 parts of blue salt.
1 eq. ferric acetate + 3 eq. ferrocyanide of potassium gave 45 parts of blue salt.
1 eq. ferric citrate + 3 eq. ferrocyanide of potassium gave 60 parts of blue salt.

The effect of mass was also tried. The addition of more ferrocyanide of potassium to a mixture of one equivalent of ferric nitrate and three of the ferrocyanide produced no appreciable increase of colour. With the citrate, however, the following numerical results were obtained:—

Ferric citrate.	Ferrocyanide of potassium.	Blue salt produced.	Ferric citrate.	Ferrocyanide of potassium.	Blue salt produced.
1 equiv. +	3 equivs.	88	1 equiv. +	9 equivs.	113
1 equiv. +	6 equivs.	107	1 equiv. +	15 equivs.	120

These numbers give rise to the curve in Plate IX. fig. 5.

Acetate or citrate of potash added to a mixture of ferric nitrate and ferrocyanide of potassium in oxalic acid, produces no perceptible change at the moment of mixing; but a decrease of colour becomes apparent after a few minutes, and continues, becoming more and more marked for some hours.

The ferrocyanide then bears a similar testimony to the truth of BERTHOLLET's position to what the ferric sulphocyanide and gallate do.

Ferric Comenamate.

Single equivalents of comenamic acid ($C_{12}H_5NO_8+4HO$) were mixed with single equivalents of the different ferric salts. With the nitrate, chloride, and sulphate it gave a most intense purple, with the citrate a wine-red solution, and with the acetate a precipitate.

The three purple solutions were about equally deep in colour; they were unaffected by the addition of any amount of iron salt, but were reddened by the addition of comenamic acid. The mixture containing the citrate was uninfluenced in the character of the tint, and almost so in the depth of it, by the addition of any amount of either the acid or the iron salt. It was quite evident from this, that comenamic acid has a great tendency to combine with sesquioxide of iron in place of water, and that it is capable of forming two distinct compounds. Indeed it was found that one equivalent, or less, of comenamic acid uniformly gave with one equivalent of nitrate of iron a deep bluish-purple compound; and that two equivalents, or more, gave a wine-red compound; whilst any proportion intermediate between one and two equivalents gave an intermediate tint.

Comenamate of potash and of ammonia gave similar results to comenamic acid itself; but the colour produced with the citrate at least was not so deep. The following were the ratios:—

1 eq. ferric citrate + 1 eq. comenamic acid gave . . . 5 parts of red salt.
1 eq. ferric citrate + 1 eq. comenamate of ammonia gave 4 parts of red salt.
1 eq. ferric citrate + 1 eq. comenamate of potash gave . 2·9 parts of red salt.

The addition of citrate of iron to a mixture of single equivalents of it and comenamate of potash caused an increase of colour, but no amount turned the solution purple.

Comenamic acid then is able to overcome the great affinity of citric acid for ferric oxide only so far as to produce the more acid salt. The purple comenamate was reddened instantly by citrate of potash, yet a large addition of that substance did not wholly destroy the colour.

That comenamic acid has not so great an affinity for sesquioxide of iron as to be unaffected by the presence of nitric, hydrochloric or sulphuric acid, was easily demonstrated. The following experiment illustrates the action of such a substance. To a solution consisting of single equivalents of ferric chloride and comenamic acid, successive portions of hydrochloric acid were added. The original mixture was bluish-purple.

Hydrochloric acid added.	Colour of mixture.
3 measures.	Bluish purple, but paler by an amount equiv. to 25 measures of water.
6 measures.	Bluish purple, but paler by an amount equiv. to 40 measures of water.
9 measures.	Bluish purple, but paler by an amount equiv. to 54 measures of water.
15 measures.	Visibly redder, and paler by an amount equiv. to 80? measures of water.
21 measures.	Red purple.
31 measures.	Pink.
55 measures.	Still perceptibly pink.

The affinity then of comenamic acid for sesquioxide of iron, though very great, is influenced both by the nature and by the quantity of other substances present in the same solution.

Ferric Bromide.

Experiments were also made on the ferric bromide. The iron salts were employed without dilution, as the bromide itself is but little redder than the chloride. Three equivalents of hydrobromic acid added to one of ferric nitrate produced a distinct red; added to the ferric citrate they produced little change in the colour. Yet the bromine has evidently a great tendency to combine with the iron; for, though the addition of a larger quantity of hydrobromic acid to the nitrate did perceptibly increase the colour, a maximum effect seemed attained when only about twelve equivalents were added. The addition of twelve equivalents in the case of the citrate produced likewise a red tint similar to that from the nitrate. Bromide of potassium did not redden the ferric citrate. Numerical results could not be obtained on account of the paleness of the colour.

If sesquioxide of iron be dissolved in hydrobromic acid, a very deep red solution is obtained, which is scarcely affected in colour by the addition of any potash salts in any quantity, unless, indeed, they decompose it with the formation of a red precipitate. Strong citric acid even has little effect upon it. It gives an intense blue with ferrocyanide of potassium in oxalic acid. This compound, however, is not a true ferric salt; it is an oxybromide.

The Ferric Salts in general.

Effect of mass of solvent.—In connexion with these experiments on ferric salts, it became a matter of interest to ascertain whether changes in the mass of water itself had any influence on the composition of the salts contained in these coloured solutions.

The only methods which occurred to me of obtaining an answer to this inquiry, were, to ascertain whether dilution caused any greater or less decrease of colour in some substances than in others of the same tint; and whether the decrease of colour by dilution was uniform in the same salt, by whatever mixture it might be produced.

It has frequently been noticed that a red solution of ferric sulphocyanide is reduced by the addition of water more than the simple dilution seemed capable of accounting for, and more than the red meconate is. In examining this matter, it seemed desirable first to ascertain how far it might be the fact. For this purpose two solutions were taken; the one of the purest sesquisulphocyanide of iron I could obtain, the other of pure ferric meconate. They were made up to the same colour and the same volume, and were then equally diluted. It became at once evident that the addition of water produced far greater difference in the colour of the sulphocyanide than in that of the meconate; but the exact proportion could not be determined, as, although the two reds were almost identical in shade at first, the sulphocyanide assumed on dilution a yellowish, and the meconate a pink hue. Numerical results, however, were obtained from the comparison of two mixtures, the one consisting of four equivalents of sulphocyanide of potassium and one of ferric chloride, the other of four equivalents of meconate of potash to one of the chloride of iron. They were made up to the same depth of colour, each occupying 200 grain measures.

200 gr. m. of the sulphocyanide equalled in colour 200 gr. m. of the meconate.
400 gr. m. of the sulphocyanide equalled in colour 730 gr. m. of the meconate.
720 gr. m. of the sulphocyanide equalled in colour 2460 gr. m. of the meconate.
1440 gr. m. of the sulphocyanide equalled in colour 7540 gr. m. of the meconate.

The disparity here is very great, and takes place at an increasing ratio.

It seemed desirable to test, if possible, whether this diversity was due entirely to the sulphocyanide, or whether the meconate might not also be departing from the ratio of decrease in colour which mere dilution would cause. For this purpose five solutions were taken of equal bulk and of the same depth of colour. They consisted respectively of meconate of iron, a mixture of ferric chloride and sulphocyanide of potassium, port wine and water, red ink, and infusion of cochineal. These solutions, though not identical in colour, were sufficiently near for the purpose. On repeated dilution of each with equal amounts of water, they all retained the same colour relatively, except the sulphocyanide, which became yellowish and much lighter.

It may fairly be concluded then, without predicating anything as to the action of water on dry salts, that large quantities of water have no specific action on meconate of iron, but that in some way they affect the sulphocyanide. Is this a mere physical effect upon the particular colour; or does some change take place in the composition of the salt itself*? In order to test whether this action of water was influenced by the presence of other substances, red solutions of equal volume and equal depth of colour were prepared by the following admixtures:—ferric chloride with sulphocyanide of potassium in large excess; sulphocyanide of potassium with ferric chloride in large excess; ferric nitrate with sulphocyanide of potassium; the same salts with the addition of a large quantity of sulphate of potash; sulphocyanide of potassium with ferric acetate; ferrous and ferric sulphocyanide with sulphocyanide of lead; and nearly pure sesquisulphocyanide of iron. On repeated dilution with equal amounts of water these all appeared to retain the same relative colour.

It seems then, as far as this experiment can prove it, that the action of water, whatever it be, is exerted equally upon red sulphocyanide of iron, with whatever other substance it may be mixed. This removes any doubt that might have rested from this cause on some of the original experiments with ferric sulphocyanide: and the fact that those experiments were always comparative, leaves little ground for any possible objection.

An experiment was likewise tried in order to determine whether the presence of

* See Note B.

other substances had any influence in the dilution of meconate of iron. Solutions were taken of pure ferric meconate; and of mixtures of ferric chloride with meconic acid; of the same with meconate of potash, both in large excess and otherwise; of ferric nitrate with meconic acid; of the same with meconate of potash; and of ferric citrate with meconic acid. On repeated dilution with equal amounts of water, no notable difference was observed in the relative depths of colour of these several solutions, except in the case of the citrate, which, on standing for some hours after dilution, lost colour considerably. The red also was of a more pure crimson where there was nitric acid.

Pyromeconate of iron prepared by double decomposition was affected in colour by dilution in a similar manner to the meconate. Water, too, seemed to have the same effect on the blue ferrocyanide as on ammoniacal sulphate of copper. In these cases, however, the salts compared were not of precisely the same tint.

Relative strength of affinity.—Having considered the evidence borne by eight coloured and soluble ferric salts as to the truth of certain views of the laws that regulate chemical combination, we have found their testimony on the main points uniform. We may now go further, and by examining the results above given determine the relative degree of affinity exerted by the different acids for sesquioxide of iron as compared with potash. Thus it is evident that citric acid has a much stronger affinity for ferric oxide, or a weaker affinity for potash, than nitric acid has; and again, it is evident that an equivalent amount of ferrocyanide of potassium removes the greater proportion of ferric oxide from citric acid, while sulphocyanide of potassium takes only 0·194 part from the nitrate.

The following is the order of affinity of the different acids experimented with for sesquioxide of iron and an equivalent amount of potash.

Least affinity for sesquioxide of iron as compared with potash

Hydrosulphocyanic acid	1
Nitric acid	4
Hydrochloric acid	5
Sulphuric acid	7
Gallic acid	10?
Pyromeconic acid?	
Meconic acid?	
Acetic acid	20?
Hydrobromic acid	
Comenamic acid	
Citric acid	100
Hydroferrocyanic acid	170?

Greatest affinity for sesquioxide of iron as compared with potash.

The numbers in the preceding table are deduced from the experimental data, but they must be considered as only rough approximations to the truth. The notes of interrogation indicate that the means of determination were themselves open to doubt.

Effect of differences of temperature.—The experiments narrated in this paper were all performed at the ordinary temperature. The slight changes that may have taken place in that respect from one day to another were incapable of affecting visibly the coloured solutions. Much greater variations had a perceptible effect, but whether this ever arose from changes in the balance of affinities I am not prepared to say.

I now pass on to consider the testimony borne by other coloured salts, not ferric compounds, in respect to the question at issue.

Gold salts.

The bromide of gold is of an intense scarlet, whilst the chloride is of a yellow colour. Dr. G. Wilson has made use of this difference in examining the question as to whether haloid salts exist as such in solution*; which suggested to me the employment of the same salts for my own purpose. In respect to the relative merits of the two hypotheses about haloid salts, it may be as well to state that I desire to express at present no opinion. If I write *bromide of gold* or *sesquichloride of iron* for the dissolved salts, I do so because that is the ordinary nomenclature.

Pure chloride of gold free from hydrochloric acid was prepared. To a portion of this three equivalents of bromide of potassium were added. The formation of the scarlet terbromide of gold was so complete, that the addition of either of the salts employed caused, singly, too small an increase of colour to be readily appreciated. Is it to be considered, then, that the decomposition in this case has been complete? may it be represented thus—

$$Au\,Cl_3 + 3K\,Br = Au\,Br_3 + 3K\,Cl\,?$$

This was more rigidly tested by adding chloride of potassium in large excess to bromide of gold. This latter salt was prepared by dissolving gold leaf in bromine water, evaporating to dryness, and redissolving in water.

Strong chloride of potassium added.	Water added to comparative solution.	Strong chloride of potassium added.	Water added to comparative solution.
5 measures	=30 measures.	35 measures	=105 measures.
10 measures	=48 measures.	50 measures	=135 measures.
20 measures	=70 measures.	75 measures	=180 measures.

Some difficulty was felt in determining the last numbers of this experiment, from the fact that the chloride of potassium had by its great excess converted nearly the whole of the bromide of gold into the yellow chloride, or still paler double chloride.

* *Vide* Athenæum, 1839, and the Edinburgh Academic Annual for 1840, p. 187.

The gradual diminution of power of the chloride of potassium added is exhibited by the curve in Plate IX. fig. 6.

It was found that bromide of gold was reduced in colour by very small quantities of hydrochloric acid, or even of the common yellow crystals of the chloride of gold, which are, as is well known, the hydrochlorate of that salt. Bromide of gold, as might have been anticipated, was not affected in colour by admixture with *neutral* chloride of gold.

These gold salts then have afforded a good example of the influence of mass in gradually counterbalancing and overcoming a strong affinity.

Platinum salts.

Neutral bichloride of platinum and different amounts of iodide of potassium were mixed in a series of vessels, diluted to an equality of bulk, and allowed to stand some hours for the colour to develope itself properly—a precaution which in this instance was necessary. The following were the appearances noted:—

Bichloride of platinum.	Iodide of potassium.	Character of mixture.
1 equiv.+	0·5 equiv.	Very pale brown solution.
1 equiv.+	1 equiv.	Reddish brown, and opalescent.
1 equiv.+	2 equivs.	The same, but deeper.
1 equiv.+	3 equivs.	The same; some biniodide of platinum deposited on the glass.
1 equiv.+	4 equivs.	Red; opalescence slight; biniodide of platinum deposited.
1 equiv.+	6 equivs.	Bright red; scarcely any opalescence or deposit.
1 equiv.+	8 equivs.	Brighter red; no opalescence or deposit.
1 equiv.+	10 equivs.	Still brighter red.
1 equiv.+	15 equivs.	Still brighter.

The formation of the insoluble iodide of platinum renders some of these cases less distinct in their testimony than the instances previously considered. The opalescence too was doubtless owing to a minute trace of solid matter. This, however, is perfectly clear, that the two salts, though they have mutually decomposed each other, have not done so in the atomic proportions; not according to the schemes

$$Pt\,Cl_2+2KI=Pt\,I_2+2KCl$$

and

$$Pt\,Cl_2+3KI=Pt\,I_2,\,KI+2KCl.$$

It has required, in fact, about four equivalents of iodide of potassium to produce the maximum amount of the platinic iodide; and the latter terms of the series exhibit a still increasing amount of the intensely red double iodide of platinum and potassium. It may be expected that the double chloride is one of the salts produced in such a mixture*. Successive additions of a strong solution of chloride of potassium to a mixture of one equivalent of bichloride of platinum with two of iodide of potassium, were found to reduce the colour greatly, making it browner.

* See Note C.

Copper salts.

Soluble copper salts are, I believe, all of a blue colour when dissolved in a large amount of water; but a strong solution of the chloride, and of one or two others, is green. It has frequently been observed, that on the addition of strong hydrochloric acid to a concentrated solution of sulphate of copper, a green colour takes the place of blue; and it has been naturally concluded that chloride of copper was then formed. This reaction was likewise investigated.

A solution of sulphate of copper was made, containing 125 grm. of the crystals in 1000 gr. measures of water. Hydrochloric acid was taken, having a specific gravity of 1139 at 58° F., and therefore containing 28 per cent. of real acid. Hence, as may be easily calculated, equal bulks of the hydrochloric acid and sulphate of copper solutions represent eight equivalents of the former to one of the latter. A number of equal portions of the blue salt were mixed with respectively one-eighth, one-quarter, one-half, &c. of their volume of the hydrochloric acid in a series of glass tubes of the same size. The following colours resulted:—

Sulphate of copper.		Hydrochloric acid.	Colour of mixture.
1 equiv.	+	1 equiv.	Blue.
1 equiv.	+	2 equivs.	Blue with a tinge of green.
1 equiv.	+	3 equivs.	Dull green.
1 equiv.	+	4 equivs.	Dull green.
1 equiv.	+	6 equivs.	Bright green.
1 equiv.	+	8 equivs.	Bright green.
1 equiv.	+	16 equivs.	Very bright green.

The tint designated as "bright green" is of a very vivid hue, in which yellow seemed to preponderate: it arises from the formation of a hydrochlorate of the chloride of copper.

From this it is evident that single equivalents of sulphate of copper and of hydrochloric acid are not resolved wholly (nor indeed to any great extent) into chloride of copper and sulphuric acid; and that the relative mass of the two substances influences the result.

In order to observe the influence of the mass of water, the following experiments were instituted. Eight portions were taken of a saturated solution of sulphate of copper at 60° F., and were mixed with progressively increasing amounts of the hydrochloric acid solution. The colours produced were noted. They are given in column I. of the subjoined table. Each of the mixtures was then diluted with half its volume of water. The resulting shades are given in column II. Column III. represents the shades when the water was doubled; column IV. when the solutions were of three times their original volume:—

Sulphate of copper solution.	Hydrochloric acid solution.	I.	II.	III.	IV.
50 measures+10	measures.	Perfectly blue.	Perfectly blue.	As Column II., but all paler and more blue.	Pure blue.
50 measures+12·5	measures.	Greenish blue.	Blue.		Pure blue.
50 measures+15	measures.	Distinctly green.	Blue, with trace of green.		Pure blue.
50 measures+20	measures.	Clear green.	Just a shade greener.		Pure blue.
50 measures+30	measures.	Bright green.	Dull bluish green.		Pure blue.
50 measures+40	measures.	Bright green.	Green } Scarcely distinguishable.		Blue, with a trace of green.
50 measures+50	measures.	Brighter green.	Green }		Blue, with a trace of green.
50 measures+70	measures.	Still brighter.	Green }		Blue, with a trace of green.

It is not to be inferred that the sulphate of copper was in larger quantity in column IV. than in column I., for water acts according to its mass upon pure chloride of copper, converting it from a green into a blue compound. For some time I imagined that the changes of colour in the preceding mixtures of sulphate of copper and hydrochloric acid did not take place in regular gradation, but that something occurred analogous to what Bunsen discovered in his experiments; yet, after repeated endeavours to fix the apparent points of transition, I arrived at the conclusion that they might arise merely from the great difficulty of comparing greens of different characters.

To seven portions of the standard solution of sulphate of copper, each measuring 50 parts, were added respectively 10, 20, 30, 40, 50, 70, and 100 parts of a saturated solution of chloride of sodium. There resulted a series of tints passing gradually from blue to almost pure green, without any sudden transition.

A strong solution of chloride of zinc added to a solution of sulphate of copper also produced a greenish colour, which increased as more chloride was added.

Knowing the disposition of oxide of lead and acetic acid to combine, it occurred to me that chloride of lead might decompose the acetate of copper very readily. Accordingly, two equal portions of the blue acetate were mixed with equivalent amounts of chloride of lead and chloride of sodium in solution; and it was indeed found that the former caused a greater diminution of the colour than the latter did. In this experiment much water was necessarily employed, but chloride of copper always gives a far paler blue solution than an equivalent amount of the acetate does.

These reactions with copper salts bear additional testimony, therefore, to the truth of the previous views*.

Molybdous salts.

As the molybdous fluoride gives a purple, and the chloride a green solution, these salts offered another means of testing whether complete or partial decomposition ensued on the mixing of binary compounds. Molybdous oxide was dissolved in hydrofluoric acid, and the resulting purple solution was treated with hydrochloric acid. It changed gradually to a greenish blue; and, on adding more hydrochloric acid, to a positive green. Time entered as an appreciable element into this change.

* See Note D.

The converse of this experiment was also tried. The molybdous oxide dissolved in hydrochloric acid of a green colour. The addition of hydrofluoric acid to this gave at the first moment a rich purple, which was immediately succeeded by a white precipitate, insoluble in any excess of hydrofluoric acid, but readily soluble in hydrochloric acid with the reproduction of the green.

Manganese salts.

Intermediate between the protoxide of manganese and the non-basic oxides, there exists a brownish-red salifiable compound, of the formula $Mn_3 O_4$. It is described in Gmelin's Handbook under the designation "manganoso-manganic oxide." Its solution in hot phosphoric acid or cold oil of vitriol is red, but it dissolves in other acids with a deep brown colour.

I prepared the sulphate and hydrochlorate of this base, and found that the addition of hydrochloric acid in excess caused a change in the colour of the sulphate from red to reddish brown, and eventually brown; while, on the other hand, the addition of sulphuric or phosphoric acid in excess to a solution of the brown chloride converted it into the red salt. Thus it appears that the oxide in question has no such affinity for either one of these acids, but that it is displaced more or less by the other.

Blue Gallate of Iron.

Gallic acid, when added to iron salts, is apt to strike a deep blue colour, from the formation of a very stable compound of the organic acid with both the basic oxides of iron at once.

A portion of this compound was produced by allowing gallic acid to stand for eighteen hours with the hydrated sesquioxide. The effect of successive portions of a strong solution of sulphate of soda was tried.

Sulphate of soda added.	Water added to comparative solution.	Sulphate of soda added.	Water added to comparative solution.
5 measures	=10 measures.	25 measures	=36 measures.
15 measures	=24 measures.	35 measures	=46 measures.

Another blue solution was prepared by mixing solutions of gallic acid and of green vitriol that had been exposed to the air. The effect of sulphuric acid was tried.

Sulphuric acid added.	Water added to comparative solution.	Sulphuric acid added.	Water added to comparative solution.
1 measure	= 9 measures.	10 measures	= 63 measures.
2 measures	=18 measures.	14 measures	=105 measures.
4 measures	=34 measures.	16 measures	=136 measures.
6 measures	=46 measures.	18 measures	=174 measures.
8 measures	=55 measures.	20 measures	=204 measures.

These numbers give the singular curve in Plate IX. fig. 7. The mixture is of far too complicated a character to admit of a full understanding of the cause of the remarkable change in ratio.

These two experiments suffice to prove the influence upon the production of this blue gallate, both of the nature and quantity of other substances present in the solution at the same time.

Quinine salts.

In his elaborate paper "On the Change of Refrangibility of Light*," Professor STOKES has shown that various acid salts of quinine exhibit that remarkable internal dispersion of light which is now known by the name "fluorescence." He mentions the acid sulphate, phosphate, nitrate, acetate, citrate, tartrate, oxalate, and hydrocyanate, as giving rise to the phenomenon; while quinine dissolved in hydrochloric acid did not present any such appearance. He found, moreover, that the addition of hydrochloric acid, or chloride of sodium, to one of the fluorescent salts destroyed the colour. Hence he concluded, and no doubt correctly, that in these cases muriate of quinine was formed; and to obviate the objection that possibly the non-fluorescent salt in solution might be a sort of double salt, in which the quinine was combined with the hydrochloric and the other acid in atomic proportion, he devised the following elegant experiment. To a strong warm solution of neutral sulphate of quinine, which displays no fluorescence, a very small quantity of hydrochloric acid was added; it produced the blue appearance: more hydrochloric acid was added; the blue was destroyed. This seemed intelligible only on the supposition that the small quantity of acid first added displaced an equivalent amount of sulphuric acid, which, combining with the undecomposed sulphate, formed the acid salt which displays fluorescence to such a remarkable degree; and that the larger quantity of hydrochloric acid decomposed this again, setting free the sulphuric acid, and leaving the quinine in solution as hydrochlorate.

From the manner in which Professor STOKES describes and comments on these experiments, it is evident that he imagined (as most others would have done) that the decomposition was perfect, and that, in the particular experiment just mentioned, every particle of the quinine existed in the solution in the form of hydrochlorate, on account of the stronger affinity of that acid for the base. He states, moreover, that "even sulphuric acid is incapable of developing the blue colour in a solution of quinine in hydrochloric acid." If this be true, it evidently militates against the conclusions that double decomposition does not take place perfectly in solution, unless aided by the insolubility or volatility of one or more of the compounds produced, and that great mass counterbalances weak affinity. Accordingly, I repeated the experiments *quantitatively*, and performed some additional ones†.

* Philosophical Transactions, 1852.

† Since writing the above, my attention has been drawn to a paragraph in Professor STOKES's second paper (Philosophical Transactions for 1853, p. 394), in which he remarks that the neutral hydrochlorate of quinine is not absolutely non-fluorescent, as first stated, and that the hydrocyanate is like the hydrochlorate.

Solutions of known strength were prepared of neutral sulphate of quinine and of hydrochloric acid. The quinine salt was employed warm, and it exhibited only a trace of fluorescence. The solutions were mixed in definite proportions, and the amount of visible fluorescence was noted.

Sulphate of quinine.	Hydrochloric acid.	Character of fluorescence.
1 equiv. +	0·5 equiv.	A deep blue entering far into the liquid.
1 equiv. +	1 equiv.	A more intense blue, and confined to the edges.
1 equiv. +	1·5 equiv.	Much as the preceding.
1 equiv. +	2 equivs.	Rather fainter blue.
1 equiv. +	3 equivs.	Decreasing.
1 equiv. +	4 equivs.	Still decreasing.
1 equiv. +	6 equivs.	Fainter.
1 equiv. +	8 equivs.	Still fainter.
1 equiv. +	12 equivs.	Very faint.
1 equiv. +	20 equivs.	Just visible.
1 equiv. +	30 equivs.	As above.
1 equiv. +	50 equivs.	Invisible except under the most favourable circumstances.

It will be seen at once, that the double decomposition between the sulphate of quinine and the hydrochloric acid was not perfect. Had it been so, the first line of the experiment would have been represented by the formula—

$$2(C_{44}H_{28}N_2O_4, HO, SO_3)+HCl=C_{44}H_{28}N_2O_4, HCl+C_{44}H_{28}N_2O_4, 2(HO, SO_3),$$

and the largest possible amount of fluorescence would have been obtained: while the second line would have been according to the formula—

$$C_{44}H_{28}N_2O_4, HO, SO_3+HCl=C_{44}H_{28}N_2O_4, HCl+HO, SO_3,$$

and there would have been no fluorescence visible with this or any higher proportion of hydrochloric acid. But instead of single equivalents of sulphate of quinine and hydrochloric acid giving a non-fluorescent mixture, the blueness was only then attaining its maximum. After the addition of 1·5 equivalent, more hydrochloric acid caused a gradual diminution of the amount of bisulphate of quinine; and although the presence of undecomposed bisulphate was only observed as far as fifty equivalents of acid added, yet it was doubtless only the imperfection of vision that prevented the experiment being carried further.

If this be the true method of interpreting the observed phenomena, and I imagine there can be no reasonable doubt that it is so, it shows that instead of hydrochloric acid having such an overwhelming tendency to combine with quinine, it is scarcely so strong in its affinity for that base as sulphuric acid is. Chloride of sodium was found to have even less power of decreasing the blue colour than an equivalent amount of the acid has.

There is a slight source of error in the experiment just detailed, arising from the constant dilution of the liquid by the addition of the hydrochloric acid solution. This dilution, however, was very trifling, since strong acid was employed with a view to obviate it as far as possible. It will be evident that it cannot affect the general conclusion; or rather, that if this source of error had not existed, the general conclusion would have been somewhat more strikingly brought out.

The influence of mass is very apparent in the experiment just described, but in order to observe it in the opposite direction, a considerable quantity of sulphuric acid was added to liquids where one equivalent of sulphate of quinine had been mixed with sufficient hydrochloric acid or chloride of sodium to render the fluorescence just invisible. The blue instantly became apparent in each case.

In order to examine this matter still more closely, some neutral hydrochlorate of quinine was prepared by dissolving the organic alkali in the acid, and evaporating gently to dryness. Its solution diluted showed a mere trace of blue, which was removed on the addition of a drop or two of free hydrochloric acid. Portions of this solution were mixed with all the acids mentioned by Stokes as giving fluorescent compounds with quinine. The addition of sulphuric, nitric, phosphoric, acetic or oxalic acid, instantly reproduced the blue colour in a very marked manner. Citric or tartaric acid added in very large excess also produced the blue, but it was faint. I failed to detect any change on the addition of a considerable amount of hydrocyanic acid.

Similarly, a solution of sulphate of soda in considerable excess was added to an acid solution of hydrochlorate of quinine. A very perceptible amount of blue made its appearance. This also is in perfect consonance with what might theoretically be expected, and indicates that not only had the commixture of sulphate of soda converted a portion of the hydrochlorate of quinine into sulphate, but the free hydrochloric acid had decomposed some of the sulphate of soda, liberating sulphuric acid, which had combined with the quinine salt to form the bisulphate. That a mixture of neutral sulphate of quinine with sulphate of soda does not give the blue tint, unless some free acid be added, was verified by previous experiment.

Stokes also states that iodide or bromide of potassium added to a solution of bisulphate of quinine, or the acid phosphate, destroys the fluorescence. On examining these reactions, I found that these haloid salts behaved precisely as the chloride did. A solution containing bisulphate of quinine had its blue tint gradually diminished by the addition of either the bromide or the iodide of potassium; and, where the fluorescence had been thus rendered barely if at all perceptible, it was restored on the addition of dilute sulphuric acid.

This accumulation of evidence all goes to prove that quinine follows the same laws as the substances previously considered.

Other fluorescent Organic Substances.

The beautiful fluorescence exhibited by a solution of the inner bark of the horse-chestnut is due to the presence of the vegeto-alkali æsculine in a free state. When combined with hydrochloric acid it loses its peculiar optical properties, which are restored again on the addition of ammonia*. It occurred to me, that though ammonia is capable of displacing æsculine from its combinations, æsculine ought to be able to decompose more or less the ammoniacal salt. Accordingly a large quantity of hydrochlorate of ammonia was added to a solution of horse-chestnut bark, and it certainly did reduce the blue more than a similar amount of water did in a comparative experiment, while the addition of ammonia to the mixture revived it.

Tincture of stramonium owes its fluorescent properties also to a free alkaloid, and the blue is similarly destroyed by hydrochloric acid, and restored by ammonia. The action of hydrochlorate of ammonia was tried in like manner with this tincture, and it seemed to give the same results as in the case of horse-chestnut bark; but they were more decided.

Compound Ethers.

Compound ethers may be regarded as organic salts in which certain compound radicles take the place of metals, and it is to be expected that they will follow the same general laws as the binary compounds that have been previously examined. Their insolubility in water precludes the use of aqueous solutions of the substances intended to act upon them, but alcohol affords a convenient medium for the reaction, and when this is employed, results are obtained which are perfectly analogous to those already described in the case of different metallic salts. The following experiment may be taken as an example. A large excess of oxalic acid was dissolved in alcohol along with acetic ether, warmed, and allowed to stand for some hours. The mixture was then submitted to gentle distillation, and there passed over acetic ether mixed with alcohol and with oxalic ether, as was proved by decomposing the distillate by hydrate of potash, and testing the resulting salt in the usual manner, when both acids were detected in considerable quantity. The oxalic acid had therefore displaced a certain amount of acetic acid; but though it existed in such large excess that it began to crystallize out, it had not displaced the whole.

I have not examined these reactions among compound ethers to any considerable extent, nor have I performed any of the experiments quantitatively, because I was aware that Professor Williamson (to whom this branch of the subject more particularly belongs) has been pursuing some investigations in the same direction.

GENERAL INFERENCES FROM THE PRECEDING EXPERIMENTS.

The concurrent testimony of the diversified experiments here detailed is in favour of the view, that when two binary compounds are mixed in solution, there ensues a partition of the two electro-positive between the two electro-negative elements, according to certain proportions regulated both by the difference of strength in the affinities, and by the relative quantities of the different bodies. The reverse of this does indeed appear at first sight to be the case in certain instances, as for instance,

* See the paper of Stokes already referred to.

when equivalent portions of sulphocyanide of potassium and ferric citrate are mixed, or of chloride of gold and bromide of potassium. But it makes all the difference whether there be a small, though inappreciable, quantity of the other salt formed at the same time, or whether the decomposition be absolute; and a consideration of the whole series of experiments, and of the influence of mass in these very instances, will leave, I think, a strong conviction on the mind that such cases differ from the others only in degree, and that if we possessed the means of observing minuter differences of colour we should find evidence of traces of the original salts still remaining. But of this each reader will form his independent judgment. Among those instances where evidently four salts were produced by the mixture of two, the following substances took part in the reaction:—

Iron (in both basic conditions), gold, platinum, mercury, copper, zinc, lead, molybdenum, manganese, baryta, lime, potash, soda, ammonia, hydrogen, ethyl, quinine, æsculine (?), base in stramonium (?).

Sulphuric, nitric, phosphoric, hydrochloric, hydrobromic, hydriodic, hydrofluoric, hydrosulphocyanic, hydroferrocyanic, acetic, oxalic, citric, tartaric, gallic, meconic, pyromeconic, and comenamic acids.

It must be borne in mind, that when, in studying the mutual action of AB and CD, we have determined the laws according to which A and D combine, we have equally ascertained them in reference to C and B; that is to say, to take a particular instance, if we find, on mixing ferric nitrate and sulphocyanide of potassium, that ferric sulphocyanide is formed in certain proportions according to the relative force of affinity and mass, we have determined this also in respect to the nitric acid and the potash. We know indeed that for every portion of ferric sulphocyanide produced, an exactly equivalent portion of nitrate of potash must be formed, so that in fact the long curve in Plate VII. fig. 1 will express the amount of nitrate of potash in the experiment equally well with that which it primarily represents. And not only this, but in any such mixture, where we know the original amounts of the two salts, and the amount of any one of the four into which they are resolved, we have the data for determining the amounts of the other three likewise. This may be illustrated from the reaction just alluded to. Suppose (which is about the truth) that one equivalent of ferric nitrate mixed with three equivalents of sulphocyanide of potassium produce one-fifth of an equivalent of ferric sulphocyanide, the following is the only formula which can represent the reaction. The amounts are multiplied by five to avoid decimals:

$$5(Fe_2O_3, 3NO_5)+15KS_2Cy=Fe_2, 3S_2Cy+12KS_2Cy+4(Fe_2O_3, 3NO_5)+3(KO, NO_5),$$

or more simply,

$$5AB+5CD=AD+4AB+4CD+CB.$$

Of course this method of reckoning is inapplicable where polybasic acids are concerned.

TESTIMONY FROM OTHER CHEMICAL PHENOMENA.

There are many chemical phenomena, beside those connected with colour, which bear testimony respecting the question whether two salts in solution resolve themselves into four.

The testimony of precipitation.—The idea that when double decomposition occurs, the acids and bases make a perfect exchange, arose doubtless from what is constantly observed when a precipitate ensues. In that case A combines wholly with D, and C with B. Yet this will be the inevitable result under the one theory as well as under the other. A mixture of single equivalents of nitrate of baryta and sulphate of potash may be taken as an illustration. Here, as has frequently been shown, if Berthollet's views be correct, at the first moment of mixing a portion of the baryta combines with sulphuric acid, but that compound being insoluble is instantly put out of the field of action, and the resulting mixture really consists of nitrate of baryta, nitrate of potash, and sulphate of potash, which of course gives rise to a redistribution of the bases and acids, and a further production of insoluble sulphate of baryta, and so on, till the amount of nitrate of baryta remaining is infinitesimally small; while at the same time the whole of the potash must necessarily combine with the whole of the nitric acid. It is scarcely necessary to observe, that this division and precipitation will take place continuously until complete; and that it may be so rapid as to elude our notice*. The fact then that precipitation when it occurs is complete, decides nothing as to the relative merits of the two theories of elective affinity. Yet there is an important difference to be noted. On Bergman's supposition, it can hardly be imagined but that cases will sometimes occur, where A has so strong an affinity for B, or C so powerful an attraction for D, that on mixing AB and CD, no interchange will take place, although AD may be an insoluble body. On Berthollet's supposition, the insoluble compound will always be wholly precipitated whenever by the interchange of acids and bases such a compound can be formed, even though it be against the preponderating direction of the affinities. Now this can be put at once to the test of experience: and what is the testimony of the thousands of double decompositions which chemists are in the habit of meeting with? Graham† says, "It is a general law to which there is no exception, that two soluble salts cannot be mixed without the occurrence of decomposition, if one of the products that may be formed is an insoluble salt." Gmelin‡, even when arguing against Berthollet's views, admits the same fact, adding, "the only case which

* Yet it is easily conceivable that when the affinity for each other of the two substances that produce the insoluble compound is very weak, the action may last some time, and become evident to our senses. Is not this actually the case when sulphate of lime in solution is added to nitrate of strontia, or carbonate of soda to chloride of calcium, or an alkaline carbonate to tartrate of yttria, or oxalate of ammonia to sulphate of magnesia, &c.?

† Elements of Chemistry.

‡ Handbook of Chemistry.

appears to present an exception is that observed by TH. SCHERER*, and this requires further examination." The case referred to is this:—if yttria and sesquioxide of iron be dissolved together in hydrochloric acid, nearly neutralized with ammonia, then treated with acetate of ammonia, and afterwards with a few drops of oxalate of ammonia, no precipitate falls, but the solution is to a considerable extent decolorized; but the addition of more oxalate of ammonia determines the formation of a white curdy precipitate. "It follows from this," says SCHERER, "that hydrochlorate and acetate of yttria can be present in a solution simultaneously with ferric oxalate, without a precipitate resulting; which indeed is very remarkable, and appears to contradict the fundamental laws of chemistry, since oxalic acid gives an insoluble precipitate with yttria." I examined this reaction. The white precipitate which was first formed on dropping oxalate of ammonia into the aforesaid mixture, was certainly dissipated on further mixing; yet on repeated trials I always found that after standing some hours a small quantity of oxalate of yttria was deposited, while the solution still retained the red colour of acetate of iron. However, there is no doubt that ferric oxalate, and the yttria salt, did coexist in the same solution; yet in order to prove that no oxalate of yttria was present along with them, it would be necessary to show that the salt in question is not dissolved by any of the other substances present in the mixture. Now it is by no means insoluble in either hydrochloric or acetic acid, and the conditions of the experiment require the acid to be in slight excess. Thus the experiment merely shows that oxalic acid left free to act on the ferric and the yttria salts, will combine with the former oxide in very much the larger proportion; a fact which is in perfect consonance with the strong affinity between that acid and the sesquioxide of iron, which has been previously remarked in this paper.

It is then a law, without a single known exception, that if AB, CD, EF, &c., by any interchange of bases and acids can possibly produce an insoluble substance, that insoluble compound does actually make its appearance. This seems to me almost conclusive evidence that the interchange always takes place originally to a greater or less degree; for I cannot believe, with one chemist of high repute, that "when bodies are brought into intimate contact, all the forces which exist, not only in themselves, but in all their possible compounds, are called into action at the same time," unless indeed it be by these compounds being actually formed.

The following experiments may illustrate more fully the truth of the explanation of complete precipitation which has been given above.

I. Strong solutions of sulphocyanide of potassium and ferric sulphate were mixed. The resulting intensely red liquid was divided into two equal parts. The one portion was largely diluted with water; and to the other portion a little strong alcohol was added, which caused the precipitation of sulphate of potash while ferric sulphocyanide was dissolved. The alcohol was poured off, and diluted with water

* POGGENDORFF, li. 470.

till of the same volume as the first portion. It was far deeper in colour, indicating evidently that the insolubility of sulphate of potash in alcohol had removed it out of the sphere of action, and had caused a much larger proportion of ferric sulphocyanide to be formed than would otherwise have been produced. Only a small quantity of alcohol was employed, and it was mixed with a large amount of water in order to obviate as much as possible the objection that the same amount of ferric sulphocyanide might appear darker in alcoholic than in aqueous solution, which is indeed the fact.

II. Another red solution was prepared by mixing sulphocyanide of potassium and ferric sulphate, and it was divided into two equal portions. To one of these hydrochloric acid was added, which of course reduced the colour somewhat. To each was then added an equal portion of neutral phosphate of soda. The acid solution remained red, though paler than before; the neutral solution became colourless, and turbid from the formation of a flocculent precipitate of ferric phosphate. That the insolubility of this salt in the neutral solution was the cause of the complete combination of the oxide of iron with the phosphoric acid, was further elucidated by adding phosphoric acid to the colourless mixture, which restored a faint red tint to the solution, doubtless because it had set free some of the sulphuric acid, which, redissolving the ferric phosphate, allowed of the formation of a small amount of the red sulphocyanide.

III. A mixture of three parts of ferric citrate, and four of ferrocyanide of potassium was prepared, and divided into two equal parts. To the one there was added a few drops of hydrochloric acid, to the other a few drops of oxalic acid. In the one case, the ferric ferrocyanide, being insoluble in hydrochloric acid, was precipitated, leaving no trace of iron in the solution; in the other case there was a blue solution, but the whole of the iron was not in the condition of ferric ferrocyanide, for the addition of more prussiate of potash caused it to become bluer. That this was due, not to the affinity of the oxalic acid for the ferric oxide, but to that of the citric acid, will be evident from the fact ascertained by the previous experiments on the ferrocyanide, that this result would not have been obtained had the nitrate been employed.

IV. It is well known that if hydrosulphuric acid gas be passed through a solution of arsenious acid in water containing a mineral acid a perfect separation of the yellow sulphuret takes place, but if there be no free acid the *solution* remains yellow. The question suggested itself—Is the whole of the arsenious and hydrosulphuric acids converted in the latter case into sulphide of arsenic and water? In order to ascertain this, a stream of sulphuretted hydrogen was passed through a solution of arsenious acid in water till it was saturated, then it was allowed to stand awhile for the excess of sulphuretted hydrogen to be dissipated, and hydrochloric acid was added. The yellow sulphide was of course precipitated, but on repeated trials I always found more or less arsenious acid still remaining in the solution.

The testimony of volatilization.—The argument that has been employed in the case of precipitation will apply *mutatis mutandis* with equal force in the case of volatilization. I am not acquainted with any exceptional instance.

The testimony of crystallization.—It will sometimes happen that certain quantities of AB and CD are mixed in an amount of water which is insufficient to keep in perfect solution AD, should the whole of A combine with the whole of D, although the salt itself is a soluble one*. In such a case, if BERGMAN's view be correct, either no AD will form, however concentrated the solution, or, should double decomposition ensue, it will form to the fullest extent possible, and may be expected to crystallize out at once with something like the rapidity with which precipitation usually takes place. If, however, BERTHOLLET's theory be a true expression of the fact, a certain amount of AD will always be formed, but it may remain dissolved in the liquid, although if the whole of A had entered into combination with D it must have separated: yet on concentration AD will make its appearance; and should this, or anything else, determine the formation of crystals, or should they ensue on the primary mixing, the crystallizable salt is *pro tanto* put out of the field of action, and a redistribution of the acids and bases will take place with further crystallization, until an equilibrium is obtained. Now the latter of these deductions describes what actually does take place, but there are several circumstances attending crystallization from a mixture of salts which are not readily explained, and which I have as yet but imperfectly investigated.

The testimony of diffusion.—Professor GRAHAM has shown† that binary compounds vary greatly in the rates at which they diffuse through water. Let it be supposed that AB and CD are mixed in one of his diffusion-cells, and that the compounds of A diffuse more rapidly than those of C. If no decomposition take place upon mixing, the amounts of A and B in the diffusate will be in equivalent proportions; and so will likewise the smaller amounts of C and D. If a complete interchange of acids and bases take place, the amount of A in the diffusate will exactly correspond with that of D, and in a similar manner C with B. If, however, A and C divide themselves between B and D, as the four compounds will be unequally diffusive, it will be very improbable that the amount of either A or C in the diffusate should happen just to correspond with the amount of either B or D. There is nothing in Professor GRAHAM's published researches that will indicate which of these is the case, nor have I made any experiments on the subject; but I entertain little doubt that the latter result would be arrived at were the matter to be investigated.

Yet diffusion will never serve as a means of determining numerically the strength of the affinities in a mixture; for, supposing the four compounds are actually produced, the more diffusive one will speedily pass away from the field of action, which

* Few if any salts are absolutely insoluble in water, but this will not affect the reasoning in a previous section, for the action of insolubility in a case of crystallization produces the same consequences as in a case of amorphous precipitation.

† Philosophical Transactions, Part I. 1850.

will necessitate a fresh distribution, and so on. Thus a state of things will ensue analogous to what is observed where one of the salts is so sparingly soluble as to separate by crystallization; great diffusibility will compensate for weak affinity; and the mutual attraction of the two components of the more diffusible salt will always be exaggerated.

The testimony of MALAGUTI'S *experiments.*—MALAGUTI* examined the present question by taking two salts, both of which were soluble in water, but only one of which was soluble in alcohol, mixing them in equivalent proportions in water, and then pouring the aqueous solution into a large quantity of alcohol. Some of the resulting salts were precipitated, others remained in solution; and the proportion in which one acid divided itself between the two bases was thence ascertained. This afforded him the data for determining what he denominates "coefficients of decomposition," a large number of which are tabulated; yet he attached no importance to the absolute value of these coefficients on account of the objection that, if there be really four salts in the aqueous solution, their proportions may change when they are thrown into alcohol. However, on considering these experiments, three important results may be arrived at:—1st, that two salts on being mixed resolve themselves into four; 2nd, that this partition takes place in a definite manner; 3rd, that the proportions of the resulting salts are independent of the manner in which the different elements were originally combined†.

The testimony of substances acted on by one of the compounds liberated in a mixture of salts.—It is to be expected that if two binary compounds be mixed, the formation of a new compound, though it remain in solution, may often be ascertained by certain chemical powers which it is capable of exerting. Instances of this are not wanting. Thus gold, as every one knows, is not attacked by hydrochloric or nitric acid singly, but is dissolved by a combination of the two; neutral potash salts of course have no action upon it; and yet gold dissolves readily in a mixture of either nitrate of potash and hydrochloric acid, or of chloride of potassium and nitric acid; whence it appears to me the conclusion may be fairly drawn, that in both mixtures the potash relinquishes a portion of the acid with which it was originally combined, or (which is the same thing) that it divides itself between the two.

Such experiments as this have no quantitative value, since the liberated substance immediately enters into a new combination, which must give rise to a fresh distribution of the different elements, and so on until no more of the active substance can be produced. A mere solvent action of the liberated body would be preferable to an action where positive chemical combination or decomposition takes place; but such cases scarcely exist. Among the actions which appear to answer this requirement most fully, is when a salt insoluble in water is dissolved in an acid, as for instance,

* "Exposition de quelques faits relatifs à l'action réciproque des sels solubles," Ann. de Chim. et de Phys. 3. t xxxvii. p. 198.

† See Note E.

ferric phosphate in hydrochloric acid; yet even here a partial decomposition in all probability ensues. The only instructive numerical results which I have obtained were by mixing a saturated solution of oxalate of lime in hydrochloric acid with various proportions of acetate of potash or soda. The hydrochloric acid combining with the alkali caused a deposition of oxalate of lime, since that salt is not soluble in the acetic acid that was liberated at the same time. The following were the results, every separate term of which was of necessity a separate experiment, though conducted at the same time and under similar circumstances.

Acetate of soda series.		Acetate of potash series.	
Salt added.	Oxalate of lime deposited.	Salt added.	Oxalate of lime deposited.
3 measures.	0·075 grm.	20 measures.	0·359 grm.
6 measures.	0·10 grm.	40 measures.	0·345 grm.
9 measures.	0·11 grm.	80 measures.	0·410 grm.
12 measures.	0·13·5 grm.	120 measures.	0·445 grm.
18 measures.	0·14·5 grm.	160 measures.	0·503 grm.
30 measures.	0·11 grm.	240 measures.	0·564 grm.
45 measures.	0·14·5 grm.		
90 measures.	0·19 grm.		

Notwithstanding certain irregularities in these series of numbers, it is sufficiently evident in both instances that the amount of oxalate deposited increased with the amount of acetate added, though not in direct ratio. In the series of experiments with the potash salt each portion of the original hydrochloric acid solution contained 0·730 grm. of oxalate of lime, therefore even in the last term of the series, where 240 measures of acetate were added, there still remained 0·166 grm. in solution. Plate IX. fig. 8. exhibits these results.

Supposed exceptions and limitations.—With this mass of evidence, and that of a very diversified character, the question arises,—Are we justified in concluding that the principles, which are so general, are *universal* in their application? Are there no exceptions? Is there no limitation?

As to exceptions, in the whole range of my experiments upon this subject, I have never met with a single instance of two substances having so strong an affinity for one another, that they combined to the exclusion of other bodies of like kind and present in the same solution, even if in large excess*. Sometimes this rests not on demonstrative but upon moral evidence, as for instance when sulphocyanide of potassium and dissolved ferric ferrocyanide are mixed, where unquestionably the amount of ferric sulphocyanide produced must be quite inappreciable, yet that some is produced may be safely inferred I think from the fact, that sulphocyanide of potassium does give a red with the ferric acetate, and acetate of potash is capable of decomposing the ferric ferrocyanide to a well-marked extent.

* Oxybromide of iron certainly appeared to resist even citric acid; but then it is not a binary compound. Anomalous results too were sometimes obtained on examining the solubility of such substances as phosphate of iron in mixtures of salts and acids, but the phenomena were always of an obscure character.

During the controversy that ensued after the publication of BERTHOLLET's treatise, many reactions were brought forward to prove the falsity of his views. Most of these were directed against certain positions of the French philosopher which were certainly untenable, while others were founded on a misapprehension of the question at issue. Those which appear the most formidable against the conclusions arrived at in this paper are, that boracic acid, or carbonic acid, or hydrosulphuric acid, are incapable of decomposing in the least degree sulphate of potash, or any analogous salt; and that chloride of sodium is not affected at all by iodine. The proof of these statements rests in each instance upon the testimony of blue litmus paper. In the first case the vegetable colour is not reddened; which is supposed to prove that no sulphuric acid has been liberated; yet if any had been set free there must have been formed at the same instant an equivalent amount of borate, or carbonate, or hydrosulphate of potash, each of which has an alkaline reaction, and would have restored the blue, or rather prevented the litmus from reddening. So in the case of the common salt and iodine (where by the way only one base is concerned), the chlorine, supposing it liberated, would not have bleached the litmus, but would have combined at the moment of its separation with some of the iodine present to form the terchloride of iodine which has a neutral reaction. That very little decomposition does take place in these instances I have no doubt, but that there is actually none is not proved.

There is however one difficulty not so easily overcome. Water is a binary compound, and it might be expected that on mixing a hydrated acid and base, or on dissolving in water a salt, such as nitrate of potash, or nitrate of ammonia, a certain amount of both the acid and the base would remain in combination merely with water. That it is not so, is proved by the fact that solutions of the salts just named do not give up any portion of their volatile base or acid, even on boiling. Has a limit to the action of the general law been here arrived at? Is water an exception standing by itself? Or is there not an assumption in supposing that water is not an integral part of the constitution of every salt when in a state of aqueous solution? To these queries I do not feel myself in a position as yet to make a reply.

Conclusions.

The general conclusions arrived at in this paper may be summed up as follows:—

I. Where two or more binary compounds are mixed under such circumstances that all the resulting bodies are free to act and react, each electro-positive element arranges itself in combination with each electro-negative element in certain constant proportions.

II. These proportions are independent of the manner in which the different elements were originally combined.

III. These proportions are not merely the resultant of the various strengths of affinity of the several substances for one another, but are dependent also on the mass of each of the substances in the mixture.

IV. An alteration in the mass of any one of the binary compounds present alters the amount of every one of the other binary compounds, and that in a regularly progressive ratio; sudden transitions only occurring where a substance is present which is capable of combining with another in more than one proportion.

V. This equilibrium of affinities arranges itself in most cases in an inappreciably short space of time, but in certain instances the elements do not attain their final state of combination for hours, or even days.

VI. The phenomena that present themselves where precipitation, volatilization, crystallization, and perhaps other actions occur, are of an opposite character, simply because one of the substances is thus removed from the field of action, and the equilibrium that was first established is thus destroyed.

VII. There is consequently a fundamental error in all attempts to determine the relative strength of affinity by precipitation; in all methods of quantitative analysis founded on the colour of a solution in which colourless salts are also present; and in all conclusions as to what compounds exist in a solution drawn from such empirical rules as that "the strongest base combines with the strongest acid."

NOTES ADDED AFTER THE PAPER HAD BEEN READ.

Note A.

That two or more solutions of the same salt in the same solvent and of equal depth of colour are of the same strength, requires no proof. Hence I apprehend no objection can be raised against the conclusion, that the gross amounts of salt dissolved in the different solutions are directly proportional to their volume. But it may be objected, that though this is true of the solutions when diluted to an equality of colour, it is not necessarily true of the solutions before they were diluted, for the solvent may exercise some chemical action on the coloured salt, absolutely increasing or diminishing its quantity. Should such be the case, it appears to me actually the most correct plan of proceeding to reckon the result when the solutions of the coloured salt are of equal strength, that is to say, when the solvent is in each case in the same proportion to the dissolved salt; for the disturbing influence of the solvent is thus practically got rid of, by its reduction to an equality in all the solutions compared.

There is however a more serious objection, namely, that the solvent may act differently on the coloured salt, according to the nature or the quantity of the colourless salts present at the same time in the several solutions, or that these salts may act differently according to the amount of solvent with which they are conjoined. That this may be the case to a slight extent is very possible, but the experiment recorded on page 202, shows that it was too inconsiderable to be appreciated in the cases there submitted to examination. As this is an important matter, I have repeated the experiment in a great variety of ways, and have satisfied my mind that the amount of error arising from this cause must be quite insignificant—at any rate as far as the ferric sulphocyanide is concerned. The slight differences that do occur are rather in the character than in the intensity of the colour.

Note B.

Acting upon a suggestion of Professor Stokes, I examined more fully whether this apparent diminution of colour on dilution might possibly be due to some peculiarity in the manner in which this salt in different states of dilution absorbs certain rays and transmits others, and whether it exhibited the same depth of colour when the amount of dilution was compensated for by the thickness of the stratum through which the light passed. In two flat-bottomed beaker glasses were placed two portions of the same solution of ferric sulphocyanide, and beneath them at the distance of a few inches was laid a sheet of white paper. The solutions appeared of the same depth of colour, as I looked down through them. On adding water to one of these and still looking down through the whole liquid, it appeared far lighter in colour, and on the addition of much water it assumed the yellowish tint. Meconate of iron, on the contrary, when looked through in the same manner, presented the same depth of colour, whatever amount of water was added to it.

The ferric sulphocyanide appears then to be really acted on by water, in a manner somewhat analogous perhaps to that in which chloride of copper and several similar salts are.

Note C.

An experiment was subsequently made by adding different proportions of iodide of potassium to a solution of the double chloride of platinum and potassium, in hopes of obtaining a result undisturbed by the separation of any solid matter. Such however was not the case. On mixing single equivalents a precipitate instantly resulted; and a deposit was gradually formed when the iodide of potassium was added in proportions of 2, 3, 4, or 5 equivalents to one of the double chloride. These solutions, as also those containing a larger amount of iodide, were intensely red. It was evident that the decomposition was far more complicated than

$$PtCl_2, KCl+2KI=PtI_2+3KCl \text{ and } PtCl_2, KCl+3KI=PtI_2, KI+3KCl.$$

Note D.

Changes in the state of combination of an element may be rendered visible by a change in the intensity of a colour, even where no change in its character occurs. Thus oxide of copper dissolved in acetic acid gives a much more intense blue than when the same amount is dissolved in sulphuric acid. This fact was taken advantage of in the following experiment, which affords additional evidence of the truth of my main deduction. Sulphuric acid was added to a solution of acetate of copper; it reduced the colour greatly. The experiment was reversed, acetic acid being added to a solution of sulphate of copper; it deepened the colour, but considerable excess of the acid was required to make a very evident difference.

Note E.

After my paper had been sent to the Royal Society, I observed the notice of M. Margueritte's "Recherches sur les affinités chimiques" in the Comptes Rendus, xxxviii. 304. He too has examined Berthollet's views, but by means of some ingenious experiments totally different from those which suggested themselves to me. His results are in perfect harmony with my conclusions, and are just such as might have been deduced from the proposition of Berthollet which is quoted near the commencement of my paper. He finds, for instance, that on dissolving chloride of sodium in a saturated solution of chlorate of potash, it will take up more chlorate of potash; which he naturally considers to be due to the formation of the more soluble chlorate of soda together with chloride of potassium. Again, chloride of ammonium is precipitated from its saturated solution by a very small quantity of nitrate of ammonia, but this does not take place when chlorate of potash has been mixed with it. He has obtained many analogous results. The conclusion he arrives at is, "Lorsque par le mélange de deux sels qui ont satisfaits à la loi d'insolubilité, il peut se former un sel plus soluble que le moins soluble d'entre eux, l'action de l'eau en détermine toujours la formation dans certaines limites." It will be seen that this conclusion is a particular case comprehended in my more general one. With some of his deductions, however, I cannot agree. He appears to have misunderstood Berthollet, perhaps because that chemist himself is not always consistent. He speaks also of an actual "affinity of the solvent," and of the "force of solubility," and the "force of insolubility," as though they were two efficient physical forces.

On repeating some of M. Margueritte's experiments quantitatively, I have obtained interesting results; but they are not described now, as I have not yet seen his more extended memoir.

IX. *On the existence of an element of Strength in Beams subjected to Transverse Strain, arising from the Lateral Action of the fibres or particles on each other, and named by the author the 'Resistance of Flexure.'* *By* William Henry Barlow, *Esq., F.R.S.*

Received February 23,—Read March 29, 1855.

It has been long known, that under the existing theory of beams, which recognizes only two elements of strength, namely, the resistances to direct compression and extension, the strength of a bar of cast iron subjected to transverse strain cannot be reconciled with the results obtained from experiments on direct tension, if the neutral axis is in the centre of the bar.

The experiments made both on the transverse and on the direct tensile strength of this material have been so numerous and so carefully conducted, as to admit of no doubt of their accuracy; and it results from them, either that the neutral axis must be at, or above, the top of the beam, or there must be some other cause for the strength exhibited by the beam when subjected to transverse strain.

In entering upon this question, it became necessary to establish clearly the position of the neutral axis, and the following experiments were commenced with that object; but they have led to others, which are also described herein, and which establish the existence of a third, and a very important element of strength in beams.

I was desirous that the experiments for determining the position of the neutral axis should be made on such a scale and in such a manner as to place this question beyond doubt; and with this object the following means were adopted:—

Two beams were cast, 7 feet long, 6 inches deep, and 2 inches in thickness; on each of which were cast small vertical ribs at intervals of 12 inches: these ribs were one-fourth of an inch wide, and projected one-fourth of an inch from the beam. In each rib nine small holes were drilled to the depth of the surface of the beam, for the purpose of inserting pins attached to a delicate measuring instrument; the intention being to ascertain the position of the neutral axis by measuring the distance of the holes in the vertical ribs when the beam was placed under different strains. The measuring instrument consisted of a bar of box-wood, in which was firmly inserted, at one end, a piece of brass, carrying a steel pin; and at the other end a similar piece of brass carrying the socket of an adjusting screw. The adjusting screw moved a brass slide, in the manner shown in Plate XII. which carried another pin similar to that inserted in the box-wood bar, at the other end of the instrument. The instrument was first made entirely of brass; but the effects of expansion from the heat of the hand were so sensible, that the wooden bar was substituted. The pins on the instrument fitted loosely into holes in the beam; and the mode of using the instrument was, to bring the pins up by means of the screw against the side of the holes with a certain degree of pressure, which, with a little practice in using the instrument, was attained with considerable accuracy.

Two beams were employed in order to avoid errors which might arise from accidental irregularities in the metal. The head of the adjusting screw was graduated to 100 divisions, and the screw had 43·9 threads to the inch, so that one division was equal to $\frac{1}{4390}$th of an inch.

The measurements were, in all cases, taken by the outsides of the pins of the measuring instrument; and when the instrument read zero, the actual distance of the outer sides of the two pins was $\frac{51661}{4390}$ inches, so that the constant number 51661 being added to the micrometer readings gives, in each case, the total distance in terms of $\frac{1}{4390}$th of an inch. The form and dimensions of these beams are given in Plate XIII.

The measurements were taken four times in each position of the beam, and the error of measurement did not generally exceed from one to two divisions; but if in the four observations an error amounting to more than four was found, it was corrected by remeasurement.

The numbers given in the following Tables are the micrometer readings, and the *means* of four observations in each case. In these experiments more than 3000 measurements were taken; but to avoid unnecessary figures, only the more prominent results are given.

Table No. I. contains the measurements of the centre division of the first beam under eight different conditions.

Table No. II. contains similar measurements of the second beam.

In the first experiment it was found that, when the beam was inverted, the measuring instrument appeared to bear upon a different part of the holes, so that a direct comparison between the distances, in the beam erect and inverted, cannot be made with the same accuracy as the comparisons of different strains upon the beam when in the same position. The first beam had been subjected to strain for the purpose of testing the measuring instrument previous to these experiments being made; but the second beam had not; and it will be seen that the effect of the strains in the latter case caused a permanent lengthening of the beam. The same strain was frequently applied afterwards, but I could not observe any increase of this effect. There was certainly a further apparent lengthening of both beams; but I ascertained that this arose from a slight wearing of the working parts of the measuring instrument, from the great number of measurements taken. In both experiments the beam was measured, first, in an erect position; and secondly, inverted; but in the Tables, the measurements of the same parts of the beam are placed opposite each other, so that they may be compared throughout with greater facility.

Determination of the Neutral Axis.

Measurements of the First Beam.

Beam erect.					Beam inverted.									
No. 1. At rest previous to being strained.	Difference.	No. 2. Strained with 7373 lbs. on the end, equal to 14,746 lbs. on the centre.	Difference.	No. 3. Weight taken off, condition the same as No. 1.	No. 4. Beam reversed, bearing its own weight on the centre.	Difference.	No. 5. Strain of 2893 lbs. on the end, equal to 5786 lbs. on the centre.	Difference.	No. 6. Strain of 5133 lbs. on the end, equal to 10,066 lbs. on the centre.	Difference.	No. 7. Strain of 7373 lbs. on the end, equal to 14,706 lbs. on the centre.	Difference.	No. 8. Weight taken off, condition the same as No. 4.
Micrometer readings.		Micrometer readings.		Micrometer readings.	Micrometer readings.		Micrometer readings.		Micrometer readings.		Micrometer readings.		Micrometer readings.
2208	+70	2278	−67	2211	2210	−33	2177	−25	2152	−26	2126	+71	2197
2186	+55	2241	−53	2188	2187	−25	2162	−16	2146	−21	2125	+53	2178
2095	+36	2131	−33	2098	2103	−20	2083	−12	2071	−13	2058	+36	2094
2127	+14	2141	−13	2128	2129	−10	2119	− 8	2111	− 5	2106	+21	2127
2110	− 5	2105	+ 5	2110	2117	− 2	2115	− 1	2114	+ 2	2116	——	2116
2052	−21	2031	+13	2054	2060	+ 5	2065	+ 6	2071	+12	2083	−20	2063
2095	−39	2056	+42	2098	2101	+15	2116	+14	2130	+19	2149	−37	2112
2052	−58	1994	+68	2052	2056	+21	2077	+21	2098	+29	2127	−60	2067
2101	−73	2028	+76	2104	2111	+28	2139	+26	2165	+34	2199	−73	2126

Note.—The extensions are marked + ; the compressions are marked —.

Determination of the Neutral Axis.

Measurements of the Second Beam.

Beam erect.								Beam inverted.				
No. 1.	Difference.	No. 2.	Difference.	No. 3.	Difference.	No. 4.	Difference.	No. 5.	Difference.	No. 6.	Difference.	No. 7.
At rest previous to being strained.		Strain of 8000 lbs. on centre.		Strain of 16,000 lbs. on centre.		Weight removed.		Strain of 8000 lbs. on centre.		Strain of 16,000 lbs. on centre.		Weight removed.
Micrometer readings.		Micrometer readings.		Micrometer readings.		Micrometer readings.		Micrometer readings.		Micrometer readings.		Micrometer readings.
1633	+37	1670	+65	1735	−89	1646	−44	1602	−56	1546	+97	1633
1525	+28	1553	+47	1600	−63	1537	−24	1513	−46	1467	+67	1534
1481	+21	1502	+34	1536	−44	1492	−19	1473	−28	1445	+42	1487
1442	+11	1453	+21	1474	−23	1451	−10	1441	−12	1429	+22	1451
1392	+ 2	1394	+ 7	1401	− 1	1400	+ 1	1401	——	1401	+ 4	1405
1375	−10	1365	− 9	1356	+18	1374	+17	1391	+11	1402	−17	1385
1338	−18	1320	−24	1296	+44	1340	+20	1360	+27	1387	−35	1352
1257	−27	1230	−37	1193	+64	1257	+31	1288	+43	1331	−57	1274
1248	−42	1206	−46	1160	+85	1245	+44	1289	+57	1346	−78	1268

Note.—The extensions are marked +; the compressions are marked −.

Considering the very minute quantities which had to be measured, and the numerous causes of disturbance to which observations of so much delicacy were liable, such as changes of temperature or want of perfect uniformity in the dimensions or texture of the beams, the results, as shown by the column of differences, exhibit more regularity than could have been expected; and they point out the position of the neutral axis, as the centre of the beam, in a manner so decided, as to remove all further doubt upon this subject, not only in the smaller strains, but in the larger ones also; which, in the case of the second beam, were carried to about three-fourths of the breaking weight.

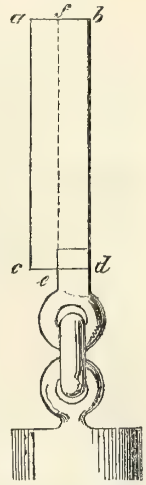
It will be observed also that the extensions and compressions increase in an arithmetical ratio from the centre to the extreme upper and lower sides of the beam.

These experiments having established the fact that the neutral axis is in the centre of a rectangular beam, and that its position is not sensibly altered by variations in the amount of strain applied, it becomes evident that if there were no other elements of strength than the resistances to direct extension and compression, the well-known formula

$$W=\frac{2adf}{3l}$$

should give the breaking weight when f is equal to the smaller of these two resistances, which in cast iron is the tensile resistance. But the weight so calculated is less than half the actual strength of the beam.

In considering this question, I was forcibly struck by the circumstance, that, in applying the law of "*ut tensio sic vis*" to contiguous fibres, under different degrees of tension and compression, the effect of lateral adhesion is omitted, and each fibre is supposed to be capable of taking up the same degree of extension and compression from the same force as if it acted separately, and independently of the adjoining fibres. But it is well known as a practical fact, that there is a powerful lateral action which tends to modify the effect of unequal strains.



If, for example, a bar, *abcd*, have a strain applied at *efdb*, the portion *defb* will not be extended so much as it would be if separated from *acef*, unless an equal strain is applied to the portion *acef*. And if a portion of a bar cannot be extended in proportion to the force applied to it, unless the contiguous part is equally strained, it follows that the outer portions of a beam subjected to transverse strain will not be extended in proportion to the force applied, because the part nearer the neutral axis is not equally strained. The measurements made for obtaining the position of the neutral axis afford direct evidence on this point.

In the first beam, a strain of 5786 lbs. caused an extension of twenty-eight divisions of the micrometer; the points measured were $\frac{11}{12}$ths of the depth of the beam. The extension at the outer fibres was therefore $28 \times \frac{12}{11} = 30$ divisions. The micrometer reading before the strain was applied was 2111, and the total distance of the points measured was $2111 + 51661 = 53772$. The effect of the strain caused therefore an extension of $\frac{30}{53772} = \frac{1}{1792 \cdot 4}$ of the length. The beam was 7 feet 4 inches long, 6 inches deep, and 2 inches thick; and as

$$W = \frac{2adf}{3l}$$

$$f = \frac{3lW}{2ad}$$

$$\text{or } f = \frac{3 \times 88 \times 5786}{2 \times 12 \times 6} = 10{,}608 \text{ lbs.};$$

so that, with a strain of 10,608 lbs. at the outer fibres, the extension produced was $\frac{1}{1792 \cdot 4}$ of the length.

But in referring to the experiments made by Mr. Hodgkinson, it will be seen that a force of 10,538, applied by direct tensile strain, extends cast iron $\frac{1}{1056}$th of its length, being nearly double that exhibited by the beam.

In the second beam, a weight of 8000 lbs. (from the mean of two results) produced an extension of forty divisions, which at the extreme fibres will be $40\frac{12}{11} = 44$ divisions.

The mean reading of the micrometer, previous to the strain being applied, was 1439; therefore the extension was

$$\frac{44}{51661 + 1439} = \frac{1}{1207}.$$

The strain at the outer fibres produced by this weight was 14,666 lbs.; so that 14,666 lbs. to the inch caused an extension of $\frac{1}{1207}$th of the length.

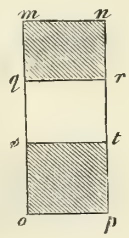
But referring again to Hodgkinson's experiments on direct tensile strain, a weight of 14,793 lbs. produced an extension of $\frac{1}{645}$th of the length; which is again nearly double that produced by the same strain when excited by a weight applied transversely.

From these and other considerations I was led to think it probable that the effect of the lateral action of the fibres or particles of a beam, tending to modify the effect of the unequal strains and opposite forces, and thus diminishing the amount of extension and compression which would otherwise arise, constituted in effect a *resistance to flexure*; and it will be found that the following experiments fully confirm the existence of this resistance as an additional element of strength in beams; and that it explains the apparent anomaly in the amount of tensile resistance when excited by direct and by transverse strains.

Assuming the probability of a resistance, acting independently of, or in addition to, the resistance of direct tension and compression, and varying with the flexure, it occurred to me that it might be exhibited experimentally by casting open girders of the forms shown figs. 2, 3 & 4, having the same sectional area in the upper and lower ribs; the same number of vertical ribs, but the distance between the horizontal ribs, and consequently the deflections of the girders, different.

In these girders the neutral axis would necessarily be (like that of the solid beam) in the centre, and the sectional area of the ribs subjected to tension and compression being the same in each, the circumstances under which rupture would ensue would be similar, except in the amount of flexure.

The formula for the strength of a girder of this form is as follows:—



Let a=the united area contained in the upper and lower ribs;
a'=the intervening space;
d=the total depth;
c=the distance between the upper and lower ribs;
l=the length of bearing;
W=the breaking weight;
and F=the force required to produce rupture in the extreme fibres or particles.

Then $a+a'$=the total area of the rectangle m, n, o, p,

$$W=\frac{2dF}{3l}(a'+a)-\frac{2ca'}{3l}\times\frac{cF}{d};$$

or

$$W=\frac{2F}{3l}\left\{(a'+a)d-\frac{a'c^2}{d}\right\},$$

$$W=\frac{2Fa}{3l}\left(d+c+\frac{c^2}{d}\right).$$

The formula may also be obtained by calculating the moments in the usual way. Using the same letters as before, we have, for the distance of the centres of compression and extension,

$$\frac{2}{3}\left(d+\frac{c^2}{d+c}\right).$$

The force acting when F is the strain which breaks the outer fibre, will be

$$\frac{F+\frac{Fc}{d}}{2}=F\frac{\left(1+\frac{c}{d}\right)}{2}.$$

Hence
$$\frac{W}{2}\times\frac{l}{2}=\frac{2}{3}\left(d+\frac{c^2}{d+c}\right)\left(\frac{1+\frac{c}{d}}{2}\right)\frac{aF}{2};$$

or
$$W=\frac{2Fa}{3l}\left(d+c+\frac{c^2}{d}\right).$$

The value of W being obtained by experiment in each case, we have from the formula

$$F=\frac{3lW}{2a\left(d+c+\frac{c^2}{d}\right)};$$

and if the strength depended only on the direct tensile power of the material, F should in each case be constant, and equal to the direct tensile resistance; but if, in addition to this, there existed another element of strength in the resistance occasioned by the lateral adhesion and varying with the flexure, the value of F would be found, in every case, greater than the tensile resistance, and to increase when the flexure increased.

Four beams were cast of each form, of which the details, the exact dimensions, deflections, and breaking weights are given in the Appendix. The results were as follows, obtained from the mean of four experiments on each form of girder:—

Description of beam.	Total depth of beam.	Sectional area of the two ribs.	Distance between the ribs.	Deflection with nine-tenths of breaking weight.	Breaking weight.
	in.	in.	in.	in.	lbs.
Form No. 2	2·51	1·98	·54	·510	2468
Form No. 3	3·00	2·00	1·00	·401	3119
Form No. 4	4·00	1·98	2·03	·301	4339

The value of F being derived from each of these results by the formula

$$F=\frac{3lW}{2a\left(d+c+\frac{c^2}{d}\right)}.$$

	Deflection.	Value of F.
Form No. 2	·510	35386
Form No. 3	·401	31977
Form No. 4	·301	28032

The tensile strength of the metal obtained from the mean of eight experiments, given in the Appendix, was 18,750 lbs.; here, therefore, was decided evidence, first, that the value of F exceeded the tensile strength in all three forms, and that it increased with the increase of flexure.

In connexion with the above-described experiments, I made four others on solid beams having the same sectional area and length as the open girders; and the mean of the four gave a breaking weight of 1888 lbs. Obtaining the value of F from these experiments, we have,—

Deflection with nine-tenths of breaking weight.	Value of F.
·670	41709 lbs.

which again exhibits an increase in the value of F, with an increase in the deflection.

The foregoing experiments having shown that in girders containing the same depth of metal, the resistance arising from the lateral action of the particles depended on the amount of the flexure, I thought it desirable to make other experiments to ascertain how this resistance varied in girders having the same total depth, and consequently nearly the same deflection, but with different depths of metal in the girder. For this purpose beams were cast of the forms Nos. 5, 6 and 7, each 4 inches deep, and with the upper and lower ribs $1\frac{1}{2}$ inch by $\frac{3}{4}$ inch, the ribs being placed as shown in the figures, so that the depth of the metal in No. 5 was twice as great as in Nos. 6 and 7.

Four beams were cast of each form,—the exact dimensions and breaking weights are given in the Appendix,—and the mean results were as follows:—

Description of beam.	Depth of beam.	Depth of metal.	Sectional area.	Deflection.	Breaking weight.
Form No. 5	4·04	3·01	2·320	·322	5141
Form No. 6	4·04	1·48	2·230	·310	5147
Form No. 7	4·07	1·56	2·380	·262	6000

Obtaining the value of F from these experiments, and comparing them with beam No. 4, which had the same total depth, we have—

	Deflection.	Depth of metal.	Value of F.
Form No. 5	·322	3·01	37408
Form No. 4	·301	1·97	28032
Form No. 7	·262	1·56	27908
Form No. 6	·310	1·48	25271

These experiments did not afford so complete a comparison as the former series, because the intervals between the vertical ribs were not equal, nor in the same proportion to the depth of metal, the effect of which would be to vary to some extent the form of the curve of deflection. Nevertheless, they show in an equally decided manner, that when the deflection is the same the resistance increases when the depth of metal in the beam is increased.

The foregoing experiments have therefore elicited three facts as regards beams formed of two parallel bars separated at given intervals by vertical ribs:—

First, that in every case the resistance, or the value of F, is greater than that due to the tensile resistance of the metal.

Secondly, that with the same depth of metal in the beam, and the same distance of bearing, the resistance is greater when the deflection is greater.

Thirdly, that with the same deflection and the same length of bearing, the resistance is greater when the depth of metal in the beam is greater.

And it follows from these results, that there is an element of strength depending on the amount of deflection in connexion with the depth of metal in the beam, or in other words, dependent upon the degree of flexure to which the metal forming the beam is subjected.

The existence of an element of strength in addition to the resistances to direct tension and compression being clearly proved by these experiments, it becomes interesting to ascertain the law under which it varies, in the form of beams experimented upon.

Now if from the value of F, the tensile strength of the metal is deducted, it will be found that the remainder maintains nearly a constant ratio in each case to the depth of the metal in the beam multiplied by its deflection. It would appear, therefore, that the total resistance, or the value of F, is composed of two quantities; one being constant and limited by the resistance to direct tension, and the other varying directly as the degree of flexure to which the metal forming the beam is subjected.

The applicability of this simple law may be tested by the results of the experiments, as follows:—

Let φ=the resistance to flexure in the solid beam at the time of rupture;
and let D=the depth,
δ=the deflection,
f=tensile resistance,
and F=total resistance.

Then in the solid beam

$$f+\varphi=F;$$

and let F', D' and δ', represent the total resistance, depth of metal, and deflection of any other of the beams; then, the lengths being equal, if the resistance arising from the lateral action varies as the depth of metal into the deflection,

$$F'=f+\varphi\frac{D'\delta'}{D\delta}.$$

The value of φ may be determined from this equation, applied to each of the experiments, in two ways; first, by supposing f to be a constant quantity; and secondly, by supposing f and φ to have a constant ratio.

By the first mode, the whole of the errors of observation and irregularities of the strength of the metal would be accumulated in φ. By the second method, these irregularities will be divided between the values of f and φ.

Adopting therefore the second method, let 1 to m represent the ratio of f to φ:

then $$f=m\varphi,$$

and $$m\varphi+\varphi\frac{D'\delta'}{D\delta}=F';$$

or $$\varphi=\frac{F'}{m+\frac{D'\delta'}{D\delta}},$$

which ought to be a constant quantity in all the experiments.

We cannot obtain the deflections at the line of rupture, but they may be assumed to be proportional to the deflections with $\frac{9}{10}$ths of the breaking weights in each case.

Now the value of F in the solid beam was found to be 41,709 lbs.; and the value of f, from the experiments on direct tension, was 18,750 lbs.: and as in the solid beam $$f+\varphi=F,$$

φ will be 22,959 lbs.,

and the ratio of φ to f will be as 1 to ·81.

For the purpose of comparison, I have deduced the value of f and φ, in solid beams, from the experiments of Mr. Hodgkinson on ten different descriptions of metal; the results of which are given in the following Table:—

Description of iron.	Transverse strength of bar 1 inch square and 54 inches between the supports.	Tensile strength per square inch.	Value of $f+\varphi$ from the formula $w=\frac{2ad(f+\varphi)}{3l}$.	Value of φ from the formula $w=\frac{2ad(f+\varphi)}{3l}$, $\varphi=\frac{3lw}{2ad}-f$.
	lbs.	lbs.	lbs.	lbs.
Carron iron No. 2, cold blast	476	16,683	38,556	21,873
Carron iron No. 2, hot blast	463	13,505	37,503	23,998
Carron iron No. 3, cold blast	446	14,200	36,126	21,926
Carron iron No. 3, hot blast...............	527	17,755	42,687	24,932
Devon iron No. 3, hot blast	537	21,907	43,497	21,590
Buffery iron No. 1, cold blast	463	17,466	37,503	20,037
Buffery iron No. 1, hot blast	436	13,434	35,316	21,882
Coed-Talon iron No. 2, cold blast	413	18,855	33,453	14,598
Coed-Talon iron No. 2, hot blast	416	16,676	33,696	17,020
Low Moor iron No. 3, cold blast	467	14,535	37,827	23,292
Means	464	16,502	37,616	21,114

The mean ratio of φ to f in these metals appears to be as 1 to ·78. The metal used in my experiments was a mixture consisting of two-thirds of South Staffordshire No. 3, hot blast pig, and one-third old metal recast. As compared with Mr. Hodgkinson's experiments, its strength accorded nearly with that of the Carron iron No. 3, hot blast.

The mean ratio of φ to f, obtained from Mr. Hodgkinson's experiments, being as 1 to ·78, and from the experiments herein detailed being as 1 to ·81, we may consider f to be four-fifths of φ; and therefore

$$m=\cdot 8.$$

Using this ratio, the values of φ and f, derived from the formula

$$\varphi=\frac{F}{m+\frac{D'\delta'}{D\delta}}$$

and

$$f=\varphi m,$$

as applied to each of the experiments, are given below :—

No. 1. $\varphi=\frac{41709}{\cdot 8+\frac{2\cdot 012\times\cdot 670}{2\cdot 012\times\cdot 670}}=23{,}171$ lbs., $f=18{,}537$ lbs.

No. 2. $\varphi=\frac{35386}{\cdot 8+\frac{1\cdot 97\times\cdot 510}{1\cdot 348}}=22{,}904$ lbs., $f=18{,}323$ lbs.

No. 3. $\varphi=\frac{31977}{\cdot 8+\frac{2\cdot 01\times\cdot 401}{1\cdot 348}}=22{,}890$ lbs., $f=18{,}312$ lbs.

No. 4. $\varphi=\frac{28032}{\cdot 8+\frac{1\cdot 97\times\cdot 301}{1\cdot 348}}=22{,}606$ lbs., $f=18{,}085$ lbs.

No. 5. $\varphi=\frac{37408}{\cdot 8+\frac{3\cdot 01\times\cdot 322}{1\cdot 348}}=24{,}626$ lbs., $f=19{,}501$ lbs.

No. 6. $\varphi=\frac{25270}{\cdot 8+\frac{1\cdot 48\times\cdot 310}{1\cdot 348}}=22{,}167$ lbs., $f=17{,}734$ lbs.

No. 7. $\varphi=\frac{27908}{\cdot 8+\frac{1\cdot 56\times\cdot 262}{1\cdot 348}}=25{,}302$ lbs., $f=20{,}242$ lbs.

These results, though not exhibiting complete regularity, are sufficiently uniform to indicate that the assumed law of the variation of this resistance is a close approximation to the truth. It will be observed also, that Nos. 2, 3, 4 and 6, give a smaller value of φ than Nos. 1, 5 and 7, which probably arises from the difference in the proportion which the distance between the vertical ribs bears to the depth of the metal; a circumstance which would affect, to some extent, the form of the curve of deflection.

In the formula $\varphi=\frac{F'}{m+\frac{D'\delta'}{D\delta}}$, $\frac{D'\delta'}{D\delta}$ represents the ratio of the depth of metal in each beam multiplied by its deflection, to the depth of metal in the solid beam multiplied by its deflection. But the deflections, as might have been expected from known laws, were nearly in the inverse ratio of the total depths of each girder; therefore the degree of flexure, and consequently the resistance to flexure in each, will be nearly as the depth of metal divided by the total depth of the girder, and we are thus enabled to obtain a formula for computing, approximately, the breaking weights of these girders, without first ascertaining their deflection.

Using the same letters as before, we have, for the resistance due to tension,

$$\frac{2a}{3l}\left(d+c+\frac{c^2}{d}\right)f;$$

and for the resistance to flexure,

$$\frac{2a}{3l}\left(d+c+\frac{c^2}{d}\right)\frac{\phi D}{d};$$

and consequently, for the united effect of the two resistances,

$$W=\frac{2a}{3l}\left(d+c+\frac{c^2}{d}\right)\left(f+\frac{\phi D}{d}\right).$$

I shall therefore conclude these observations by comparing the breaking weights computed for tensile resistance alone, and those obtained from the formula which includes the resistance to flexure, with the actual breaking weights obtained by the experiments, taking the value of f=18,750 lbs., and ϕ=23,000 lbs.

Description of beam or girder.	Breaking weight if the resistance depended on direct tensile strength.	Breaking weight computed by the formula, including the resistance to flexure.	Breaking weight as obtained by the experiments.
	lbs.	lbs.	lbs.
No. 1	849	1890	1888
No. 2	1308	2567	2468
No. 3	1808	3287	3084
No. 4	2912	4659	4353
No. 5	2578	4935	5141
No. 6	3819	5533	5147
No. 7	4031	5919	6000

The accordance exhibited by the computed and the actual breaking weights, evinces the general accuracy of the formula, as applied to this form of beam; while these results, compared with those computed for direct tensile force alone, show how large a proportion of the strength of cast iron, when subjected to transverse strain, is due to the resistance arising from the lateral action.

It will also be seen that comparisons of the relative strengths of different forms of section, calculated, as has been customary, on the assumption that the resistances are constant forces, or governed by a constant coefficient, must be entirely fallacious.

It was my intention to have included in this paper a similar investigation as to the position of the neutral axis, and the amount of the resistance arising from lateral action of the fibres in wrought iron; but as the experiments will take some time to complete, and as the facts elicited in reference to cast iron are of sufficient importance to render it desirable that they should be made known, I will reserve the examination of wrought iron for the subject of another communication.

Girder No. 1.

	Experiment No. 1.	Experiment No. 2.	Experiment No. 3.	Experiment No. 4.
	inches.	inches.	inches.	inches.
Depth	2·015	2·02	2·073	2·040
Thickness	·975	·98	1·030	·990
Area of section	1·965	1·98	2·135	2·020
Weight applied, lbs.	Deflection.	Deflection.	Deflection.	Deflection.
40	·015	·013	·014	·014
376	·145	·115	——	——
600	·203	——	——	——
712	·280	·233	·264	·244
936	·330	——	——	——
1160	·490	·420	·397	·414
1608	·725	·625	·579	·614
1664	Broke·755	·655	——	——
1720	......	·680	·629	·659
1832	......	·737	·679	·734
1888	......	Broke	·699	·764
1916	......	......	——	Broke
1944	......	......	·734	
2000	......	......	·762	
2028	......	......	·774	
2056	......	......	·789	
2084	......	......	Broke	
Breaking weight, lbs.	1664	1888	2084	1916
Deflection with nine-tenths of breaking weight, inches	·643	·667	·699	·670

Girder No. 2.

	Experiment No. 1.	Experiment No. 2.	Experiment No. 3.	Experiment No. 4.
	inches.	inches.	inches.	inches.
Total depth	2·54	2·53	2·49	2·50
Depth between upper and lower ribs	·56	·55	·51	·55
Area of top rib	1·00	1·00	·97	·98
Area of bottom rib	1·01	1·00	·99	·97
Weight applied, lbs.	Deflection.	Deflection.	Deflection.	Deflection.
40	·009	·007	·007	·007
712	——	·132	·134	·137
804	·199	——	——	——
1292	·304	——	——	——
1516	——	·302	·319	·312
1740	·414	——	——	——
1852	——	·372	——	——
1964	·489	·397	·426	·433
2076	——	·427	——	——
2188	Broke	·445	·479	·487
2300	......	·479	·526	·532
2412	......	·512	Broke	·550
2524	......	·542	......	Broke
2636	......	·575		
2748	......	Broke		
Breaking weight, lbs.	2188	2748	2412	2524
Deflection with nine-tenths of breaking weight, inches	·489	·532	·482	·516

Girder No. 3.

	Experiment No. 1.	Experiment No. 2.	Experiment No. 3.	Experiment No. 4.
	inches.	inches.	inches.	inches.
Total depth	3·02	3·00	3·00	3·00
Depth between upper and lower ribs	·98	1·00	1·01	1·01
Area of top rib	1·03	1·02	·97	1·01
Area of bottom rib	·99	·98	1·01	·97
Weight applied, lbs.	Deflection.	Deflection.	Deflection.	Deflection.
40	·006	·005	·005	·005
712	——	·085	·085	·085
844	·113	——	——	——
1516	·216	·185	·197	·195
1740	·248	——	——	——
2188	·328	——	·297	·293
2300	——	·295	——	——
2524	·388	——	——	——
2636	·418	——	·363	·375
2748	·433	·377	——	——
2972	·483	·410	·423	Broke
3028	Broke	·425	·438	
3084	......	·435	·452	
3112	......	·437	Broke	
3224	......	Broke		
Breaking weight, lbs.	3028	3224	3112	2972
Deflection with nine-tenths of breaking weight, inches	·435	·402	·397	·371

Girder No. 4.

	Experiment No. 1.	Experiment No. 2.	Experiment No. 3.	Experiment No. 4.
	inches.	inches.	inches.	inches.
Total depth	3·99	4·00	3·99	4·01
Depth between upper and lower ribs	2·00	2·03	2·05	2·04
Area of top rib	1·00	·97	·98	·98
Area of bottom rib	1·00	·99	·98	1·01
Weight applied, lbs.	Deflection.	Deflection.	Deflection.	Deflection.
40	·002	·002	·002	·003
712	·047	·040	·048	·058
1516	·104	·097	·102	·108
1964	·134	——	——	——
2188	·161	·155	·155	·148
2636	·199	·197	·185	·183
3084	·227	·227	·223	·218
3420	——	·259	——	——
3532	·269	·267	·255	·253
3756	·299	·282	·285	——
3980	·317	·312	·300	·303
4092	·329	·320	·307	——
4148	·336	·322	·313	——
4204	Broke	·327	Broke	——
4260	......	Broke	......	·333
4316	......	......	......	——
4400	......	......	.....	·343
4428	......	......	......	——
4745	......	......	......	Broke
Breaking weight, lbs.	4204	4260	4204	4745
Deflection with nine-tenths of breaking weight, inches	·297	·293	·282	·331

Girder No. 5.

	Experiment No. 1.	Experiment No. 2.	Experiment No. 3.	Experiment No. 4.
	inches.	inches.	inches.	inches.
Total depth	4·02	4·05	4·05	4·04
Depth between upper and lower ribs	1·04	1·04	1·04	1·00
Area of top rib	1·125	1·16	1·14	1·22
Area of bottom rib...............	1·162	1·13	1·15	1·20
Weight applied, lbs.	Deflection.	Deflection.	Deflection.	Deflection.
712	Deflections uncertain from imperfect fastening of the cord.	——	——	——
1516		——	——	——
2188		——	——	——
2290		·133	·148	·138
2636		——	——	——
2885		·173	·182	·178
3084		——	——	——
3445		·213	·221	·223
3532		——	——	——
3980		——	——	——
4005		·268	·270	·259
4428		——	——	——
4565		·313	·320	·308
4652		——	——	——
4705		·323	·335	——
4845		·348	·350	·335
4876		——	——	——
4927		——	——	——
4985		·348	Broke	·340
5008		——	......	——
5050	Broke	——	......	——
5125	......	Broke	......	·355
5265	......	......	......	·365
5405	......	......	......	Broke
Breaking weight, lbs.............	....	5125	4985	5405
Deflection with nine-tenths of breaking weight, inches	......	·321	·313	·331

Girder No. 6.

	Experiment No. 1.	Experiment No. 2.	Experiment No. 3.	Experiment No. 4.
	inches.	inches.	inches.	inches.
Total depth	4·02	4·05	4·03	4·06
Depth between upper and lower ribs	2·52	2·55	2·56	2·61
Area of top rib	1·13	1·18	1·08	1·10
Area of bottom rib	1·13	1·09	1·11	1·10
Weight applied, lbs.	Deflection.	Deflection.	Deflection.	Deflection.
712	Deflections uncertain from the same cause as Experiment 1, No. 5.	—	—	—
1516		—	—	—
2188		—	—	—
2290		·130	·138	·138
2636		—	—	—
2885		·168	·186	·175
3084		—	—	—
3445		·205	·220	·222
3532		—	—	—
3980		—	—	—
4005		·251	·263	·272
4428		—	—	—
4565		·300	·313	·313
4652		—	—	—
4845		·315	Broke	·350
4876		—	......	—
4988		—	......	·365
5100		—	......	—
5125	......	Broke	......	·378
5212	Broke	......	......	—
5265	......	......	......	·382
5405	......	......	......	Broke
Breaking weight, lbs.	......	5125	4845	5405
Deflection with nine-tenths of breaking weight, inches	......	·298	·293	·340

Girder No. 7.

	Experiment No. 1.	Experiment No. 2.	Experiment No. 3.	Experiment No. 4.
	inches.	inches.	inches.	inches.
Total depth	4·05	4·10	4·08	4·05
Depth between upper and lower ribs	2·50	2·51	2·51	2·52
Area of top rib	1·19	1·26	1·21	1·16
Area of bottom rib	1·19	1·19	1·17	1·16
Weight applied, lbs.	Deflection.	Deflection.	Deflection.	Deflection.
2290	·105	·105	·095	·090
2885	·115	·130	·120	·125
3445	·150	·160	·140	·160
4005	·185	·185	·180	·182
4565	·217	·215	·215	·210
5125	·255	·250	·235	·237
5405	·272	·267	—	—
5685	Broke	·285	·270	·272
5825	......	·292	—	Broke
5965	......	·305	Broke	
6105	......	·310		
6245	......	·320		
6385	......	·330		
6525	......	Broke		
Breaking weight, lbs.	5685	6525	5965	5825
Deflection with nine-tenths of breaking weight, inches	·252	·297	·253	·246

Summary of the Experiments on Transverse Strength, giving the mean results.

	Depth.	Sectional area.	Distance between the ribs.	Breaking weight.	Deflection with nine-tenths of breaking weight.
	in.	sq. in.	in.	lbs.	in.
Form of beam No. 1.	2·015	1·965	......	1664	·643
	2·020	1·980	......	1888	·667
	2·073	2·135	......	2084	·699
	2·040	2·020	......	1916	·670
Mean	2·012	2·025	......	1888	·670
Form of beam No. 2.	2·54	2·01	·56	2188	·489
	2·53	2·00	·55	2748	·532
	2·49	1·96	·51	2412	·482
	2·50	1·95	·55	2524	·516
Mean	2·51	1·98	·54	2468	·510
Form of beam No. 3.	3·02	2·02	·98	3028	·435
	3·00	2·00	1·00	3224	·402
	3·00	1·98	1·01	3112	·397
	3·00	1·98	1·01	2972	·371
Mean	3·01	2·00	1·00	3084	·401
Form of beam No. 4.	3·99	2·00	2·00	4204	·297
	4·00	1·96	2·03	4260	·293
	3·99	1·96	2·05	4204	·282
	4·01	1·99	2·04	4745	·331
Mean	4·00	1·98	2·03	4353	·301
Form of beam No. 5.	4·02	2·287	1·04	5050	......
	4·05	2·290	1·04	5125	·321
	4·05	2·290	1·04	4985	·313
	4·04	2·420	1·00	5405	·331
Mean	4·04	2·322	1·03	5141	·322
Form of beam No. 6.	4·02	2·26	2·52	5212	......
	4·05	2·27	2·55	5125	·298
	4·03	2·19	2·56	4845	·293
	4·06	2·20	2·61	5405	·340
Mean	4·04	2·23	2·56	5147	·310
Form of beam No. 7.	4·05	2·38	2·50	5685	·252
	4·10	2·45	2·51	6525	·297
	4·08	2·38	2·51	5965	·253
	4·05	2·32	2·52	5825	·246
Mean	4·07	2·38	2·51	6000	·262

Experiments on Direct Tension.

Number of experiment.	Sectional area at the place of fracture.	Last weight supported.	Weight with which the bar broke.	Remarks.
	inches.	lbs.	lbs.	
1.	1·0506	18,560	18,840	A small air-bubble.
2.	1·0557	19,680	19,960	A small air-bubble.
3.	1·0100	21,360	21,500	A small air-bubble at corner, very small.
4.	1·0364	16,320	16,320	Honey-combed.
5.	1·0301	17,440	17,440	Sound.
6.	1·0403	16,320	17,440	A small air-bubble.
7.	1·0150	21,640	21,920	Sound.
8.	1·0200	22,200	22,470	Sound.
Mean	1·0323	19,190	19,486	

Mean greatest weight supported, per inch . . . . . 18,590 lbs.
Mean weight which broke the bar, per inch . . . . 18,876 lbs.

Considering the actual breaking weight to be between these two, and rather nearer the latter, when due allowance is made for the small air-bubbles, the mean breaking weight may be taken at 18,750 lbs. per square inch.

X. *On the Development of Striated Muscular Fibre in Mammalia. By* WILLIAM Ş. SAVORY, *Tutor and Demonstrator of Anatomy of St. Bartholomew's Hospital Medical College. Presented by* JAMES PAGET, *F.R.S.*

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THE descriptions which have been hitherto given of the development of muscular fibre may be included in a few words. Little or nothing of importance has been discovered beyond the original researches of VALENTIN and SCHWANN. Concerning these SCHWANN says, "In order briefly to recapitulate our researches into the generation of muscle, the process may be thus stated. Round cells, furnished with a flat nucleus, are first present, the primary cells of muscle. These arrange themselves close together in a linear series; the cells thus arranged in rows coalesce with one another at their points of contact; the septa by which the different cell-cavities are separated then become absorbed, and thus a hollow cylinder closed at its extremities, the secondary cell of muscle, is formed, within which the nuclei of the original cells, from which the secondary cell has been formed, are contained, generally lying near together on its wall. This secondary cell then passes through all the stages of a simple one. It expands throughout its entire length, whereby the nuclei are further removed from one another, and sometimes even become elongated in the same direction. A deposit of a peculiar substance, the proper muscular substance, takes place at the same time upon the inner surface of the cylinder, by which the cavity is at first narrowed and at length completely filled. The cell-nuclei lie external to this substance, between it and the cell-membrane of the secondary cell.

"The transverse striæ in the voluntary muscles become more manifest, and the deposited substance is more distinctly seen to be composed of longitudinal fibres, as the fœtus advances in age. The nuclei are gradually absorbed. The cell-membrane of the secondary muscle-cell remains persistent throughout life, so that each primitive muscular fasciculus is always to be regarded as a cell*."

This account has been generally accepted by those who have since written on the subject. Dr. MARTIN BARRY, however, must be excepted. He entertains a very different view of the origin of muscular fibre, and declares that the cells, by the coalescence of which the tubes are at first formed, are really blood-cells. He says that the blood-corpuscles apply themselves to one another in a linear series. By degrees

* Microscopical Researches into the Accordance in the Structure and Growth of Animals and Plants, translated from the German of Dr. TH. SCHWANN, by H. SMITH, 1847, p. 141.

the appearance of a cylinder is produced, which becomes more perfect as the partitions between the cells disappear*.

These observations have not been confirmed. M. Lebert has more recently studied the development of striated muscular fibre chiefly in the heart of the chick †. The following is a short abstract of his peculiar description of the process. "The rhythmical contractions of this organ become very manifest and regular towards the thirty-sixth hour of incubation; nevertheless, it is at this time composed of nothing else than organo-plastic globules or elementary cells imbedded in a granular blastema. Between the fourth and fifth days of incubation are seen in the midst of the mass of globular particles certain elongated subcylindrical bodies, sometimes grouped together in a reticular manner; these bodies being the first rudiments of the muscular fibres, not merely in the heart but also in the other striated muscles, are designated by M. Lebert 'myogenic cells.' Between the seventh and eighth days the 'organo-plastic globules' undergo a considerable diminution, and the muscular substance presents a more complete development. A longitudinal striation shows itself in the contents of the cylinders, which seems partly due to the grouping of the granular particles of which these contents consist; the transverse striations do not show themselves until some time afterwards.....The organo-plastic globules which at first separated the primitive cylinders gradually disappear; the cylinders approach one another, and before the end of embryonic life they are found to be grouped into fasciculi‡."

When many months since I began to study the development of muscular fibre, it was with the hope of being able to add something to what appeared to be a very imperfect explanation of the process. The results of these investigations I have endeavoured to communicate in this paper. It may be as well to state at once that the conclusions I have arrived at are, for the most part and in all important points, completely at variance with the account which is generally believed to be the true one. I can therefore only meet the doubts with which I must expect the following statements to be received, with the assurance that they have not been hastily or carelessly advanced, and by an appeal to the verdict of future inquiry.

These observations have been made for the most part upon fœtal pigs, but they have been confirmed by repeated examinations of fœtal calves, lambs, goats, rabbits, rats, and human embryos.

In very small embryos, specimens of muscular tissue for microscopical examination are most conveniently obtained from the dorsal region. If therefore a portion

* Philosophical Transactions, 1840, 1841. † Annales des Sciences Nat. Juin 1849, Mars 1850.

‡ Principles of Human Physiology, by Dr. Carpenter, 1853, pp. 305, 306. I have recently read a paper by Dr. Holst, giving a short account of some observations by Reichert and himself on the development of muscular fibre. From some of the engravings which accompany the paper, he has probably seen, although I think misinterpreted, the condition of the fibre in what I have described as the second and third stages of development.

of the tissue immediately beneath the surface by the side of the vertebral column of a fœtal pig, from 1 to 2 inches in length, be examined, there will be seen, besides blood-corpuscles in various stages of development, nucleated cells and free nuclei, or cytoblasts scattered through a clear and structureless blastema in great abundance. These cytoblasts vary in shape and size; the smaller ones, which are by far the most numerous, being generally round, and the larger ones more or less oval. Their outline is distinct and well-defined, and one or two nucleoli may be seen in their interior as small, bright, highly-refracting spots; the rest of their substance is either uniformly nebulous or faintly granular, Plate X. fig. 1.

The first stage in the development of striated muscular fibre consists in the aggregation and adhesion of these cytoblasts, and their investment by blastema so as to form elongated masses. In these clusters the nuclei are not at first generally arranged in a single series; but two, three, or even more occasionally lie side by side in apparent disorder. Almost, if not quite as soon as these cytoblasts are thus aggregated into these long masses, they become invested by the blastema, and this substance at the same time appears to be considerably condensed, so that the outlines of the nuclei become almost or completely obscured. The fibre thus appears to be irregularly cylindrical or somewhat flattened. It is so opake that its interior is no longer to be plainly discerned, and its surface is rough and uneven. The appearance of the muscular fibres at this early period of their development is represented in Plate X. fig. 2. It often happens, that here and there, where a nucleus has not become completely invested by condensed blastema, a portion of its well-defined dark margin may be observed standing out in the circumference of the mass. These early fibres measure about $\frac{1}{3000}$th of an inch in diameter. It is almost impossible to estimate their length, as they are so readily broken in being prepared for the microscope; hence the lengths of the masses presented to view vary exceedingly. It has been said that the cytoblasts become invested by a layer of apparently condensed blastema, immediately after or at the same time that they aggregate and adhere together. In some cases, before the nuclei come into contact, this external investment may already be discerned forming around them, giving to them occasionally the appearance, if not very carefully examined, of nucleated cells, and this perhaps may have led to the original description of Valentin. But this investment of the nuclei previously to their aggregation is not common: as a very general rule, the nuclei meet each other free as cytoblasts.

In order to obtain a fair view of these early fibres, a portion of tissue should be selected from a perfectly fresh specimen, for if the embryo be kept longer than a few hours, the masses break up and disappear; and this disintegration is much accelerated by placing them in water, for the investing blastema appears to be readily dissolved and the nuclei separate. Indeed the tissue is so moist at this period, that, when prepared for the microscope, the addition of water is scarcely necessary. The action of the weaker acids also rapidly dissolves the investing blastema and sets the cytoblasts free. However, when the fœtal tissue is treated with water, there is a period just before the fibres break up when they become much more transparent, and the nuclei may be seen in their interior clustered together irregularly, and in absolute contact; this fact, that the nuclei are in actual contact, is a most important one, for it proves very conclusively that they do not occupy the interior of cells. I have often earnestly sought for, but never yet detected, the appearance so generally figured in books of a muscular fibre as a string of nucleated cells; on the contrary, the nuclei, when they approach each other, are free cytoblasts, and they may be clearly seen to be in absolute contact.

The nucleated cells which are seen in the blastema are no more concerned in the formation of muscular fibre than are the blood-cells. With regard to their purpose, it is to be observed that they are chiefly found in the layer of tissue investing the embryo, and if this exterior layer be carefully scraped off from the back of the embryo, and a portion of tissue be examined below the surface, nucleated cells, as compared with free cytoblasts, will be extremely rare.

These nuclei, thus aggregated and invested, next assume a much more regular position. They fall into a single row with remarkable regularity, and the surrounding substance at the same time grows clear and more transparent, and is arranged in the form of two bands bordering the fibre and bounding the extremities of the nuclei, so that they become distinctly visible, and the fibre at this stage presents the appearance represented in Plate X. fig. 3. The nuclei have now become decidedly oval and very closely packed together, side by side, so closely indeed that they appear as if compressed. Thus they form a single row in the centre of the fibre with their long axes lying transversely, and their extremities bounded on either side by a thin, clear pellucid border of apparently homogeneous substance. No structure can be discerned between the nuclei; they lie in close contact, except towards their extremities, and even appear as if pressed together. It is here to be remarked that this position of the nuclei is a strong additional argument against the supposition that they are or were contained in cells. If so, what explanation can be offered to account for the present arrangement? It occasionally happens, especially towards the extremities of a fibre, that some irregularity in the position of the nuclei may be discerned. They have not fallen into their places, but still remain as an irregular cluster, dilating the fibre by a separation of the lateral bands to a corresponding extent, Plate X. fig. 4. More frequently, in the course of a fibre an occasional nucleus is seen, which instead of lying transversely is placed obliquely.

The bands of tissue forming the borders of the fibre and bounding the extremities of the nuclei, at first thin and pellucid, soon increase in thickness by the addition of surrounding blastema to their external surface. They increase in breadth, and this increase is due to the addition of fresh material upon their exterior, and not to a deposit upon their inner surface, for the extremities of the nuclei are not encroached upon, and the outline of the fibre, which is at first even and well-defined, soon becomes rough and irregular, obviously from the addition of fresh material, Plate VII. fig. 5. Sometimes these bands appear defective or broken at intervals, or altogether wanting for some little distance. This no doubt is the result of the manipulation required for the microscope. I have occasionally seen fibres, which, either from the same cause or from some defect in their formation, possessed only one lateral band; the extremities of the nuclei being invested on one side only.

If a portion of muscular tissue be examined which has been roughly dissected, or which has been taken from a fœtus previously kept for a few hours in water, fibres will often be seen which are more or less broken up. Some will be found with the lateral bands separated from or partly stripped off the nuclei and variously twisted. The nuclei themselves will be wanting in many places, or considerably disturbed. These and other derangements, the result of violence, assist to explain the construction of the fibres at this period of their development.

It is interesting to compare the arrangement of the nuclei in this stage of development of muscular fibre in Mammalia with the position of the nuclei in the fully-developed fibre of some of the lower classes. Mr. Bowman, in his paper in the Philosophical Transactions, has noticed that in the muscular fibre of many insects and of some reptiles, the corpuscles are disposed along the central axis of the fibre with remarkable regularity, and in some instances, as in the *Tipula* (Harry Longlegs), which he figures, the corpuscles are thus arranged with their long axis transversely. (See fig. 6 *a*, and compare it with figs. 3 and 7, Plate X.) The central row of nuclei is well shown in the fibres of the leg of the common Blue Bottle Fly (*Musca vomitoria*) after the addition of a weak acid. In insects, the nuclei are very often visible in the fibre without any previous preparation. I have sometimes distinctly observed them in fibres from the leg of the *Tipula*, and in many other examples.

The fibres next increase in length and the nuclei separate; the nuclei remain no longer in close contact; small intervals appear between them, and as they have more space they increase in width and become more nearly circular. The spaces between the nuclei rapidly widen, until at last they lie at a very considerable distance apart. At the same time the fibre decreases very considerably in diameter, and the cause of this is sufficiently obvious. As the nuclei part from each other and as the spaces between them increase, the bands which they separated fall in,—approach each other and ultimately coalesce*. This fact is placed beyond doubt by the examination of some fibres in which these changes are in progress, Plate X. fig. 8. At one extremity of a fibre may be seen the oval nuclei packed closely together, and bordered at their extremities by the substance of the fibre; further on they may be seen separating and becoming rounder, the bands at the same time are beginning to approxi-

* For example, in a specimen of muscular tissue from a fœtal pig $3\frac{1}{2}$ inches long three fibres were measured. In the first the nuclei were almost in contact, and the width of the fibre was $\frac{1}{1800}$th of an inch; in the second the nuclei were $\frac{1}{1500}$th of an inch apart, and the width of the fibre was $\frac{1}{2000}$; in the third the nuclei were $\frac{1}{1000}$th of an inch apart, and the width of the fibre was $\frac{1}{3000}$.

mate, so that the fibre is at this part much narrower; and still further on the nuclei are more completely separated; they are again oval, with their long axis now in the direction of the fibre; the borders have united, and the diameter of the fibre is consequently very considerably diminished. This falling-in of the lateral bands, as the nuclei separate, and their ultimate coalescence, affords evidence for believing, that when they were separated, there could have existed but very little intermediate substance between them amongst the nuclei.

As a general rule, the changes just described proceed uniformly throughout the entire fibre. But in most specimens some fibres will be found in which development has advanced more rapidly towards one extremity than at the other, and these serve admirably to illustrate the changes which occur.

Soon after the nuclei have separated some of them begin to decay. They increase in size; their outline becomes indistinct; a bright border appears immediately within their margin; their contents become decidedly granular; their outline is broken and interrupted; and presently an irregular cluster of granules is all that remains, and these soon disappear. With regard to these groups of granules, however, it is worthy of remark, that as the fibres continue to increase in length, and the remaining nuclei are still further separated, they are extended longitudinally and the granules become more scattered. It sometimes happens that an irregularity may be observed in the separation of the nuclei. Here and there two or more occasionally remain in contact, as if adherent.

This lengthening of the fibre, and consequent separation of the nuclei, is due to an increase of material, and not to a stretching of the fibre*, for the lateral bands, although they grow firmer, do not decrease in width as they approach each other; they preserve their size. Occasionally, indeed, they appear a little narrower, as if stretched, but this is rare, and is no doubt due to manipulation.

The fact is, as these lateral bands fall in and coalesce, their breadth undergoes no apparent alteration. They remain separated for a time at those parts of the fibre where the nuclei are, but they ultimately join, and the nuclei lie imbedded between them.

The changes described above are generally most obviously marked, and are therefore more readily traced in those fibres which are formed at the earliest period. In those of later growth, the lengthening of the fibres does not commence so early, or proceed so rapidly, and is therefore not so obvious, for the nuclei usually decay before they are separated to any extent. Indeed, in some of the fibres this separation of the majority of the nuclei seems scarcely to occur at all; while but a small interval exists between them, and while their oval form is still preserved, most of them perish in their places, with their long axis still lying transversely; and the position they occupied is marked for a time by clusters of granules extended transversely at frequent intervals along the course of the fibre. (See Plate X. fig. 9*a* & *c*.)

In these fibres, however, the same essential changes occur. They subsequently

* As SCHWANN described it, Untersuchungen, &c.

lengthen and decrease in diameter. The only difference is, that owing to the later period at which this occurs, the majority of the nuclei perish previously, and the distinction between the lateral bands and the central portion of the fibre grows obscure; hence the process is not so obvious, because the separation of the nuclei, when they remain, is a most remarkable feature. But the extension of the clusters of granules longitudinally, the wide separation of the granules from one another, and the remarkable decrease in the diameter of the fibre, are not to be mistaken. (See Plates X. & XI. fig. 9 *d*, *e*, *f* & *g*.)

It is to be observed, that the longer the lengthening of the fibres is delayed, the thicker do the lateral bands become. The addition of fresh material from without, which, when the fibre is elongating, is consumed in maintaining the bands at their original width, now adds very much to their breadth, so that these fibres often attain a very considerable diameter. On the contrary, in those fibres in which the lengthening occurs at an earlier period and is more rapidly accomplished, the lateral bands being as yet very thin, the fibre thus formed is very narrow. Indeed, they are sometimes narrower in the intervals between the nuclei than are the nuclei themselves; so that, where these are situated, the fibre is necessarily bulged to contain them. This appearance may, however, be caused by stretching the fibre while preparing it for the microscope.

The striæ first become visible at this period. A faint indication of their appearance may be sometimes observed in the lateral bands, almost as soon as these are fully formed; and as these bands approximate, the striæ become more plainly marked, and often contrast strongly with the intermediate and apparently homogeneous central portion of the fibre. In any case they can always be readily detected at the time when the distinction between the lateral bands and axis disappears.

The striæ are first discerned immediately within the margin of the fibre, and gradually pass towards the centre. When they first appear, generally the longitudinal but sometimes the transverse lines are most plainly marked. A few streaks are usually seen here and there along different portions of the fibre, and these gradually extend and blend together, giving to the fibre for a time an irregular streaked appearance, until at length they are seen throughout its entire substance, but for a long time they remain most prominently marked towards the margins. When first formed they are certainly much smaller than at a subsequent period. The striæ are much finer, and many more exist in a given space than at a more advanced period of development.

When this stage is completed, the fibre presents a very uniform appearance throughout its entire length, Plate XI. fig. 10. It appears as a narrow flattened band*. The nuclei that remain are seen at tolerably regular intervals in the substance of the

* A form which in insects is often permanently maintained.—Bowman, Phil. Trans.

It is impossible not to remark the close similarity between the appearance of striated muscular fibre at this period of its development and the permanent condition of the highest form of organic muscular fibre.

fibre, generally at an equal distance from either margin. They are large and oval, with their long axis in the direction of the fibre. Between these, widely scattered clusters of granules are seen—the remains of those which have perished, and which soon disappear. The striæ, although faintly and delicately marked, are sufficiently obvious, especially towards the circumference.

The fibre now commences to increase in size, and its development is continued by means of the surrounding cytoblasts, which are very numerous amongst the fibres. These may be seen to become attached to its exterior, and then invested by a layer of the surrounding blastema. Thus, as it were, nodes are formed at intervals on the surface of the fibre. In some specimens the adherent nuclei may be seen attached to the fibre at very regular distances, but in many cases no such uniformity can be detected. Generally, however, the nuclei are so near to each other, that the investing material of one, as it spreads, becomes blended with that of its neighbour, and so a continuous layer of fresh material of greater or less extent is added to the exterior of the fibre. This is at first clear and pellucid, like the original border of the fibre when first formed, and presents a striking contrast to the present substance of the fibre. It is at this period readily detached by a little rough manipulation, but it soon becomes intimately connected and indefinitely blended with the exterior of the fibre. The striæ and other characters of the adjacent portions of the fibre are soon acquired; the nuclei at the same time gradually sink into the substance of the fibre, and an ill-defined elevation, which soon disappears, is all that remains. (See Plate XI. fig. 11, *a*, *b* & *c*.)*

Sometimes these changes occur around a single nucleus which is isolated from any other. More frequently, the blastema surrounding many nuclei, attached at short intervals along the lengths of the fibre, becomes blended into a uniform layer; and this is often of such considerable extent, that it appears as a regular band lying along the whole length of the fibre. With a little dissection these bands may generally be detached, and they then appear as small accessory fibres, lying by the sides of the others. (See Plate XI. fig. 13 *b*).

Between the extreme conditions all intermediate variations may be found; sometimes the investments of different nuclei adhere more firmly to the exterior of the fibre than to each other, but it more frequently happens that as they coalesce they are more readily separated together from the fibre than from each other.

If a portion of muscular tissue be examined at the period when the fibres are increasing by the addition of fresh nuclei, there will be seen in the field, besides a number of free nuclei, many nuclei floating about invested by blastema, and having the appearance of caudate cells. These are nuclei which have adhered to the fibre and become coated, and afterwards detached by violence.

* I think that some sketches published by VIRCHOW in his 'Archiv für Pathologische Anatomie und Physiologie und für Klinische Medicin,' Siebenten Bandes, Erstes Heft, must have been drawn from muscular fibres in this stage of their development.

All the changes which have been described may often be traced in the same specimen: first, the attachment of nuclei to the exterior of the fibre; secondly, their investment by blastema; thirdly, the gradual sinking of the nuclei into the substance of the fibre, the corresponding subsidence of the elevation, the development of striæ, &c.

Authors generally describe and figure, after VALENTIN and SCHWANN, fœtal fibres in which the nuclei in the interior are represented as bulging the fibres or prominent on the surface. I believe such descriptions and representations to have been drawn from fibres which were growing and developing by the addition of fresh nuclei, in the manner I have described.

Fœtal fibres rarely possess the same diameter throughout their entire length; they are seldom uniform in this respect. Now this variation is easily explained by their mode of increase. Those portions to which fresh material has been recently added are for a time increased in diameter, but each inequality gradually disappears. It might be imagined that this condition is due to pressure, and undoubtedly this source of error is very liable to arise; but the variation may be constantly observed, even when the fibre floats freely in the field.

Lastly, the substance of the fibre becomes contracted and condensed. The diameter of a muscular fibre from a fœtus towards, or at the close of intra-uterine life, is considerably less than the diameter of a fibre at a much earlier period.

As the fibre acquires its more perfect characters, so its substance becomes condensed: it diminishes in size. This is continually counteracted to some extent by the addition of fresh material and nuclei; but notwithstanding this, the size of the fibre continues for a time to decrease, so that the diameter of a muscular fibre at birth is considerably less than it was during a much earlier period of its existence *.

That this decrease in the size of the fibre is due to a condensation of its substance, and not to stretching or any like cause, seems proved,—1st, by the relative position of the nuclei, for the decrease in breadth is not accompanied by a corresponding separation of the nuclei; and 2ndly, by the fact, that as the fibre decreases it becomes much less transparent. The nuclei in its interior grow much more obscure, and they are at length concealed by the increasing density of the muscular substance.

At the time of birth muscular fibres present considerable variety in size and other characters. The great majority measure in the pig from $\frac{1}{4000}$th to $\frac{1}{2000}$th of an inch in width, but many will be found beyond either of these extremes. The striæ are very plainly marked, and in many almost as large as in the adult fibre; in some, however, they are much finer. In most of the fibres the nuclei in their interior are obscured or quite hidden by the density of their substance, but in others they are still

* Thus, for example, taking average specimens:—

Diameter of fibre of pig at time of birth, $\frac{1}{3000}$th of an inch.

Diameter of fibre of pig 5 inches long, $\frac{1}{2000}$th of an inch.

visible. Nuclei may be seen in all stages of their progress passing from without into the substance of the fibre, and hence the majority appear placed at or near the border of the fibre. Many granules are visible, scattered irregularly throughout the fibre, Plate XI. fig. 12.

After birth the fibres enlarge with a considerably increased rapidity; they very soon attain a large size, and ere long reach the adult condition; but for some time the variation in the diameter of different fibres is very obvious. In the pig, the striæ (sarcous elements?) attain their full size two or three weeks after birth.

In the development of muscular fibre, then, the following stages may be traced:—

Aggregation of cytoblasts, and their investment by surrounding blastema.

Their regular arrangement into a single series. Formation of lateral bands.

The lengthening of the fibre and separation of the nuclei. Approximation of bands. The appearance of striæ.

The further development of the fibre and its growth, by the addition of fresh substance to its exterior, by means of the surrounding nuclei.

Condensation of the fibre.

Now it may have been understood from the foregoing description, that these several stages do not follow one another as a simple consecutive series. On the contrary, two of the above processes are generally proceeding at the same time. For instance, during the whole period of development, fresh material is being continually added to the fibre, at first independently, and afterwards by means of additional nuclei; while the nuclei are rearranging themselves into a single row, fresh blastema is constantly added to that which already invests them. The bands bordering the nuclei are continually increasing by the addition of fresh material; so that, as previously explained, the longer the separation of the nuclei is delayed, the thicker do they become; and in those fibres especially in which many of the nuclei perish before they have separated, the lateral bands attain a very great breadth, and the fibres a very considerable diameter.

So also, as already stated, the fibres at a subsequent period continue for some time to decrease in diameter, although fresh material is being added by means of additional nuclei.

As a general rule, all fresh material which is added to the fibre before the original nuclei separate, is attracted independently of fresh nuclei, but that which is added after the original nuclei have separated or become disintegrated, is by means of the additional nuclei which are attached to the exterior of the fibre. The exceptions occur in those fibres in which the lengthening appears delayed much beyond the usual time, and until after the majority of the original nuclei have disappeared. (See Plate XI. fig. 13.) In these cases nuclei in great abundance are often seen attached to the surface of the fibre. They are frequently so abundant as to be absolutely in contact, as if the same number of nuclei which would have attached themselves to the fibre had it elongated, are now crowded together into the smaller space*.

From the period of its first formation, the substance of the fibre gradually increases in strength and firmness as development advances. This fact is well illustrated in fibres which have been preserved some time in different fluids. The more advanced the development of the fibre, the less readily is its texture destroyed; therefore it is much more difficult to preserve the fibres in their earlier period than in the later stages of development. If fibres which are elongating, and in which the nuclei are separating, are kept for some time in a preservative liquid, and by-and-by examined, it will be noticed, that while the structure of those fibres in which the nuclei have separated to a considerable extent is still perfect and distinct, the substance of those in which the nuclei are still in contact is broken up and confused; and it will be observed that there is a close correspondence between the extent to which the nuclei are separated and the integrity of the fibre.

The rate of development of different fibres in the same muscle is by no means uniform, and some fibres are doubtless formed at a later period than others. If, for instance, the muscular tissue of a pig, between 2 and 3 inches long, be examined, many fibres will be found in very different degrees of development. The majority are in the second, or passing from the second into the third stage, presenting the single row of closely arranged nuclei bordered by lateral bands, while others will be seen in the first stage of formation as elongated groups of nuclei, partially invested by blastema. In any specimen numerous fibres will generally be seen in various degrees of development within certain limits. Even in the same fibre in different portions of its length, this variation may often be detected (see Plates X. and XI. figs. 14, 8 and 9 *g* and *i*), and in fibres at a more advanced period of development, and even for some time subsequently to birth, a considerable difference in size may be often observed. Generally speaking, there appears to be more uniformity in the rate of development of different muscles than of different fibres of the same muscle.

Of the development of the sarcolemma I have little or nothing to say, except that according to my observation it is not formed in the manner originally described by Schwann and more recently by Kölliker.

I have never been able to determine satisfactorily the exact period of its first appearance, nor do I imagine that it is possible to do so. It is probably formed gradually, and only attains its more perfect characters at an advanced period of development.

To obtain a clear idea of the structure of fœtal fibres, and of the changes that occur at different periods of their development, specimens should be examined both

* Perhaps this was the case in the instance which Mr. Bowman has alluded to and represented. It was a specimen taken from the chrysalis of a tiger-moth.—Phil. Trans. 1840.

from perfectly fresh embryos and from others which have been kept for some hours. The changes which occur in the appearance of the fibres during the earlier periods of development, after the embryo is removed from the parent, are very remarkable, and these changes are hastened by the presence of water. The blastema in which the fibres are imbedded, and from which they are formed, is clear, transparent, and apparently structureless, but the material which is recently added to the exterior of the fibre is rather opake and obscure; and the obscurity is increased by the roughness of the surface which refracts the light irregularly, and very often gives a glistening aspect to the fibre. As this blastema is developed into the substance of the fibre it clears up, it grows much more transparent, and its general characters and arrangement can be readily investigated. Now the blastema, when recently added to the exterior of the fibre, and which has not yet assumed the structure of the fibre, adheres but slightly and is readily detached, so that the fibres, which, when fresh, are quite obscure, by simple preservation for a short time, and more especially if placed in water, part with their more recent investment of blastema, and perhaps nuclei, and become clear and transparent. Hence, during the first stage of development, by simple preservation, and still more rapidly and effectually by the action of water, the fibre is altogether broken up. The delicate material investing and binding the nuclei together is easily dissolved, the nuclei separate, and no appearance of the early fibre remains. It is very common, however, in such a case, to see three, four or more nuclei attached together and in close contact, floating freely about, the recently added blastema having been detached from their surface.

At a later period of their development, the action of water not only assists in removing the recently added and undeveloped layer of blastema which obscures the structure of the fibre, but it penetrates the interior, and renders its entire substance much more clear and pellucid, and the nuclei within more transparent.

The Nuclei, which are such important agents in the development and maintenance of muscular fibre, and indeed of tissues generally, claim much more consideration than a passing notice*. These nuclei, when free as cytoblasts, may be observed in various stages of development. The younger ones are round or slightly oval, possessing a dark and distinct outline: their substance has a nebulous or faintly granular aspect. They very generally contain one or two well-marked nucleoli, which appear either as dark or as bright highly refracting spots, according to the focus at which they are viewed. The average diameter of these cytoblasts is from $\frac{1}{5000}$th to $\frac{1}{4000}$th of an inch; the smallest are less than $\frac{1}{6000}$th.

They increase in size as age advances, and at the same time they become more

* If the preceding description of the development of muscular fibre be a true one, it is another striking example of the truth and accuracy of the opinions which Mr. Paget has expressed with regard to the relations and functions of nuclei, and their extreme importance.—Lectures at the Royal College of Surgeons, May 1847. Lecture V. Surgical Pathology.

decidedly oval, generally measuring in their long axis somewhat more than $\frac{1}{2000}$th ($\frac{1}{1700}$th), and in their short axis $\frac{1}{3000}$th. Their outline becomes broader and paler, and is less distinctly defined; their contents grow more plainly granular, and between their margin and contents, that is immediately within their softened outline, a clear bright ring may be observed.

Their characters may in some respects be more easily investigated, and the changes they undergo can be more obviously traced as they lie imbedded in the substance of muscular fibre. The walls of the nuclei are firm and far more resisting than the substance of the fibre; for when the fibre is stretched and narrowed between them, the nuclei remain unaltered in shape and bulge the fibre at the part where they are situated; and it may be constantly observed that when young fibres float freely in liquid they never bend at the points where the nuclei are situated, but always at some part between them. A sudden bend or twist is often seen just beyond the extremity of a nucleus, but the nucleus itself preserves its shape, and renders that portion of the fibre which it occupies comparatively firm.

As the nuclei lie in the substance of the fibre they increase considerably in size, and become fainter in appearance. The change in their shape and outline, and the bright border immediately within their circumference, are generally distinctly traced; they seem at last to be filled with granules, and as these are more plainly developed their walls become broken and obscure, until at last their position is marked by an irregular cluster of granules, which remain very visible for a short time and then rather suddenly disappear.

These granules have many of the characters of minute oil-globules; they appear either as dark or as bright highly refracting spots, with a thick, dark and well-defined border, according to the focus at which they are viewed. When the tissue is preserved in spirit for some time, many of them fuse together into larger globules, which in aspect and other characters completely resemble globules of oil. They are not readily acted on by ether, but the granules disappear in fibres that have been preserved for a short time in it. Except at the latter periods of development, the nuclei can be readily discerned in the substance of the fœtal fibres without any previous preparation, unless the structure of the fibre is obscured by the deposit of fresh and undeveloped material upon its surface. Towards the close of fœtal life, as the tissue grows more dense, the nuclei are discerned with greater difficulty; they become obscured by the substance of the fibre; in either case, if the tissue be preserved for a day or two, or if placed for a short time in water, the muscular substance is rendered much clearer and the nuclei become distinctly visible. The same plan of simple maceration in water succeeds so well, even with muscular fibres at a far more advanced period of development, that the nuclei they contain may be thus beautifully shown. In a day or two the substance of the fibre becomes clear and transparent and the nuclei appear; the advantage of this simple plan is, that the nuclei remain unaltered: they preserve their natural appearance. The addition of weak acids, which is recommended for the purpose of bringing the nuclei into view, always affects their appearance by altering their shape to an extent proportioned to the strength of the re-agents. They shrink, their outline becomes more or less disturbed, and they appear shrivelled; hence many of the descriptions and drawings of these nuclei are unfaithful. An artificial appearance, the effect of re-agents, is described and figured as the natural one.

However much the shape and other characters of these nuclei may be affected by various re-agents, they are not easily destroyed. When muscular tissue is preserved in alcohol, they shrink, but remain for a long time; at last, however, they become indistinct, they appear like large and irregular oil-globules, and finally break up. By the prolonged action of sulphuric ether, the nuclei become irregular in outline and somewhat shrivelled, and their substance presents a uniform clear and glistening appearance.

These and similar effects appear in great measure due to time, and this ultimate disintegration of the nuclei, under almost all circumstances, is the chief obstacle to the preservation of fœtal muscular fibre; but generally, as the nuclei become altered, the whole structure of the fibre undergoes a change. The striæ, when they exist, become confused and then disappear, and the interior of the fibre is occupied by globules of different size, closely resembling globules of oil.

Explanation of the Plates.

PLATES X. and XI.

Fig. 1. Cytoblasts from the tissue of the dorsal region of a fœtal pig, 1 inch in length.

Fig. 2. Portion of muscular fibres from a fœtal pig, between 1 and 2 inches in length. The first stage in the formation of muscular fibre.

Fig. 3. Muscular fibre from a fœtal pig, between 2 and 3 inches long, showing the linear arrangement of the nuclei and narrow bands.

Fig. 4. A similar specimen. The nuclei at one extremity have not fallen into their places, and the lateral bands are separated to a corresponding extent.

Fig. 5. Fibre from a fœtal pig, $3\frac{1}{2}$ inches long. Lateral bands increasing in width. Breadth of fibre $\frac{1}{1500}$th of an inch.

Fig. 6 *a*. Fibres from the upper extremity of the leg of a *Tipula*.

Fig. 6 *b*. Fibres from the upper extremity of the leg of a blue bottle fly (*Musca vomitoria*).

Fig. 7. Fibre from a fœtal pig, 3 inches long, showing separation of the nuclei.

Fig. 8 *a*. Fibre from a fœtal pig, $3\frac{1}{2}$ inches long, showing the progress of development in different portions. Breadth of fibre, where the nuclei are in contact, $\frac{1}{1800}$th of an inch; where nuclei are $\frac{1}{1000}$th of an inch apart, breadth of fibre $\frac{1}{3500}$th of an inch.

This fibre floated freely in the field.

Fig. 8 *b*. A smaller specimen from a fœtal pig, $2\frac{1}{2}$ inches long.

Fig. 9 *a*. Fibre from a fœtal pig, 4 inches long. Nuclei in progress of dissolution. Breadth of fibre $\frac{1}{1500}$th of an inch.

Fig. 9 *b*. Fibre from the same pig, extending; remaining nuclei and granules separating. Breadth of fibre $\frac{1}{2000}$th of an inch.

Fig. 9 *c*. Fibre from a fœtal pig, 5 inches long. Nuclei not separated, but perishing in their places. Breadth of fibre $\frac{1}{1100}$th of an inch; of lateral bands $\frac{1}{3100}$th of an inch.

Fig. 9 *d*. Fibre from a fœtal pig, $5\frac{1}{2}$ inches long. Nuclei broken up into clusters of granules.

Fig. 9 *e*. Fibre from the same pig, extending. Breadth of fibre $\frac{1}{1500}$th of an inch. Clusters of granules $\frac{1}{4000}$th of an inch apart.

Fig. 9 *f*. Fibre from the same pig, still further extended. Clusters of granules separating and elongating. Breadth of fibre $\frac{1}{2000}$th of an inch.

Fig. 9 *g*. Fibres from the same pig, showing different progress of development in different parts. Nuclei breaking up into granules.

Fig. 9 *h*. Fibre from a fœtal pig, $3\frac{1}{2}$ inches long. Distinction between lateral bands and centre well marked, faint appearance of striæ in bands. Breadth of fibre $\frac{1}{2000}$th of an inch. Nuclei round, $\frac{1}{4500}$th of an inch, separating, and about $\frac{1}{1500}$th of an inch apart. Breadth of bands $\frac{1}{9000}$th of an inch.

Fig. 9 *i*. Fibre from a fœtal pig, 4 inches long. Shows unequal rate of development in different portions.

Fig. 10. Fibre from a fœtal pig, $3\frac{1}{2}$ inches long. Nuclei separated; striæ visible; breadth of fibre $\frac{1}{3000}$th of an inch.

Fig. 11 *a*. Fibre from a fœtal pig, 8 inches long; showing the method of increase by external nuclei in various stages.

Fig. 11 *b*. Another fibre from the same pig.

Fig. 11 *c*. Another fibre from the same specimen. Striæ well marked. These fibres vary in diameter from $\frac{1}{5000}$th to $\frac{1}{3000}$th of an inch.

Fig. 11 *d*. Fibres from the same pig, after the addition of citric acid.

Fig. 11 *e*. Free nuclei from the same specimen.

Fig. 12. Fibres from a pig at the period of birth.

Fig. 13 *a*. Fibre from a fœtal pig, 5 inches long. Primary nuclei separating. Increase by external nuclei. Breadth of fibre $\frac{1}{2000}$th of an inch.

Fig. 13 *b*. Fibre from a fœtal pig, 4 inches long, increasing by means of external nuclei. Breadth of fibre from $\frac{1}{2000}$th to $\frac{1}{5000}$th of an inch.

Fig. 14. Fibre from fœtal pig, $2\frac{1}{2}$ inches long, showing irregular rate of development. At one extremity the nuclei have scarcely become arranged in a linear series; while at the other the nuclei have separated, and the striæ are faintly visible.

The figures just described have been drawn from the fibres of the fœtal pig.

Now with regard to the comparative dimensions of many of these fibres, it must be observed that the size of the fœtus did not always correspond to the degree of its development, and this for two reasons. First, growth and development do not always proceed at an equal or uniform rate. Secondly and chiefly, it was impossible to obtain a sufficient supply of fœtal pigs from parents of the same size. I had no means of choosing in this respect, and received specimens both from small and large sows. Moreover, I could not generally ascertain the exact age of the fœtus examined, and have, therefore, to avoid error, only spoken of its length.

These facts will explain the results of some measurements, which might otherwise appear to be contradictory.

St. Bartholomew's Hospital,
November 1854.

XI. *Researches on Organo-metallic Bodies.*—Second Memoir. *Zincethyl.*

By E. Frankland, *Ph.D., F.R.S.,*
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Received February 9,—Read March 15, 1855.

In a former memoir on organic compounds containing metals*, after describing the preparation, composition, and properties of zincmethyl, I mentioned the corresponding formation of the homologous compounds zincethyl and zincamyl; but although the composition of these latter bodies was to a certain extent fixed, by the study of the products of their decomposition in contact with water, yet the difficulty of procuring them in sufficient quantity, by digesting the iodides of methyl, ethyl, and amyl, with zinc, in strong glass tubes at high temperatures, was so great, that I could neither extend their investigation further, nor succeed in rendering even the history of zincmethyl quite complete. In pursuance, however, of the intention announced in the memoir just alluded to, I subsequently commenced studying the action of these bodies upon certain organic compounds containing chlorine and other electro-negative elements, with a view to replace these elements by the groups $C_2 H_3$, $C_4 H_5$, &c., but meeting with some unexpected results, it appeared to me highly desirable, first to complete the history of at least one of the organo-zinc compounds, before proceeding further with the substitution experiments.

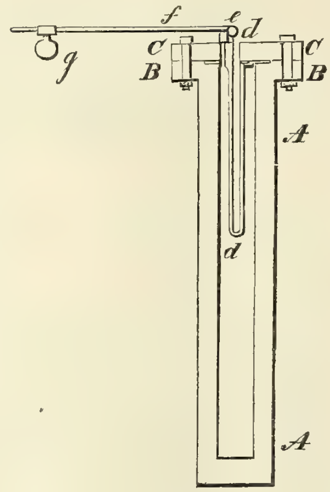
From the comparative facility with which the materials for the preparation of zincethyl can be procured, I selected this body as the subject of my experiments; and as it was necessary to prepare it in much larger quantities than heretofore, which could not be accomplished without great danger in sealed glass tubes heated in an oil-bath, I availed myself of the proximity of Mr. Nasmyth to get constructed two vessels of enormous strength, in which these operations might be conducted, on a tolerably large scale, without risk of dangerous explosions; and I take this opportunity of thanking that eminent engineer for the very kind and effectual manner in which he carried out my views with regard to these instruments, both of which are quite indispensable for the prosecution of researches with organo-metallic bodies in general, but especially with those containing zinc.

The one I will first describe is employed as a Papin's digester, and sealed glass tubes can be heated in it to any temperature below redness. It is constructed entirely of Low Moor wrought iron, and consists of a cylinder, A, A (fig. 1), closed

* Transactions of the Royal Society, 1852, p. 417.

at the bottom, and welded in one piece by the steam-hammer. This cylinder is $18\frac{1}{2}$ inches long, $\frac{5}{8}$ inch in thickness, and 3 inches internal diameter; it is furnished at top with a flanch, B, B, $1\frac{3}{8}$ inch broad and $\frac{5}{8}$ inch thick, its upper surface turned true, and having an internal annulus sunk $\frac{1}{20}$ inch below the level of the surrounding surface. The cap of the digester, C, C, is made to fit upon this flanch, with which it corresponds in thickness and diameter; it is furnished with a projecting face $\frac{1}{4}$ inch deep, fitting the mouth of the cylinder exactly. Within the circle of this projecting face, the cap is perforated by two apertures, into one of which is securely fixed the cast-iron tube *d d*, closed at the bottom, 6 inches long and $\frac{1}{2}$ inch internal diameter, forming a mercury bath for the reception of a thermometer. The other aperture, which is bouched with brass, serves as the bed of the safety valve *e*, which consists of a piece of brass wire $\frac{1}{8}$ inch diameter, slightly flattened on two sides and furnished with a head accurately ground to the surface of the cap: pressure is applied to this valve in the usual manner by the lever and weight *f*, *g*. Both the flanch and cap are perforated by four holes for the reception of four screw bolts $\frac{1}{2}$ inch in diameter, which are inserted from below, and work into nuts that can be tightly screwed up by a lever key. The whole of the pressure produced by these screw bolts is made to take effect exclusively upon the surface of a leaden washer $\frac{1}{8}$ inch thick, placed in the sunken annulus above mentioned; and thus the apparatus is made perfectly impervious to gases and vapours, even under the enormous pressure of more than 100 atmospheres. Before use, this vessel was proved, by being two-thirds filled with water, and then gradually heated up to the melting-point of lead. It has since been exposed with impunity to a still more severe test; for on one occasion, when charged with water and a glass tube filled with iodide of methyl, at a temperature of 200° C. the glass tube burst, and such was the tension of the iodide of methyl vapour, that the safety valve was instantaneously expelled, and the heavily loaded lever thrown completely over. In this digester, volatile liquids enclosed in glass tubes of large dimensions and moderate thickness of glass, may be exposed to any temperature below redness with safety. I prefer to use water in the digester, but other and less volatile liquids may of course be substituted if desired: in most experiments, however, it is important that the pressure upon the exterior of the glass tubes should not be much less than that in their interior, and this condition is generally secured by the employment of water in the apparatus.

Fig. 1.



The second digester is made of wrought copper and is of smaller dimensions, being especially designed for the preparation of large quantities of organo-zinc compounds, without the intervention of glass tubes. It consists of a wrought copper tube, A, A (fig. 2), 18 inches long, $1\frac{1}{4}$ inch internal diameter, and $\frac{1}{2}$ inch in thickness, drawn from a solid mass of the metal by a recently invented process. This tube is closed at bottom by a screw plug, and is furnished at top with a brass flanch, B, B, $1\frac{5}{8}$ inch broad and 1 inch thick, screwed upon the copper tube: the vessel thus formed is closed by the brass cap C, C, of the same dimensions as the flanch upon which it fits. The cap is furnished with a central projection 1 inch deep, fitting the copper tube, and pierced with a central aperture D, tapped to receive the screw plug E, which, with the intervention of a collar of lead, effectually closes the aperture. The cap C C is secured to the digester by three screw bolts $\frac{1}{2}$ inch in diameter, which are inserted from above and screw into the lower flanch. Perfect impermeability to gases is secured, as in the iron digester, by a sunken annulus and a ring of lead. When it is desirable to collect the gases evolved during any operation in this digester, the plug E is replaced by a carefully made stopcock, to the nozzle of which a gas-delivery tube can be attached, when the reaction is completed. This digester is heated by means of a cylindrical copper oil-bath placed in a gas-stove, as shown in fig. 3. The gas-stove consists of a strong wrought-iron framework, A, A, around which is fixed the sheet-iron cylinder B, B closed at the bottom, but furnished with a draught regulator C, and contracted at top by a ring of sheet iron so as just to admit the cylindrical oil-bath D, D, the flanch of which rests upon the upper extremities of the wrought-iron framework, which are turned inwards for this purpose. The sheet-iron cylinder is surrounded by another of polished tin plate to prevent the too rapid radiation of heat; there is an interval of half an inch between the two cases, and both are pierced with holes at E for the exit of the products of combustion. A $\frac{1}{4}$-inch copper pipe, F, pierced with eighteen or twenty small apertures, forms the gas-burner. By this arrangement, it is easy to maintain an almost constant degree of heat for any length of time. The temperature is ascertained by the thermometer f, immersed in the oil-bath through an aperture bored in the cap and flanch of the digester for this purpose. This gas-stove is also used for heating the iron digester, but without the intervention of the oil-bath. In this copper digester, which is capable of resisting enormous pressure, it is easy to prepare four or five ounces of zincethyl at one operation.

Fig. 2.

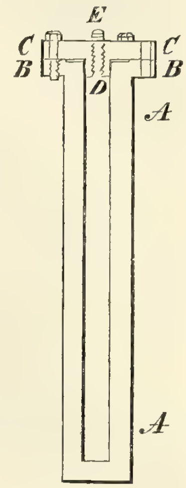
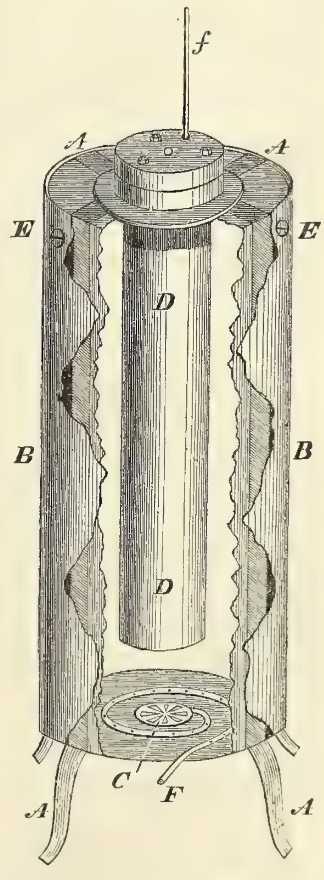


Fig. 3.



The zincethyl used in the experiments detailed below, was prepared in the following manner. The copper digester, well cleaned, dried, and heated to about 150° C., was charged with four ounces of finely granulated zinc, which had been previously heated to the same temperature, for at least half an hour, in order to get rid of every trace of moisture, the presence of which, in the ingredients used for the preparation ot organo-zinc compounds, must be most carefully avoided, as all these compounds are instantaneously decomposed by contact with water into oxide of zinc and hydride of the organic radical: extraordinary care therefore, in freeing the materials perfectly from moisture, is amply repaid in the increased quantity of the product. As soon as the hot zinc had been introduced into the digester and slightly rammed down, the cap C, C was securely screwed into its place, the plug E inserted, and the apparatus allowed to become cold. Two ounces of iodide of ethyl, by measure, were then mixed with an equal bulk of ether*, which had been well washed and subsequently distilled several times from anhydrous carbonate of potash. About 100 grains of anhydrous phosphoric acid were then added to this mixture, which was well agitated and allowed to stand for half an hour: the phosphoric acid gradually cohered into a porous gummy mass, from which the ethereal mixture could be readily decanted and poured into the digester through a funnel inserted in the aperture D. The plug E being then firmly screwed down, the digester was placed in the oil-bath, and maintained at a temperature of about 130° C. for twelve or eighteen hours: at the end of this time traces only of iodide of ethyl remained undecomposed.

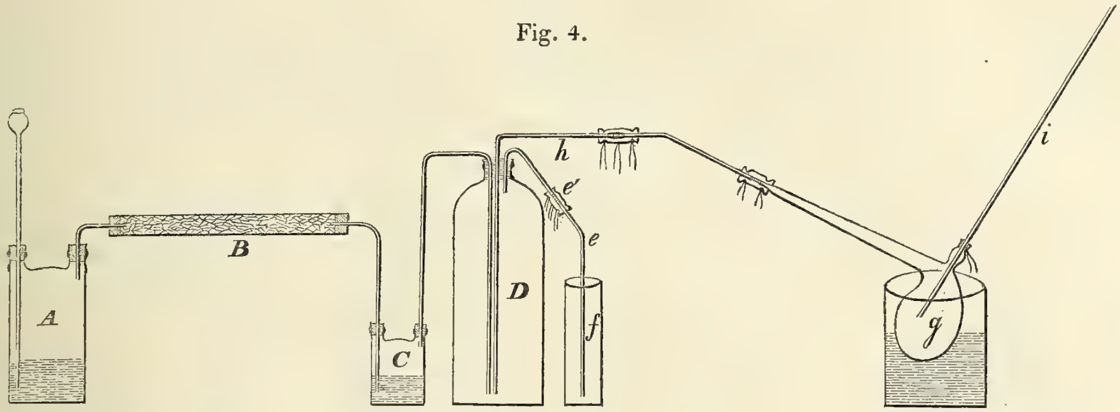
After the apparatus had been allowed to cool, the screw E was loosened, to allow a quantity of gas, principally hydride of ethyl, to escape; if all the materials were carefully freed from moisture, the quantity of gas was insignificant, but if moisture were present it was much larger, being derived, as above stated, from the decomposition of zincethyl. When the gas had escaped, the screw was removed and replaced by a cork and bent tube, conveniently arranged for the distillation of the volatile contents of the digester.

As zincethyl is spontaneously inflammable in air and instantaneously decomposed by water, it is necessary that the vessels into which it is distilled should be kept constantly filled, either with dry hydrogen, nitrogen, or carbonic acid gas; the latter I have found most convenient in practice. The maintenance of an atmosphere of this gas can be easily effected by the following arrangement of apparatus, fig. 4. A is a Woulf's bottle for the generation of the carbonic acid, which streams through a chloride of calcium tube B, and Woulf's bottle C containing concentrated sulphuric acid, before entering the reservoir D, which should be of about two quarts' capacity, and contain a stratum of concentrated sulphuric acid about one inch deep. From this reservoir the dry gas can be allowed to escape, either by the tube *e* into the vessel *f*, or through the retort *g* by the tube *h*.

* I have already pointed out that the admixture of ether almost entirely prevents the elimination of gases during the reaction (Journ. Chem. Soc. vol. ii. pp. 293 and 298), and Brodie (Journ. Chem. Soc. vol. iii. p. 409) has since shown that it greatly facilitates the production of zincethyl, nearly the whole of the iodide of ethyl being transformed into this body.

When all is prepared for the distillation of the zincethyl, the reservoir D being filled with carbonic acid, and the tube *i* from the digester (which latter still remains

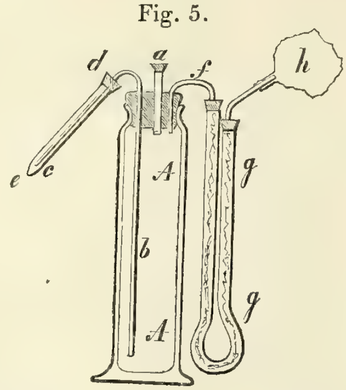
Fig. 4.



in the oil-bath and gas-stove) inserted into the tubulure of the retort *g*, the stream of carbonic acid, which has hitherto flowed through *f*, is cut off at *e'* by a caoutchouc valve, and is thus made to flow through *g*, making its escape between the tubulure and the tube *i*. After the carbonic acid has taken this route for a sufficient length of time to ensure the complete expulsion of air from *g*, a piece of sheet caoutchouc is passed round the tubulure of *g* and the tube *i*, so as to make the union gas-tight, whilst at the same moment an assistant opens the caoutchouc valve *e'*; thus allowing the carbonic acid again to escape through *f*. The retort *g* is thus connected with a reservoir kept constantly filled with pure and dry carbonic acid, which effectually excludes atmospheric oxygen and moisture from *g* during the distillation. Heat being now applied to the digester, by means of the gas-stove and oil-bath, ether begins to distil over, so soon as the temperature of the digester exceeds the boiling-point of that liquid. The first half-ounce is nearly free from zincethyl and might be collected apart, but I generally prefer to receive the whole product of the distillation in one vessel. As the temperature rises to 140° or 150° C. the product becomes more and more rich in zincethyl, but a heat of 190° C. is required to bring over the last portions of this body, a large quantity of which is apparently in some form of combination with the iodide of zinc, and can only be expelled at a very much higher temperature than the boiling-point of zincethyl. During the last stages of the distillation, there is a slow evolution of gas, due to the decomposition of a small portion of the zincethyl by the high temperature required for its complete expulsion.

After the whole of the volatile products have passed over, the tube *i* must be carefully withdrawn from *g* and immediately replaced by a thermometer passing through a well-dried cork. The beak of the retort, disconnected from the tube *h*, must now be inserted air-tight into a suitable tubulated receiver, previously filled with dry carbonic acid, the tubulure of which is connected with the carbonic acid receiver D, and the rectification may now be commenced. The liquid begins to boil at about 60° C., but the thermometer gradually rises until about three-fourths have passed over, when it becomes stationary at 118° C. The receiver must now be changed, the retort being first allowed to become quite cold. On the subsequent application of heat, the whole of the remaining liquid distils over at 118° C. This is pure zincethyl, and may be received in a peculiarly constructed vessel, from which it can be conveniently expelled in small quantities as required.

Fig. 5.



This vessel, which is a kind of syringe glass, is represented by fig. 5. A, A is a tall glass cylinder with a slightly contracted mouth, well fitted with a sound and thoroughly dry cork coated with gutta percha varnish. Through this cork pass three tubes: one *a*, terminates just below the cork and projects about 2 inches above the mouth of the cylinder; it is about a quarter of an inch internal diameter, and serves to receive the drawn-out beak of the retort *g* (fig. 4) during the distillation of the pure zincethyl: when the distillation is completed, *a* is closed by a cork covered with sealing-wax. Another tube, *b*, one-sixteenth of an inch bore, passes to the bottom of A; it is bent above at an acute angle, and, after passing through a small cork *d*, terminates at *c* in a capillary orifice. The cork *d* serves to close another small tube, *e*, which encases *b*, and protects its capillary orifice from the atmosphere. The third tube, *f*, connects the upper part of A, A with a chloride of calcium tube *g*, *g*, to the opposite extremity of which the thin caoutchouc globe *h* is attached. By removing the tubular cap *e* and pressing *h*, which is previously filled with carbonic acid, it is thus easy to transfer any required quantity of the liquid in A to any other vessel.

To determine the composition of zincethyl, a quantity was transferred into a small glass bulb with a very narrow mouth, previously filled with dry carbonic acid. A number of smaller glass bulbs, capable of containing about 5 decigrammes of the liquid and having long capillary tubes attached, were then filled with carbonic acid by being repeatedly heated and cooled in an atmosphere of this gas; these bulbs were then partially filled in the usual manner with zincethyl from the larger bulb above mentioned, and the increase of weight being noted, they served for the following quantitative determinations:—

I. ·2578 grm., burnt with oxide of copper and oxygen, gave ·3653 grm. carbonic acid and ·1915 grm. water.

II. ·2443 grm., similarly treated, gave ·3495 grm. carbonic acid and ·1799 grm. water.

III. ·3139 grm., slowly decomposed in a graduated tube filled with slightly acidulated water, gave ·2045 grm. oxide of zinc and ·1459 grm. hydride of ethyl, as deduced from the annexed observations:—

	Observed volume.	Height of inner column of water.	Temperature.	Height of barometer.	Corrected volume at 0° C. and 760 mm pressure.
Volume of Hydride of Ethyl	cub. cent. 113·38	mm 178·5	15·0° C.	mm 766·0	cub. cent. 108·23

These determinations agree with the formula

$$C_4 H_5 Zn,$$

as is seen from the following comparison :—

	Calculated.		Found.			
			I.	II.	III.	Mean.
C_4	24·00	39·01	38·64	39·02	44·93 (C and H)	38·83
H_5	5·00	8·13	8·25	8·15		8·20
Zn	32·52	52·86	——	——	52·27	52·27
	61·52	100·00			97·20	99·30

In experiment No. III. the volume of hydride of ethyl is considerably less than that required by the above formula, owing no doubt to its solubility in the acidulated water used as the confining fluid; in fact a similar error, to a less extent, occurred in the corresponding experiment with zincmethyl*; but to place the cause of error beyond all doubt, the following additional determination was made. Into a graduated eudiometer tube, filled with mercury, an indefinite quantity of zincethyl, enclosed in a glass bulb, was introduced, the capillary stem of the bulb being broken inside the eudiometer tube. A few drops of water were now passed up into the same tube, and the apparatus was exposed to a heat gradually raised to the boiling-point of zincethyl, the conversion of which into hydride of ethyl and oxide of zinc was thus perfectly effected, whilst a few drops of water only exerted an absorptive action upon the former. After the apparatus had again become cold, the following observations were made :—

	Observed volume.	Temperature.	Difference of mercury level.	Height of barometer.	Corrected volume at 0° C. and 760 mm pressure.
Volume of Hydride of Ethyl	cub. cent. 114·62	13·5° C.	mm 135·0	mm 763·2	cub. cent. 88·61

The oxide of zinc, carefully dissolved from the interior of the eudiometer and from the mercury, by dilute hydrochloric acid, re-precipitated as basic carbonate and then ignited, weighed ·1603 grm.

These numbers correspond almost exactly with those which ought to be obtained from zincethyl :—

* Philosophical Transactions, 1852, p. 431.

	Calculated.		Found.
C_4H_5 . . .	29·00	47·14	47·33
Zn	32·52	52·86	52·67
	61·52	100·00	100·00

At ordinary temperatures zincethyl is a colourless, transparent, and mobile liquid, refracting light strongly and possessing a peculiar odour, rather pleasant than otherwise, and therefore differing greatly from that of zincmethyl. Its specific gravity is 1·182 at 18° C. Exposed to a cold of −22° C., it exhibited no tendency to become solid. Zincethyl boils at 118° C., and distils unchanged. The specific gravity of its vapour is 4·259, according to the following determination by GAY-LUSSAC's method:—

Weight of zincethyl used	·3103 grm.
Observed volume of vapour	106·0 cub. cent.
Temperature	148° C.
Height of inner column of mercury	129·0 mm.
Height of barometer	764·8 mm.
Volume of residual gas* at 0° C. and 760 mm pressure . . .	2·78 cub. cent.
Corrected volume of vapour at 0° C. and 760 mm pressure .	56·07 cub. cent.
Specific gravity of vapour	4·259

Zincethyl vapour, therefore, consists of one equivalent of zinc vapour and one volume of ethyl gas, the two being condensed to one volume:—

1 volume of ethyl gas	2·0039
1 equivalent of zinc vapour	2·2471
1 volume of zincethyl vapour	4·2510
Found by experiment	4·259

This vapour volume of zincethyl is highly remarkable, and almost compels us to conclude, that the vapour volume of the double atom of zinc is only equal to that of oxygen, instead of corresponding with the volume of hydrogen, in accordance with the generally received supposition. Zincethyl, therefore, appears to belong to the so-called water type, and to consist of two volumes of ethyl and one volume of zinc vapour, the three volumes being condensed to two: for if we were to assume that an equivalent of zinc occupies the same vapour volume as an equivalent of hydrogen, we

* This residual gas consisted of hydride of ethyl derived from the decomposition of a portion of the zincethyl by traces of moisture adhering to the tube or mercury, the total removal of which appears to be impossible. The space occupied by this hydride of ethyl would obviously be exactly double that occupied by the zincethyl, from which it was derived. The observed and corrected volume of zincethyl vapour (57·46 cub. cent.) was consequently too great by $\frac{2·78}{2}=1·39$ cub. cent., and hence the true volume of ·3103 grm. of zincethyl vapour was, as above stated, 56·07 cub. cent.

should then have the anomaly of the combination of equal volumes of two radicals being attended by condensation. The determination of the specific gravity of zincethyl vapour supplies us with the first datum respecting the vapour volumes of the class of metals to which zinc belongs. It has hitherto been assumed, that the vapour volume of these metals is equal to that of hydrogen; but the vapour volume of zincethyl goes far to contradict this assumption.

Although zincethyl is remarkable for the intense energy of its affinities, which place it nearly at the head of the list of electro-positive bodies, yet it does not appear to be capable of forming any true compounds with electro-negative elements, its reactions being all double decompositions in which the constituents of the zincethyl separate. When a few drops of zincethyl, diluted with ether to prevent inflammation, are passed into a mercurial eudiometer containing dry atmospheric air, a rapid absorption of oxygen takes place, with the formation of a white amorphous solid composed of zinc, ethyl, and oxygen. This reaction, which is also common to zincmethyl and zincamyl, led me to suppose, that, like cacodyl, these bodies combined directly with oxygen*; but the results of the action of oxygen upon zincethyl detailed below, prove that no such compound is formed; the white body being ethylate of zinc, and not an organo-metallic compound, in the strict sense of the term.

Action of Oxygen upon Zincethyl.

When zincethyl is brought into contact with oxygen or atmospheric air, it instantly ignites, burning with a brilliant blue flame, fringed with green, and evolving dense clouds of oxide of zinc; a cold body held in this flame, becomes coated with a black deposit of metallic zinc, surrounded by a white ring of oxide of zinc. In this rapid action of oxygen, all the elements of zincethyl are attacked, and the products are oxide of zinc, water, and carbonic acid; but if the zincethyl be placed in a vessel filled with dry carbonic acid and immersed in a freezing mixture, oxygen gas may be slowly admitted without inflammation taking place. At first the absorption of oxygen is rapid and attended with the production of white fumes, which fall to the bottom of the vessel in the form of a white powder; but the action soon becomes greatly retarded, owing to the formation of a solid crust upon the surface of the zincethyl, protecting the latter from further contact with oxygen, so that after the lapse of weeks, or even months, the oxidation is very imperfect. During the progress of the oxidation, a small quantity of a black substance, resembling finely-divided metallic zinc, is deposited from the liquid; but the amount of this deposit is so small, that I did not succeed in obtaining more positive evidence of its nature.

As this mode of oxidizing zincethyl in a pure state was found to be incapable of yielding results from which the nature of the action could be readily deduced, it was modified by mixing the zincethyl with about three times its bulk of anhydrous ether, and then submitting it, as before, to a slow current of oxygen. At first the absorp-

* Philosophical Transactions, 1852, p. 431.

tion of the gas took place rapidly, and was accompanied, as in the case of pure zinc-ethyl, with the formation of white fumes; these however soon ceased, and a white precipitate began to be deposited; but, by frequent agitation of the vessel, the formation of impervious crusts was prevented, and thus the oxidation, though slow, was continuous, and became complete in four days. During the latter part of the oxidizing process, and after the disappearance of the white fumes, a considerable quantity of inflammable gas, having the properties of hydride of ethyl, was evolved. When the oxidation appeared complete, the current of dry oxygen was replaced by one of atmospheric air, which was continued for two days longer, until the whole of the ether had volatilized. The solid product left in the flask was then transferred to a well-stoppered bottle placed in a closed receiver over sulphuric acid. It presented the appearance of a white porous amorphous substance, light and friable, possessing a very slight, but peculiar and agreeable ethereal odour. Heated in a close vessel it suffered no change until the temperature reached 90° C., when a sudden and almost explosive generation of volatile matters occurred, leaving a dirty yellow-coloured solid residue, which suffered no further change up to 150° C., but which, before attaining a red heat, evolved a considerable quantity of gas burning with a blue flame. Exposed *in vacuo* over sulphuric acid for twelve hours and then submitted to analysis, the product of oxidation yielded the following results:—

I. ·6285 grm., burnt with oxide of copper, gave ·5840 grm. carbonic acid and ·3027 grm. water.

II. 1·1257 grm. gave 1·0534 grm. carbonic acid and ·5363 grm. water.

III. ·9635 grm., treated with dilute hydrochloric acid, in which it dissolved, was precipitated boiling by carbonate of soda, and the precipitated basic carbonate of zinc washed, dried, and ignited: it yielded ·5040 grm. oxide of zinc.

IV. 1·1205 grm., cautiously ignited with access of air, gave ·5849 grm. oxide of zinc.

These results lead to the following per-centage numbers:—

	I.	II.	III.	IV.	Mean.
C	25·34	25·52	——	——	25·43
H	5·35	5·29	——	——	5·32
ZnO	——	——	52·58	52·20	52·39
O	——	——	——	——	16·86
					100·00

These figures correspond with no probable formula, and evidently denote the white product of the oxidation of zincethyl to be a mixture and not a pure substance. It is worthy of remark, however, that the relative atomic proportion of carbon and hydrogen is nearly the same as in ethyl:—

Analysis No. I.

Atoms of Carbon : atoms of Hydrogen = 4 : 5·06.

Analysis No. II.

Atoms of Carbon : atoms of Hydrogen = 4 : 4·98.

As this substance is scarcely soluble in either ether or absolute alcohol, and is decomposed by water as well as by heat, there appeared no possibility of separating its proximate constituents, and I therefore sought for some further clue to its nature in the products of its decomposition. A preliminary experiment showed that water acted upon it energetically, the solid became transitorily coloured yellow, a peculiar odour was developed, similar to that produced when iodine is dissolved in a solution of a caustic alkali, and the water became impregnated with alcohol, for on being treated with acetate of potash and sulphuric acid, it gave the characteristic odour of acetic ether. After washing the solid residue of this reaction with a small quantity of water, and drying *in vacuo*, it yielded a mere trace of carbonic acid on ignition with oxide of copper, and was found to be pure oxide of zinc. This behaviour afforded strong evidence, that one of the chief constituents of oxidized zincethyl is ethylate of zinc,

$$ZnO\,C_4\,H_5\,O.$$

In order to ascertain more clearly the change which contact with water produced in the oxidized product, a portion of the same specimen, which yielded the above analytical results, was saturated with water and exposed over sulphuric acid *in vacuo* until it ceased to lose weight. Submitted to analysis it then yielded the following results :—

V. ·6336 grm., burnt with oxide of copper and oxygen, gave ·1393 grm. carbonic acid and ·1412 grm. water.

VI. ·6678 grm., cautiously ignited with access of air, gave ·4766 grm. oxide of zinc.

These results agree with the formula

$$ZnO, HO, C_4\,H_3\,O_3 + 6ZnO\,HO,$$

as will be seen from the following comparison :

	Calculated.		Found.
C_4	24·00	6·04	5·99
H_{10}	10·00	2·51	2·48
7ZnO . . .	283·64	71·33	71·37
O_{10}	80·00	20·12	20·16
	397·64	100·00	100·00

Although the occurrence of acetate of zinc and hydrated oxide of zinc in the above atomic proportions is probably only accidental, yet a comparison of these results with those of analyses Nos. I., II., III. and IV. prove that the action of water upon the oxidized product derived from zincethyl, consists in the replacement of ethyl by hydrogen or ether by water, and therefore confirms the conclusion drawn from its quantitative reactions, that it contains ethylate of zinc, which is no doubt decomposed by water in a manner quite analogous to the decomposition of the ethylates of potash and soda,

$$\left.\begin{matrix} ZnO\,C_4\,H_5\,O \\ 2HO \end{matrix}\right\} = \left\{\begin{matrix} C_4\,H_5\,OHO \\ ZnO\,HO. \end{matrix}\right.$$

When we consider the mode in which the oxidation of the zincethyl was effected, and the ease with which ether passes into acetic acid in the presence of free oxygen, there can scarcely be a doubt that the product of that oxidation consisted of a mixture of ethylate of zinc, acetate of zinc, and hydrate of zinc,

$$C_4\,H_5\,O\,ZnO$$
$$C_4\,H_3\,O_3\,ZnO$$
$$HO\,ZnO,$$

and that the following equations explain their formation, as well as that of hydride of ethyl, which was observed to escape during the latter stages of the oxidation :—

$$C_4\,H_5\,Zn + O_2 = C_4\,H_5\,O\,ZnO$$
$$C_4\,H_5\,Zn + C_4\,H_5\,O + O_4 = ZnO\,C_4\,H_3\,O_3 + C_4\,H_5\,H + HO$$
$$C_4\,H_5\,ZnO + 2HO = ZnO\,HO + C_4\,H_5\,OHO.$$

I conceive that the primary action of the oxygen is expressed by the first equation, and that this action alone goes on, so long as the vapours of zincethyl continue to diffuse themselves into the atmosphere of the flask in which the reaction takes place, and instantaneously absorb the oxygen as fast as it is admitted; but that so soon as that stage of the process is arrived at, where the zincethyl no longer forms vapour, which occurs when the white fumes cease, then the reaction expressed in the second equation commences, for it was proved, that from this time, free oxygen and ethyl vapour coexisted in the flask; but as it is difficult to conceive, that hydrated oxide of zinc can exist in the presence of zincethyl, it is probable that the reaction expressed by the third equation only occurs after the whole of the zincethyl has been oxidized.

Analyses Nos. I., II., III. and IV. indicate that the product of the oxidation of zincethyl has the following per-centage composition :—

Ethylate of zinc	68·28
Acetate of zinc	16·70
Hydrate of zinc . . .	15·02
	100·00

The results of the separate analyses Nos. I., II., III. and IV. agree very nearly with those which would be yielded by such a mixture, as is seen from the following comparison:—

	Calculated.	I.	II.	III.	IV.	Mean.
C . .	25·52	25·52	25·30	——	——	25·43
H . .	5·27	5·29	5·35	——	——	5·32
ZnO .	52·47	——	——	52·58	52·20	52·39
O . .	16·74	——	——	——	——	16·86
	100·00					100·00

Nevertheless, the behaviour of the body obtained by first saturating the oxidized product with water, and then drying over sulphuric acid *in vacuo*, convinces me that this body contains some other ingredient in addition to basic acetate of zinc; for on gradually heating it to 100° C., it exhaled a peculiar ethereal odour and became of a bright yellow colour; this ingredient is present, however, in small quantity only, and I have not succeeded in isolating it; possibly it is either aldehyde, or some product of the action of zincethyl upon acetic acid.

The most interesting reaction in the oxidation of zincethyl, and that to which I therefore directed my chief attention, is the formation of ethylate of zinc; not as regards the new compound of ether and zinc itself, but on account of the extraordinary *modus operandi* of the oxygen in its production. That ethyl, a body low down in the electro-positive series, should in this way unite with oxygen in the presence of a large excess of the intensely electro-positive zincethyl, is so remarkable and unexpected an occurrence, that such a view of the reaction could not be credited, unless supported by additional evidence more conclusive than that already advanced; especially as the formation of ethylate of zinc admits of an equally simple explanation on the supposition, that the ether it contains is derived from that employed to dilute the zincethyl, and that the carbohydrogen of the zincethyl is evolved either in its integral state, or transposed, as is usually the case, into hydride of ethyl and olefiant gas, thus—

$$C_4H_5Zn+O+C_4H_5O=C_4H_5O\,ZnO+C_4H_5.$$

The oxidation of pure zincethyl would have decided this important question; but as it is impossible to obtain a perfect oxidation of pure zincethyl, this method could not be adopted. A solution of zincethyl in benzole would probably have united the requisite conditions for a successful experiment; but as I had no pure benzole at my command, and as the corresponding reactions of iodine, bromine and sulphur upon zincethyl furnish the key to this action of oxygen, I did not prosecute the inquiry further in this direction.

Action of Iodine upon Zincethyl.

Iodine acts upon zincethyl with great energy, and even with the evolution of heat and light, if a few grains of each be suddenly brought into contact. The violence of the action may, however, be conveniently moderated to any desired extent by the intervention of ether, in which either ingredient, or both of them, may be dissolved. If the iodine and ether be perfectly anhydrous, and the solution of zincethyl in ether cooled below 0° C., the action is unattended by the evolution of any gas; but if the temperature be allowed to rise above 10° C., a small amount of gas is given off near the close of the operation. To ascertain the nature of this reaction, a quantity of zincethyl, dissolved in the ether which had distilled with it from the copper digester, was placed in a flask and cooled to −20° C. A saturated solution of iodine in ether, both carefully freed from moisture, was then cautiously added in small quantities at a time, and with constant agitation of the contents of the flask, care being taken that the temperature never rose above 0° C. The addition of the iodine solution was continued until the intensity of the action had so far moderated as to allow of the iodine being used in the solid state, and it was then added in fine powder until the slight coloration of the liquor indicated that this element was present in excess and the action completed. The ethereal liquid remained transparent during the whole operation, and no gas was evolved. The completion of the reaction was also indicated by the solution of zincethyl ceasing to effervesce when a drop of it was brought in contact with water.

One portion of the solution thus saturated with iodine was allowed to evaporate spontaneously over sulphuric acid *in vacuo*. A white crystalline residue remained, and the sulphuric acid became filled with minute crystals of iodine, whilst on opening the receiver it evidently contained the vapour of iodide of ethyl. After exposing the white crystalline residue to a slow current of dry air for some hours, to remove the remaining traces of iodide of ethyl, it was ignited with oxide of copper, when it yielded traces only of carbonic acid and water; it was in fact pure iodide of zinc. The second portion of the solution was distilled from a water-bath; the distillate consisted of ether and iodide of ethyl; on being frequently washed with water, dried over chloride of calcium and redistilled, it yielded a large quantity of pure iodide of ethyl, which boiled at 72° C., and gave the following analytical results:—

·7253 grm., burnt with oxide of copper, gave ·4155 grm. carbonic acid and ·2093 grm. water. These numbers correspond with the formula

$$C_4H_5I.$$

	Calculated.		Found.
C_4 . . .	24·00	15·40	15·61
H_5 . . .	5·00	3·21	3·21
I	126·88	81·39	——
	155·88	100·00	

The action of iodine upon zincethyl may therefore be expressed by the following equation:—

$$\left.\begin{matrix} C_4H_5Zn \\ II \end{matrix}\right\} = \left\{\begin{matrix} C_4H_5I \\ ZnI. \end{matrix}\right.$$

This result completely establishes the interpretation given above of the action of oxygen upon zincethyl, so far as regards the production of ethylate of zinc; but as a further confirmation, I ascertained the actual amount of iodine which was required for the complete decomposition of a certain quantity of zincethyl. The weight of the latter could only be ascertained approximatively by determining the weight of the iodide of ethyl from which it was derived. Taking this maximum weight as the basis of calculation, and assuming that every trace of iodide of ethyl had been converted into zincethyl, and the latter brought into contact with the iodine without any loss, then each atom of zincethyl took up 1·8 atom of iodine, a number which evidently confirms the above equation, when we take into account the numerous sources of loss of zincethyl incidental to the operations through which it passed.

Action of Bromine upon Zincethyl.

The action of bromine upon zincethyl is exceedingly violent, and attended by dangerous explosions even when an ethereal solution of bromine is added to a dilute ethereal solution of zincethyl cooled to $-15°$ C.; but the reaction can be conveniently effected by suspending a tube containing bromine in a flask half-filled with a solution of zincethyl in ether. The bromine vapour is gradually absorbed by the zincethyl solution until the latter is saturated. The products of the reaction are bromide of ethyl and bromide of zinc, and are therefore perfectly analogous to those yielded by the action of iodine upon zincethyl:

$$\left.\begin{matrix} C_4H_5Zn \\ Br\,Br \end{matrix}\right\} = \left\{\begin{matrix} C_4H_5Br \\ Zn\,Br \end{matrix}\right.$$

Action of Chlorine upon Zincethyl.

Zincethyl burns spontaneously with a lurid flame in an atmosphere of chlorine gas; the zinc and hydrogen are converted into chlorides, whilst carbon is deposited in the form of soot. I have not studied the products of a more moderate action, as it is difficult to bring the materials together without too great an elevation of temperature. There can be no doubt, however, that the moderated action of chlorine would be analogous to that of bromine, and that the products would be chloride of ethyl and chloride of zinc—

$$\left.\begin{matrix} C_4H_5Zn \\ Cl\,Cl \end{matrix}\right\} = \left\{\begin{matrix} C_4H_5Cl \\ Zn\,Cl. \end{matrix}\right.$$

Action of Sulphur upon Zincethyl.

Carefully dried flowers of sulphur have only a slight action upon an ethereal solution of zincethyl, but the application of a gentle heat suffices to produce a brisk reaction; the sulphur gradually disappears, a white flocculent precipitate is formed, and a strong odour of sulphide of ethyl is developed. The chief product of this reaction is the double sulphide of ethyl and zinc (mercaptide of zinc), which is produced as follows:

$$\left.\begin{matrix} C_4\,H_5\,Zn \\ S\,S \end{matrix}\right\}=C_4\,H_5\,S+Zn\,S.$$

There is also formed a little free sulphide of ethyl, according to the following equation:

$$\left.\begin{matrix} C_4\,H_5\,Zn \\ S\,S \end{matrix}\right\}=\left\{\begin{matrix} C_4\,H_5\,S \\ Zn\,S. \end{matrix}\right.$$

The action of sulphur upon zincethyl is strikingly seen, if vulcanized india-rubber joints be used in the apparatus for the preparation of zincethyl; such joints, even when exposed only to the diffused vapour of zincethyl, soon become covered with pustules, which swell up to a large size and burst with slight explosions, until the caoutchouc is completely disintegrated.

This remarkable behaviour of zincethyl in contact with the electro-negative elements, cannot fail to have an important influence upon our views of the condition of bodies at the moment of chemical change,—a subject so ably discussed by Brodie *, whose ingenious views, I consider, receive a new support from the reactions detailed in the foregoing pages. This behaviour also strikingly confirms the suggestions I ventured to make in a former memoir †, relative to the moleculo-symmetrical form of the organo-metallic compounds. In the inorganic combinations of zinc this metal unites with one atom only of other elements, a very instable peroxide, not hitherto isolated, being the only exception. The atom of zinc appears therefore to have only one point of attraction, and hence, notwithstanding the intense affinities of its compound with ethyl, any union with a second body is necessarily attended by the expulsion of the ethyl.

Action of Water upon Zincethyl.

I have already mentioned ‡ that water and zincethyl suffer mutual, and almost instantaneous decomposition, when brought into contact with each other, being transposed into oxide of zinc and hydride of ethyl,

$$\left.\begin{matrix} C_4\,H_5\,Zn \\ HO \end{matrix}\right\}=\left\{\begin{matrix} C_4\,H_5\,H \\ ZnO. \end{matrix}\right.$$

I have, in fact, already made use of this reaction as a means of analysing zincethyl, and it is therefore only necessary for me here to prove the composition of the gas evolved. For this purpose, a quantity of zincethyl was gradually decomposed by water, and the gaseous product collected. It had all the physical and chemical properties of the hydride of ethyl I have already described §, and yielded the following analytical results:

* Philosophical Transactions, 1850, p. 789.

† Ibid. 1852, p. 438.

‡ Philosophical Transactions, 1852, p. 436.

§ Journ. of Chem. Soc. vol. ii. p. 288; vol. iii. p. 341.

I. Fuming sulphuric acid did not reduce the volume of the gas.

II.	Pressure of gas used	31·04
	Pressure after admission of oxygen .	346·80
	Pressure after combustion	268·92
	Pressure after absorption of CO_2 . .	207·11

These numbers agree very nearly with those yielded by the combustion of pure hydride of ethyl, one volume of which consumes 3·5 volumes of oxygen and generates two volumes of carbonic acid.

Vol. of combustible gas.		Vol. of oxygen consumed.		Vol. of carbonic acid generated.
31·04	:	108·65	:	61·81
1	:	3·5003	:	1·991.

Zincethyl is similarly decomposed by the hydrated acids and by the hydrogen compounds of chlorine, bromine, iodine, fluorine and sulphur.

I cannot close this memoir without thanking my late assistant, Mr. C. J. Tuffnell, for the aid he rendered me in several of the determinations mentioned above; his extreme accuracy and delicacy of manipulation enabled him to afford me very valuable assistance.

XII. *On the Anatomy of* Nautilus umbilicatus, *compared with that of* Nautilus Pompilius. *By* John Denis Macdonald, *R.N., Assistant-Surgeon of H.M.S.V. 'Torch,' commanded by Lieut.* William Chimmo, *R.N., tender to H.M.S. 'Herald,' Captain* H. M. Denham, *R.N., F.R.S., commanding the Expedition to the South Seas. Communicated by Sir* William Burnett, *K.C.B. &c.*

Received February 22,—Read March 22, 1855.

Her Majesty's Steam-Vessel 'Torch' having visited the Isle of Pines in the month of July 1854, one of the officers had the good fortune to pick up a recent specimen of *Nautilus umbilicatus* on the outer reef off Observatory Island. The creature had most probably been thrown up by the waves, and remained within a ledge of coral when the spring tide receded. The natives frequently find *Nautili* entrapped in this way, but we could not prevail upon them to bring us the recent animals, although a liberal remuneration was offered.

The specimen forming the subject of the present paper was alive when brought on board, but it was too much exhausted to exhibit any active movements when placed in a vessel containing sea-water. On touching the tentacula they curled up, or moved about irregularly, and the muscular fibres of the funnel lobe contracted slowly, without however producing respiratory currents.

A considerable portion of the posterior part of the hood appeared to have been eaten away by some predaceous enemy, but in other respects the animal was perfect.

On comparing the *Nautilus Pompilius* with *Nautilus umbilicatus* in the recent state, besides the remarkable differences existing between their respective shells, one is struck with the deep position of the latter animal when fully retracted, the space between the upper surface of the hood and the lip of the shell being so considerable, that in the lateral view no part of the creature is visible. On the other hand, the animal of *N. Pompilius* completely fills the chamber of occupation, and many of the tentacula, with a large portion of the hood, rise above the peristome of the shell, so that when the soft parts are removed the rounded orifice of the siphuncle may be distinctly seen in the last-formed septum. Not so however with *N. umbilicatus*; on account of the great depth of the chamber in which the animal is lodged, the opening of the siphuncle cannot be seen in the empty shell. It must be remembered, however, that when a new septum is formed, the last chamber is comparatively shallow, and it continues to deepen as new additions are made to the lip of the shell until the development of another septum is necessitated.

Apart from the shells, these two species, if indeed they may be considered distinct, so closely resemble each other as to render it very much easier to trace out their similarity in corresponding parts, than to determine essential differences between them.

The specimen of *N. umbilicatus* examined proved to be a female, a fact which may serve to modify the views of those who, adopting the ingenious speculations of D'ORBIGNY with reference to the sexes of the Ammonites as indicated by the characters of their shells, apply them also to the several kinds of *Nautili* known. The numbers of the different tentacula in this example, as represented in the following table, agree sufficiently well with those of *N. Pompilius* already recorded; considering the liability of these organs to exhibit slight modifications in form, arrest of development, or supernumerary parts on one or both sides.

Numerical Table of the Tentacula.

	Left side.	Right side.	Total.
Digital tentacula	19	19	38
External labial tentacula ...	12	12	24
Internal labial tentacula ...	12	12	24
Ocular tentacula	2	2	4
Total	45	45	90

As it is not my intention to enter into an elaborate description of the whole anatomy of *N. umbilicatus,* seeing that in so doing I should be needlessly repeating all those particulars so succinctly detailed by Professor OWEN with respect to the *N. Pompilius,* I therefore purpose making some observations on the microscopic anatomy of the organs of the special senses, and the glandular follicles appended to the four vessels which convey the blood from the sinus-system to the branchiæ; and in the explanation of the figures, I shall note as they occur, all the more important matters bearing on the question as to whether the *N. umbilicatus* is to be regarded simply as a variety of *N. Pompilius,* or as an originally distinct species.

Organ of Vision.—To my former observations on the minute anatomy of the eye in *N. Pompilius* I have little more to add, as they are alike applicable to *N. umbilicatus.* I am more fully satisfied than ever that the pigmentary coating is subjacent to the retina, and that the filamentous ends of the fusiform cells of the former commingle with the finely granular and vesicular matter of the latter. I have reason to believe also that the long axes of the pigment-cells themselves are perpendicular to the surface of the retina, like the club-shaped bodies of JACOB's membrane, and parallel examples to this are found in the eyes of some marine Annelidans.

I have not been able to trace a vestige of a lens, nor do I believe that such can exist; and the only representative of a vitreous humour is a kind of viscosity which appears to protect the retina from the direct action of the sea-water. The exterior of the eye (Plate XIV. fig. 1 *a*) was marked at the back part and in the region of the pupil with blotches of the same rich brown pigment which tinted the upper surface of the hood and a few of the tentacular sheaths.

Organ of Hearing.—The thorough investigation of the various systems and organs of an animal previously almost wholly unknown, with a single mutilated specimen at the disposal of the anatomist, is attended with difficulties which can only be surmounted by the accomplished dissector. An undertaking of this kind fell to the lot of Professor Owen when he entered upon the examination of the first recent Pearly Nautilus that reached the shores of England, and the beautiful monograph resulting from the right use of this single opportunity is a lasting memorial of the genius of its author. The organ of hearing seems to have been the only matter of any importance that escaped the scrutiny of the Professor, and although I have been fortunate enough, myself, to discover the auditory capsules in the recent *N. umbilicatus*, I can easily conceive the difficulty of detecting them in a specimen long preserved in spirits, which renders the tissues opaque and ill-adapted for microscopic investigation.

The acoustic capsules in my specimen were about one-twelfth of an inch in diameter, subspherical in form, and situated at the union of the supra with the subœsophageal ganglia, but more especially connected with the short pedicle of the anterior division of the latter (Plate XV. fig. 1 *d*). I have been induced to look for them in this locality, bearing in mind the condition of the organ of hearing in most Gasteropoda, and recognizing the close affinity of the Nautilus to this order.

In every instance the supra-œsophageal ganglia occupy the cephalic region, but on account of the great length of the neck and body anterior to the visceral nucleus in Heteropoda, the subœsophageal nervous masses suffer the maximum amount of backward displacement, and the whole nervous system approximates the homogangliate type. In this extreme case the ear is still preserved in the neighbourhood of the eye, and the special centres of vision and audition are incorporated with the supra-œsophageal ganglia, from which both the auditory and optic nerves arise.

The inner wall of each ear-sac in the Nautilus is somewhat flattened, lying in contact with the nervous matter; but its more convex external surface rests in a little depression on the upper and internal border of the cephalic cartilage. It is enveloped in a kind of fibrous tissue, and filled with a cretaceous pulp consisting of minute elliptical otokonia, which, when under a high power, present a bright and strongly refracting point near each extremity. These particles vary much in size, and are sometimes curiously combined, so as to appear double, or assume the form of a star or cross, &c.

This simple auditory apparatus may be readily exposed by making an incision externally in the deep groove between the funnel lobe and the basal part of the tentacular sheaths, immediately in front of the hollow subocular process.

The cilia lining an auditory sac containing minute otokonia, are always much finer than those required to impart a rotatory motion to a single spherical otolithe. I have not observed them in the ear of the Nautilus, although I cannot for a moment doubt their existence.

Valenciennes traced three small nervous filaments into a cavity of the cephalic cartilage, which he says was filled with a homogeneous pulp, and did not contain any kind of concretions; but not having had the opportunity of perusing his original memoir on the *N. Pompilius*, in which animal he observed this structure, I cannot determine with any degree of certainty the parts to which he refers. From my own observation, I can scarcely imagine that the nature of the venous sinus, which excavates the cephalic cartilage on either side, could be mistaken by a well-informed anatomist, and yet there is much probability in the suggestion of Professor Owen, that this is the locality to which Valenciennes assigns the organ of hearing.

As our knowledge of the Mollusca advances, the localization of the special centre of audition, as well as the nature of the contents of the simple capsules, which shadow forth the vestibule of the more perfect apparatus of hearing, will prove of great importance to the natural classification of these animals. Thus, even with respect to the contents of the capsules, we find spherical otolithes present in the Heteropoda, Pectinibranchiata and several other orders; and otokonia in the Pulmonifera, and as far as I have been able to discover, except while yet in their embryonic state, in the whole of the Pteropoda, including Peron's *Phyllirhoë*, which is more closely allied to these last than to the Gasteropoda. Now as we know that the ear-chambers in the dibranchiate Cephalopoda contain single otolithes of large proportional size, although this rule may not be general in its application, it would not be unreasonable to conclude, from the evidence afforded by the Nautilus, that the auditory sacs of the extinct Tetrabranchiata were distended with minute pulverulent otokonia.

Organ of Taste.—Although a doubt may still exist as to the locality of the olfactory sense in the Nautilus, this can hardly be said of the organ of taste, the mucous membrane of the mouth is so richly supplied with sentient papillæ. These bodies are distributed in three principal groups; thus a considerable number beset a stout vertical fold of the lining membrane, extending, on either side, from the root of the tongue to the back of the pharynx. The papillæ are more numerous on the inner side and along the free border of each fold, in order to give them a greater extent of motion and render them opposable to the remaining group (Plate XV. fig. 4 *b*), which occupies the median line, between the orifice of the tongue-sac, *a*, and the commencement of the œsophagus, *e*; and although placed a little posterior to the organ usually recognized as the tongue in the higher Mollusca, they may with great propriety be named the lingual papillæ, to distinguish them from the others. All the papillæ agree, however, in their general character and minute anatomy, being either simple or compound, exhibiting much irregularity in form, and being clothed with long and delicate columnar epithelial cells, the homogeneous basement membrane enveloping a kind of areolar tissue.

Renal Organs.—It is reasonable to suppose that an animal possessing a complex digestive system for the reduction of crude animal matter, should be also furnished with renal glands, to separate from the blood those deleterious principles which must otherwise accumulate. In the Nautilus there are no organs to which this function can be more justifiably assigned than the numerous glandular follicles appended to the vessels which convey the blood from the sinus system to the branchiæ; admitting also that by altering their capacity they may serve to regulate the amount of blood circulating through the respiratory apparatus under those changes of pressure which the animal must experience in sinking to great depths and rising in its watery element. Mayer suggested that the homologues of these follicles in the higher Cephalopoda were the emunctories of the urine, and Professor Owen considers it more philosophical to conclude that the organs of so important an excretion should be present in all the class, than that they should be represented by the ink-gland and bag which are peculiar to one order. As the most satisfactory method of arriving at a just conclusion, however, I shall leave further argument for the present, and describe the minute anatomy of the organs, which may be provisionally called the renal follicles of the Nautilus.

These follicles are subcylindrical in form, somewhat dilated at the free extremity, to which is appended a folded and funnel-shaped process of membrane, which expands rather suddenly, presenting a jagged and irregular border. They open by a smooth and oral or slit-like orifice into the afferent pulmonary vessels, on each of which, as Professor Owen has observed, they are disposed in three clusters.

The outer membrane is smooth and glossy, homogeneous in structure, and sprinkled over with minute rounded and transparent bodies, probably the nuclei of cells. Beneath this layer flat bundles of fibres, apparently muscular, are traceable here and there, principally disposed in a longitudinal direction, and sometimes branched.

The lining membrane consists of a loose epithelial pavement, in many respects similar to that of the uriniferous tubules of the higher animals; the cells containing, besides the nuclei, numerous minute oil-globules, or a substance much resembling concrete fatty matter.

This membrane is thrown up into an infinite number of papillæ and corrugations, so as to augment the extent of surface considerably. The papillæ are more numerous at the inner part, or towards the attached end, and a circlet of longitudinally disposed folds radiate from the bottom of the follicles, in which a number of small pits or fenestrations is sometimes visible. The sides of these folds are wrinkled transversely, so as to present a median zigzag elevation.

The funnel-shaped membranous process above noticed is continuous with the lining membrane, consisting of an extension of the same epithelial pavement, but the cells are somewhat larger and more regular in form. The cavity of each follicle, therefore, communicates with the exterior through the centre of this process, and the aperture is thus guarded by a kind of circular valve, permitting the escape of secreted matters, but effectually preventing the entrance of fluid from without.

Now that grave doubts have been cast upon the existence of the so-called epithelial investment of the Malpighian tufts of the kidney in Vertebrata, the office of these minute vascular bodies would seem to present a solitary example of the secretion of a peculiar fluid directly from the blood, or independently of the agency of nucleated cells. The glandular follicles of the Nautilus just described appear, as it were, to go one step further, the vascular and secreting portions having so far coalesced as to be almost undistinguishable the one from the other. When the Malpighian tufts are excessively distended with the contained blood, the albuminous elements pass away with the thinner parts, and doubtless congestion of the renal follicles of the Nautilus would be attended by a similar result*.

These views may be still further supported by glancing at the relationships of the kidney, liver, and respiratory organs in the Vertebrata and Mollusca respectively. Thus, in vertebrate animals the biliary fluid is secreted from venous blood supplied by the portal system. In Mammals the secerning vessels of the kidney are chiefly arterial; but in Fishes, which possess a distinct portal system in connexion with the kidney, the urine is principally separated from venous blood, which ultimately commingles with that returning from the liver before reaching the *branchial* heart.

In Mollusca, on the contrary, the biliary secretion is furnished from arterial blood; and if the glandular follicles of the Nautilus above noticed are veritable renal organs, as they evidently appear to be, the kidney exchanges place, as it were, with the liver, lying between the great sinus system and the branchiæ, which return their blood into a *systemic* heart.

It will be perceived, therefore, that in these respects a remarkable difference exists between the Mammalia and Mollusca; but the steps of transition from the one to the other are so distinctly marked in the intervening classes, taken in their natural order, that we are enabled more fully to comprehend the nature of this disparity; the apparently anomalous position of the renal glands in Nautilus and Sepia, and indeed also in certain Gasteropoda and Conchifera, in which their function has been more satisfactorily determined, being reconciled with the relative anatomy of those organs in animals of higher grades.

The body of *N. umbilicatus* is larger and more elongated than that of *N. Pompilius*, as it occurs in the South Seas, although the specimens of the latter species brought from the Chinese Seas much exceed both in size. In the *N. umbilicatus* the longitudinal lamellæ on the median lobe of the external labial processes are divided by a wide groove into two distinct lateral sets, and the corresponding lamellæ between the internal labial processes are about seventeen in number and of considerable thickness. In *N. Pompilius* the latter lamellæ are much thinner and more numerous, and the lateral sets of the former are united together in the median line,

* On comparing these follicles with their spongy homologues in *Sepia*, for example, one cannot fail to observe in them a relationship similar to that existing between the lobulated renal organ of the Porpoise and the more condensed and perfect kidney of Man.

commencing anteriorly with an azygos transverse lamina. In both kinds, however, I have distinctly traced out the corresponding tentacula, with such minor differences as might be expected to occur in different specimens of either separately. The question here naturally arises,—Are the peculiarities observable in the descriptive and microscopic anatomy of each of sufficient importance to entitle them to be considered distinct species, or are they within the pale of that latitude which must be allowed to variety?

Any tendency in a being to revert to an original type, when such has been determined, betrays variety; but this tendency in the *Nautili* now under consideration is never manifested by the occasional occurrence of specimens presenting characters which place them intermediate between *N. Pompilius* and *N. umbilicatus*.

Having visited the Fijii Islands since my former paper on the *N. Pompilius* was written, I find that the umbilicated *Nautili* are not known to the natives, although *N. Pompilius* is very plentiful; but at Fatuna or Wallis's Island, where both are found, the people recognize the difference between them, depending upon the presence or absence of umbilical pits. Now, although particular localities, with all attending circumstances, may favour the production of varieties, yet the permanence of the distinctive characters of these *Nautili* without symptom of amalgamation, and the discovery of a female specimen of *N. umbilicatus*, strongly support the view that they are distinct species, though very closely allied.

Explanation of the Plates.

PLATE XIV.

Fig. 1. Left lateral view of *N. umbilicatus* removed from the shell.

a. Eye.

b. Hood.

c. Tentacular sheaths.

d. Digital tentacula.

ee. Ocular tentacula.

f. Subocular hollow process.

g. Funnel lobe.

h. Mantle.

i. A lobe of the mantle which rests upon the black-stained patch of the shell.

j. A process similar to the last, beneath which it lies in the hollow of the hood, being continuous with the portion or part of the funnel lobe.

k. Shell-muscle.

l. Root of siphuncle. A small vessel may be seen coursing round it inferiorly.

The following parts are indistinctly visible through the mantle :—

m. Nidamental gland.

n. Chamber of pericardium, lodging the base of the ventricle of the heart, the branchial vessels and renal follicles.

A few globules of air are represented at the upper part of the chamber, having entered it by the sub-branchial openings, first described by Professor Owen.

o. Gizzard.

p. Ovary.

Fig. 2. Simple dissection of the Nautilus, exposing all the more important parts of its anatomy, disturbed as little as possible from their natural relations.

a. The eye slit open in the vertical direction, showing the opalescent retina spread over the pigmentary layer. The subocular hollow process is seen projecting from the base of the eye-pedicle, immediately above the cephalic cartilage *f*, which appears in section.

b. Hood.

c. Tentacular sheaths.

d. Tentacula.

e. Funnel lobe. *e'*. Process corresponding to *j*, fig. 1.

f. Cephalic cartilage in section.

g. Shell-muscle.

h. Mantle. *h'*. Process corresponding to *i*, fig. 1.

i. Buccal mass.

j. Nervous ring of cephalic ganglia encircling the œsophagus and the terminal branches of the principal vessel. The long visceral nerves are seen passing in the interspace between the shell-muscles, to which a number of short radiating branches are distributed.

k. Dilated portion of œsophagus forming a kind of cross.

l. The gizzard.

m. Sacculated portion of stomach below the gizzard, receiving the large biliary duct, conveying thither the secretion of the three principal portions into which the liver is divided.

n, *n'*, *n''* respectively, the left, right and middle mass of the liver.

o. Biliary duct of the right division of the liver which is brought down from its natural position on the right side of the cross, the smaller extremity lying in contact with—

p. Peculiar glandular-looking bodies connected with the cross by muscular bundles and cellular tissue.

q. The intestine, forming two principal flexures.

r. Branchiæ.

s. Nidamental gland.

t. Anterior wall of pericardium, laid open to expose the contained organs.

t′. Posterior wall of ditto.

u, *u′*. Anterior and posterior renal follicles separated by an induplication of the anterior wall of the pericardium. This fold also extends between the anterior and posterior branchial vessels of the corresponding side which pour their blood into

v. The ventricle of the heart. The elongated posterior part of this organ extends into the abdominal cavity, through an opening in the posterior wall of the pericardium, and passing beneath the oviduct, gives rise to the principal arterial trunk, which continues its course beneath the rectum and the middle portion of the liver, around which it turns, giving off numerous branches to the neighbouring parts, as represented in the figure. It next ascends between the left principal division of the liver and the cross, and coursing obliquely along the left side of the œosophagus, it ultimately reaches the posterior part of that tube, to which it gives one or two vessels, and terminates immediately beneath the supra-œsophageal ganglia in fine branches, chiefly supplied to the buccal mass. In this course it gives off numerous lateral vessels of small size to the roof and sides of the cavity in which it lies, and a more important branch winds round the glandular bodies *p*, and is distributed to the border of the mantle.

When the posterior conical portion of the ventricle of the heart appears beneath the rectum, a small duplicature of the investing membrane is seen connecting these parts; this fold, I believe, includes one extremity of the "elongated pyriform sac," first noticed by Professor Owen as arising near the base of the aorta and ending in the venous sinus.

w. Ovarium, laid open on the right side, to expose, *x*, the calyces, of various sizes, attached along its roof.

The corrugations of the lining membrane of the calyces are proportionately large or small, bearing relation to the size of these bodies.

The ovary in this case was distended with a plastic albuminous fluid of a rich amber colour, and a considerable quantity of it found its way into the cavity of the abdomen, through a large oval opening in the anterior wall of the ovary, with a thickened puckered margin, at which the investing and lining membranes are continuous. The posterior wall of the pericardium, on the right side, acts as a valve to this opening.

Although I have observed the albuminous fluid just alluded to free in the abdominal cavity of *N. Pompilius,* I did not discover the orifice by which it must have escaped from the ovary.

z. The narrow oviduct is seen passing forwards on the right side of the rectum, and ending in a thickened and apparently glandular extremity, from the left side of which a fold of the investing membrane arches over the rectum near the termination of the long visceral nerves, lying below and in front of the reflected layer of peritoneum, which prevents the admission of the seawater with the blood bathing the cephalic ganglia, œsophagus, and buccal mass. The lateral attachment of this layer is represented by faint lines on the inner surface of the shell-muscle, extending upwards and backwards to the apex of the left hepatic mass.

PLATE XV.

Fig. 1. Dissection exhibiting the auditory sac of the right side, inside *in situ,* the cephalic ganglia, neighbouring parts, &c.

a. The eye, beneath which is seen the hollow process resembling a tentacular sheath.

b. Supra-œsophageal nervous mass, cylindroid in form, and presenting a ganglioniform enlargement at either end, from which the optic nerves are given off. The stout fibrous envelope is slit open, exposing the reversed loops of four nerves which pass onwards to the buccal mass.

c. Posterior division of the subœsophageal ganglia, giving off from its convex border a considerable number of nerves to the shell-muscles, and, on either side of the median line, two principal visceral nerves.

d. Portion of anterior division of subœsophageal ganglia, with the auditory sac lying on the outer surface of its pedicle, near the point where all the principal cephalic nervous masses of the corresponding side unite. A small depression may be noticed on the upper and inner border of the cephalic cartilage in which the auditory capsule rests.

The funnel lobe, shell-muscles, and the principal artery running upon the œsophagus, giving it a small vessel and terminating in fine branches, which pass through the nervous ring towards the buccal mass, need no references.

Fig. 2. The auditory sac: natural size.

Fig. 3. The otokonia: highly magnified.

Fig. 4 exhibits the opening of the tongue-sac *a*, the lingual papillæ *b*, and those distributed upon the buccal folds of lining membrane *c*.

d. Semi-cartilaginous substance forming the matrix of the horny jaws, and affording attachment to muscles *f*.

e. Opening of the œsophagus.

g. Sublingual processes, the inferior one being detached from the corresponding jaw.

Immediately in front of the lingual papillæ, the mucous membrane forms two small wing-like folds, lying side by side, *h*, which serve to close the opening of the lingual sac; and a linear projection extends, from between them, along the roof of the sac, being impressed by the teeth of the rachis, and the internal series of uncini on either side. The microscopic structure of the supporting cartilage of the lingual strap is perfectly similar to that of *N. Pompilius*; and with the trifling difference, that the uncini of *N. umbilicatus* are a little shorter than those in *N. Pompilius*, the lingual ribbon, and I may also include the horny jaws of both, so nearly resemble one another as to render separate illustration quite unnecessary.

Fig. 5. A cluster of papillæ enlarged to show their general form and character.

Fig. 6. Tip of one of the papillæ, highly magnified to show the investing columnar epithelium. A considerable number of these almost filiform cells have been removed in order to display the rest more satisfactorily.

Fig. 7. Several of the renal follicles detached, with a portion of branchial vessel with which they were connected: natural size.

Fig. 8. Enlarged representation of the follicles seen at fig. 7, showing the funnel-shaped membranous processes appended to their free extremity, and the openings by which they communicate with the cavity of the branchial vessel.

Fig. 9. Crystalline bodies often occurring within the follicles.

Fig. 10. A few of the fibres which occasionally present themselves, disposed in flattened and branched bundles, between the external and internal coats of the follicles.

The remaining figures represent highly magnified portions of the lining membrane.

Fig. 11. Three of the zigzag longitudinal folds taken from the bottom of a follicle.

Fig. 12. A few of the hollow papillæ from near the attached end of ditto.

Fig. 13. Outer surface of a small fragment of the epithelial lining, showing the slit-like opening of a hollow process.

Fig. 14. Five cells of the same pavement, more highly magnified, to show the nuclei surrounded by minute fatty globules, reminding one of the condition of the epithelial cells of the human kidney in the early stage of Bright's disease.

XIII. *Remarks on the Anatomy of* Macgillivrayia pelagica *and* Cheletropis Huxleyi (Forbes); *suggesting the establishment of a new Order of Gasteropoda. By* John D. Macdonald, *R.N., Assistant-Surgeon H.M.S. 'Herald.' Communicated by Sir* W. Burnett, *K.C.B. &c.*

Received November 23,—Read December 21, 1854.

Having examined the anatomy of the *Macgillivrayia pelagica* and several smaller species of pelagic Gasteropoda, not exhibiting the least similarity in the character of their shells, I found that they all presented a very close relationship to each other, in the type of their respiratory organs, and in other points of structure of less importance.

The gills in every instance seemed to be fixed to the body of the animal immediately behind the head, and did not appear to be appended to the mantle, as in the Pectinibranchiata properly so called. They were also invariably four in number, and arranged in a cruciform manner round a central point.

They were free in the rest of their extent, elongated and flattened in form, with a pointed extremity, and fringed with large flowing cilia, set in a frilled border. They were furnished, moreover, with muscular fibres, disposed both transversely and in a longitudinal direction, and exhibited great mobility when protruded, but lay side by side in the last whorl of the shell when retracted.

The auditory capsules, each containing a spherical otolithe, were closely applied to the inner and posterior part of the larger or anterior ganglion of the subœsophageal mass.

There were but two tentacula, with an eye situate at the outer side of the base of each, consisting of a globular lens with distinct optic nerve and retinal expansion. The foot was large and very mobile, but a vesicular float has only been observed in *Macgillivrayia*.

The respiratory siphon was either a simple fold of the mantle, so rolled upon itself as to form a temporary tube (*Cheletropis*), or, as in *Macgillivrayia*, the borders of the fold were united through their whole extent, only leaving a small oblique aperture at the end to reveal its true nature. The siphon in *Macgillivrayia* is about twice the length of the shell, and as in the other species examined, beset with vibratile cilia.

A lingual ribbon of good proportional length with well-marked rachis and pleuræ occurs in all the species, none presenting the character of the broad pavement of hooklets to be found in the genus *Ianthina*. Very perfect labial plates with a close file-like arrangement of dental points arm the mouth, in some instances at least, but most probably in all.

It is very remarkable, that the little animals possessing all the characters just detailed in common, should fabricate shells so very different in general form and other particulars as to permit of their arrangement into well-marked genera.

The obvious difference existing between the pectinibranchiate type of respiratory organs and that of the little Gasteropoda now under consideration, must at once afford sufficient grounds for placing the latter in a distinct order by themselves; in illustration of which, I have selected the anatomy of *Macgillivrayia pelagica* and *Cheletropis Huxleyi*, whose shells have been already described by Professor E. Forbes, and figured in Mr. Macgillivray's 'Narrative of the Voyage of H.M.S. Rattlesnake.'

The beautiful little Gasteropod named after the latter gentleman, who was its first discoverer, was originally believed to be purely Australian, but upwards of a dozen specimens were captured in the towing-net, about the latitude of Bahia, while H.M.S.V. 'Torch' was proceeding to Rio de Janeiro.

The disc of the foot when expanded was of considerable breadth, but its attachment to the body was small and situate just beneath the neck (*Trachilipoda*). Its lateral borders were united posteriorly, forming a rounded extremity, to the upper surface or heel of which the concentric horny operculum with spiral nucleus was attached, but they were notched in front, so that the angles between them and the anterior margin, which was slightly convex, were prominent and pointed a little backwards. The raphe to be noticed in the mesial line, and in fact the whole character of this part of the organ, seemed to shadow forth the transformation of the single foot of the Gasteropod into the wing-like expansions of the Pteropod. The mouth of the animal was furnished with two horizontally placed, crescentic plates, adapted for acting upon one another in breaking up food.

The lingual strap bore many points of analogy to that of the Heteropods, the single series of plates in the rachis being angular with a finely serrated border, and the pleuræ consisting each of three rows of simple uncini, or with delicate teeth on the concave border.

The eyes, which were distinctly to be seen with a common lens, were surrounded by a rose-coloured zone, giving them a remarkable appearance. The four naked branchiæ fringed with gracefully curved cilia of unusually large size, radiated from a point at the back of the head like so many feathers set in a crown, which when taken together with the glowing eyes and brown labial teeth of the little mollusk, imparted quite a singular aspect to the whole physiognomy.

On the left side of the body a tubular process of the mantle protruded from the shell, and seemed to indicate the coexistence of a respiratory chamber with naked branchiæ. The length of this siphon nearly equalled that of the foot, and its aperture was oblique, as that of a portion of the mantle rolled into a tube would naturally be.

I had not the good fortune to find the vesicular float like that of *Ianthina*, noticed by Mr. Macgillivray in the first examples taken; but in one or two successful hauls of the towing-net off the Agulhas bank, Cape of Good Hope, the little animal again made its appearance, having been lost sight of on the voyage from Rio to the Cape, and the float was found *in statu quo*, consisting of an aggregation of vesicles varying, both in number and size, in different cases. It was exceedingly delicate, and might have been easily destroyed or separated from the foot on former occasions by the force of the water rushing through the meshes of the net.

The float of the *Ianthina* has been thought to be an extreme modification of the operculum, the absence of which in this genus no doubt has given rise to the idea; but as in the little *Macgillivrayia* both operculum and float are to be found in the same individual, we must admit the latter structure to be quite independent of the former, answering a distinctly different purpose.

The following account of the anatomy of *Cheletropis Huxleyi* is drawn up nearly *verbatim* from notes made on the examination of the species early in 1853; since which time I have met with one or two others of the same genus.

Numerous specimens of this interesting little mollusk were obtained in Bass's Strait and in the South Pacific, between Sydney and Lord Howe Island.

Its shell is of a darkish neutral colour, quite transparent, very brittle, and dotted all over with minute tubercles.

The spire is of moderate length, but small compared with the last whorl of the shell, which is large and full.

The aperture is oval, terminating anteriorly in a wide canal or notch. This notch, with two others of larger size on the outer lip, and two prominent teeth intervening, impart a characteristic appearance to the shell. Leading from the posterior tooth on the outer lip, a linear thickening of the shell may be traced quite to its apex.

The operculum is of an oval form, concentric, developed round a small spiral nucleus situate near one extremity, and altogether very much resembles that of *Atlanta*, being also extremely thin, vitreous-looking and brittle. It is not very easily detached from the foot for examination, and this circumstance, taken together with its extreme minuteness, might explain why it had not been observed by Professor Forbes.

The foot when exposed is proportionably long, rounded at the anterior and pointed at the posterior extremity. The whole surface of the disc is closely speckled with deep purple pigment-cells, in the centre of which the nuclei remain bright and transparent, not being obscured by the deposit. The whole surface of the foot is thickly covered with extremely delicate and active cilia.

That portion of the mantle which in *Macgillivrayia pelagica* forms a long and perfect tube, as a respiratory siphon, is short, and the opposite edges are merely brought together, without organic union, in the present species.

The cilia arming this part are much larger than those of the foot just alluded to

I have not discovered any vestige of a float, although it is possible that such may exist normally, and be detached by the rush of water through the towing-net, or some other accident.

The branchiæ or gills are of two kinds, *i. e.* covered and naked. The covered gill, as far as I have been able to observe, is single, but of considerable length. It is beautifully pectinated and fringed with long vibratile cilia, representing doubtless the respiratory organ of the pectinibranchiate Gasteropoda.

The basis of this structure is a long and narrow strip of a tough and fibrous material, folded upon itself so as to form a series of loops, invested with a coating of epithelium richly ciliated along the free border.

The naked gills are four in number, similar both in situation and character to those of *Macgillivrayia.*

Each gill is of an oval or elongated form, presenting a thin, frilled and corrugated border, beset with long whip-like cilia, which strike the water with a lashing movement.

In the central parts muscular fibres are distinctly discernible, some disposed lengthwise and others transversely, so that the whole structure and appearance of these organs, although very small, would recall to mind the sea-mouse and the numerous other marine annelidans of that character.

The lingual strap is of considerable length, the anterior extremity lying between two club-shaped pieces of cartilage consisting of extremely minute cells. The rachis is formed of a single series of dental plates, which alternate with those of the pleuræ. The latter consist of broad quadrilateral masses, each presenting one or two principal tubercles, the most internal of which is somewhat uncinated. The posterior extremity of the tubular sac, in the floor of which these peculiar teeth are arranged, is rounded and slightly enlarged.

Besides the lingual teeth, the mouth is furnished with two file-like triturating plates, which are articulated with each other inferiorly (Pl. XVI. fig. 7). The two tentacula of each side appear as it were enclosed in one envelope, so as to form a single tactile instrument, bearing a large dark eye on its outer side, near the base. To this latter organ the tegumentary covering forms a kind of cornea, beneath which is a spherical lens, resting on a mass of black pigment, both being enclosed in a little sac; and the optic nerve, emerging from the supra-œsophageal ganglion, joins the miniature globe and expands into a retina. I have not been able to trace an opening through the pigment for the passage of luminous rays, but it is most probable that, as in the ocelli of insects, such exists at the central part, the pigment only encroaching on the sphere of the lens sufficiently to correct the aberration of light.

At some distance behind the eyes, when the neighbouring parts are carefully removed with fine needles, the auditory capsules may be distinctly seen with the microscope. They are of a rounded or oval form, containing each a beautifully transparent and highly refracting otolithe, much larger than the lens of the eye.

Explanation of Plate.

PLATE XVI.

Fig. 1. *Macgillivrayia pelagica* (about three times the natural size), as it appeared ascending the side of the vessel in which it was placed.

Fig. 2. Ditto, with foot expanded at the surface of the water, the ends of the branchiæ protruding.

Fig. 3. Ditto, the crucial gills fully exposed as the animal lay upon its side at the bottom of the vessel.

Fig. 4. The four gills slightly magnified to exhibit their character more clearly.

Fig. 5. Portion of lingual strap of *Macgillivrayia.*

Fig. 6. Ditto of *Cheletropis.*

Fig. 7. Labial plates of the latter.

Fig. 8. Portion of covered gill of same.

Fig. 9. A tentaculum, showing the anatomy of the eye.

Fig. 10. One of the acoustic capsules containing a spherical otolithe. All the objects highly magnified.

The remaining figures show the relative sizes of the shells of several species of pelagic Gasteropods, all of which are represented about twice the natural size.

Figs. 11, 13 and 14 are quite new, requiring both names and descriptions.

Fig. 12. A species of *Cheletropis,* probably not that of Huxley.

Fig. 15. *Macgillivrayia pelagica.*

The opercula of the respective shells are shown in the lowermost row, somewhat magnified, to exhibit the spiral nucleus and the lines of growth which determine their ultimate shape and character.

H.M.S. ' Herald,' Sydney,
February 11, 1854.

XIV. *Further Observations on the Anatomy of* Macgillivrayia, Cheletropis, *and allied genera of pelagic Gasteropoda. By* JOHN DENIS MACDONALD, *R.N., Assistant-Surgeon H.M.S.V. 'Torch.' Communicated by Sir* WILLIAM BURNETT, *K.C.B. &c.*

Received February 22,—Read March 22, 1855.

DURING a late voyage from Sydney to Moreton Bay, specimens of *Macgillivrayia*, *Cheletropis*, and a few other genera of minute pelagic Gasteropoda, apparently undescribed, were daily taken in the towing-net, and having embraced this opportunity of determining the actual mode of attachment and connexions of the ciliated arms, at first presumed to be naked branchiæ, I am anxious to append the following remarks to those which I have already made on this interesting subject.

In a former paper I mentioned, more particularly with reference to *Cheletropis Huxleyi*, that the gills were of two kinds, *i. e.* "covered and naked;" the former, corresponding to that of the pectinibranchiate Gasteropoda generally, I have never found absent in any of the genera; but from careful observation of the so-called naked gills of these minute animals, while yet alive in their native element, I am disposed to believe that they are chiefly employed as organs of prehension, and may also assist in natation. When these ciliated appendages are fully extended, the line of cilia is perfectly straight and uninterrupted, so that the frilled border, noticed in the previous account, is a character simply depending on the partial contraction of the longitudinal muscular fibres, a preparatory step to complete retraction of the organs. They have no connexion whatever with the mantle, but encircle the mouth, including the tentacula and eyes, communicating with each other at the base like the segments of a deeply-cleft calyx. In the accompanying figure of *Macgillivrayia*, which has been closely copied from nature, the whole scheme of arrangement is sufficiently well seen. In the specimens of this genus examined, the arms were quite transparent and marked at irregular intervals with transverse streaks of a brownish purple colour.

In the extended form they were several times the length of the shell, and like the arms of a polype, when touched, they rolled themselves up, and started back into the shell with surprising rapidity. They were also exquisitely sensitive, exhibiting short twitching movements when minute atoms suspended in the water came in contact with them.

It will be observed by reference to the figure, that the respiratory siphon is represented as a simple process of the mantle converted into a tube by the apposition of its lateral borders without organic union, which I must confess is at variance with my former views; moreover this process appeared to be much shorter than I had found it in other examples, of which those at present under consideration may be a variety, if not specifically distinct.

Before the paper already alluded to was written, although I had examined a considerable number of species, I never found more than four arms encircling the head, but I have since discovered six in a single genus with which I had been long familiar by external characters (Plate XVI. fig. 18, representing a member of this genus, shows the arrangement of its ciliated arms, &c.). The operculigerous lobe of the foot is quite cylindrical and of some length, bearing the peculiar operculum on its truncated extremity, the clawed process pointing to the left side. The sucker-disc is very small, presenting an anterior and posterior lobe, such as exist in *Atlanta*, in which they differ only in being lateral.

The two tentacula bear each an ocellus on the outer side near the base, and the ciliated arms, in every respect save number, resemble those of *Macgillivrayia* and its congeners. The clawed operculum is developed from a spiral nucleus situate near the internal or thickened border; it seems to be a weapon of defence, which is wielded with great dexterity by the little animal, which makes skips and jerks by means of its complex foot, after the manner of *Nassa* or *Strombus*.

It may be well to notice here briefly another interesting member of this diminutive tribe of Gasteropoda, very commonly met with in the South Pacific, and having an almost indefinite range. It resembles a miniature *Natica* in many points, including both animal and shell. The shell is few-whorled, with small compressed spine and ventricose mouth; the operculum paucispiral and well-marked with the lines of growth. The foot of the little creature is not unlike a broad and square-toed shoe, receiving or bearing the remainder of the animal and the shell. The shoe-upper, as it were, presents two rounded lateral lobes which lie over the anterior part of the shell, like the mentum of *Natica*.

The little animal creeps on its foot with great rapidity, appearing rather to slide along than progress by a vermicular movement, and by spreading out and hollowing this organ at the surface of the water, by the same instinct which prompts the fresh-water Lymnæad to form a ready boat of its foot, this shell-protected speck buoys up its tiny body, cast abroad, though not lost, in the ocean's immensity.

Explanation of Figures in Plate XVI.

Fig. 16. Front view of *Macgillivrayia pelagica*, with its ciliated arms (*a*) extended and encircling the mouth and tentacula (*b*). *c*. Open canal of respiratory siphon. *d*. The foot.

Fig. 17. Lateral view of ditto. *a*. Ciliated appendages. *b*. Portion of left tentaculum. *c*. Siphon. *d*. The foot. *e*. Operculum.

Fig. 18. Enlarged figure of a minute pelagic Gasteropod, presenting six ciliated arms (*a*), similar to those of *Macgillivrayia*. *b*. Tentacula, bearing each an ocellus on the outer side near the base. *c*. Rudimentary creeping disc. *d*. Operculigerous lobe. *e*. Operculum.

Fig. 19. Two transverse rows of the lingual teeth of this little animal.

Fig. 20. Front view of shell of ditto.

Fig. 21. Back view of same.

Fig. 22. The operculum, with its clawed process (internal surface).

Fig. 23. Front view of the closed shell of another minute species.

Fig. 24. Back view of ditto; the retracted arms, the ocelli, and the anterior lobes of the foot, appearing through the transparent wall of the shell*.

* Figures of the above genera were given in the illustrations of the former paper.

XV. *On a Class of Differential Equations, including those which occur in Dynamical Problems.*—Part II. *By* W. F. Donkin, *M.A., F.R.S., F.R.A.S., Savilian Professor of Astronomy in the University of Oxford.*

Received February 17,—Read March 22, 1855.

THE following paper forms the continuation and conclusion of one on the same subject presented to the Royal Society last year, and printed in the Philosophical Transactions for 1854. I have however put it, as far as possible, in such a form as to be independently intelligible.

The fourth Section (the first of this Part) contains a recapitulation of some of the most important results of the former Part, in the form of seven theorems, here enunciated without demonstration.

In the fifth Section the method of the variation of elements is treated under that aspect which belongs to it in connexion with the general subject. It is applied, by way of example, to deduce the expressions for the variations of the elliptic elements of a planet's orbit from the results of art. 30 (Part I.), on undisturbed elliptic motion; this example was chosen, partly because the resulting expressions are required in a future section, and partly for the sake of incidentally calling attention to a fallacy which has been, perhaps, often committed, and certainly seldom noticed. The same method, under a slightly different and possibly new point of view, is applied, as a second example, to the problem of the motion of a free simple pendulum, omitting the effect of the earth's rotation. I believe the methods of this paper might be advantageously employed in the treatment of that general form of the problem of a free pendulum which has been considered by Professor Hansen in his Prize Essay. I was unwilling, however, to attempt what might have turned out to be merely an unconscious plagiarism, without having seen the Essay in question, of which I only succeeded in obtaining a copy on the day of writing this preface. As I now perceive that the investigation would be quite independent, I hope to enter upon it at some future time.

The sixth Section contains some general theorems concerning the transformation of systems of differential equations of the form considered in this paper, by the substitution of new variables. The most important case consists in the transformation from fixed to moving axes of coordinates, in dynamical problems. Some of the results are, I think, interesting, and perhaps new.

The seventh and last Section contains an application of the preceding theorems, in connexion with the variation of elements, to the transformation of the differential equations of the planetary theory. This investigation, if interesting at all, will probably be so to the mathematician rather than to the astronomer. I think, however, that if the theories of physical astronomy were more frequently treated rigorously and symmetrically, apart from any approximate integrations; and if, when the latter are introduced, more care were taken to give a clear and exact view of the nature of the reasoning employed, it might be possible to draw the attention and secure the cooperation of a class of mathematicians who now may well be excused, if, after a slight trial, they turn from the subject in disgust, and prefer to expatiate in those beautiful fields of speculation which are offered to them by other branches of modern geometry and analysis.

The contents of the two last Sections are more or less closely connected with the subjects of various memoirs by other writers, especially Professor HANSEN and the Rev. B. BRONWIN. I cannot pretend to that degree of acquaintance with them which would enable me to give an exact statement of the amount of novelty to be found in my own researches. I believe it is enough to justify me in offering them to the Society; beyond this I make no claim.

Oxford, Feb. 15, 1855.

SECTION IV.

49. The following theorems were demonstrated in the former part of this essay, and are recapitulated here for convenience of reference. (As before, total differentiation with respect to the independent variable t will, in general, be denoted by accents, which will be used *for no other purpose*.)

Theorem I.—If X be a function of n variables $x_1, x_2, \dots x_n$, and if $y_1, y_2, \dots y_n$ be n other variables connected with the former by the n equations

$$\frac{dX}{dx_1}=y_1,\ \frac{dX}{dx_2}=y_2, \dots \frac{dX}{dx_n}=y_n, \quad \dots \dots \dots \quad (50.)$$

then will the values of $x_1, x_2, \dots x_n$, expressed by means of these equations in terms of $y_1, \dots y_n$, be of the form

$$x_1=\frac{dY}{dy_1},\ x_2=\frac{dY}{dy_2}, \dots x_n=\frac{dY}{dy_n}; \quad \dots \dots \dots \quad (51.)$$

and if p be any other quantity explicitly contained in X, then also

$$\frac{dX}{dp}+\frac{dY}{dp}=0 \quad \dots \dots \dots \dots \quad (52.)$$

(the differentiation with respect to p being in each case performed only so far as p appears *explicitly* in the function).

The value of Y is given by the equation

$$Y=-(X)+(x_1)y_1+(x_2)y_2+\dots+(x_n)y_n, \quad \dots \dots \dots \quad (53.)$$

where the brackets indicate that $x_1 \dots x_n$ are supposed to be expressed in terms of $y_1 \dots y_n$ (arts. 2, 3.).

Theorem II.—Suppose the function X to contain explicitly, besides the n variables $x_1 \dots x_n$, another variable t, and also n constants $a_1, a_2, \dots a_n$; and in addition to the equations (50.), let the following be assumed:

$$\frac{d\mathrm{X}}{da_1}=b_1, \dots \frac{d\mathrm{X}}{da_n}=b_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54.)$$

where $b_1, \dots b_n$ are n other constants; so that, by virtue of the $2n$ equations (50.), (54.), the $2n$ variables $x_1 \dots x_n, y_1 \dots y_n$, may be considered as functions of the $2n$ constants $a_1, \dots a_n, b_1, \dots b_n$, and t. Then if from the equations (50.), (54.), and their total differential coefficients with respect to t, the $2n$ constants be eliminated, there will result the following $2n$ simultaneous differential equations of the first order; viz.—

$$x'_i=\frac{d\mathrm{Z}}{dy_i}, \; y'_i=-\frac{d\mathrm{Z}}{dx_i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55.)$$

where Z is a function of $x_1 \dots x_n, y_1, \dots y_n$ (which will in general also contain t explicitly), and is given by the equation

$$\mathrm{Z}=-\left(\frac{d\mathrm{X}}{dt}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56.)$$

In this equation $\frac{d\mathrm{X}}{dt}$ represents the partial differential coefficient of X taken with respect to t so far as t appears explicitly in the original expression for X in terms of $x_1 \dots x_n, a_1 \dots a_n$ and t; and the brackets indicate that $a_1, \dots a_n$ are afterwards to be expressed in terms of the variables by means of the equations (50.), (arts. 5, 6.)

Theorem III.—Let the supposition that the $2n$ variables $x_1 \dots x_n, y_1 \dots y_n$ are expressed in terms of the $2n$ constants and t, be called *Hypothesis* I.; and the converse supposition that $a_1 \dots a_n, b_1 \dots b_n$ are expressed in terms of the $2n$ variables and t, *Hypothesis* II.; then will the following relations subsist:

$$\left.\begin{aligned}\frac{dx_i}{da_j}&=-\frac{db_j}{dy_i}, \; \frac{dx_i}{db_j}=\frac{da_j}{dy_i}\\ \frac{dy_i}{da_j}&=\frac{db_j}{dx_i}, \; \frac{dy_i}{db_j}=-\frac{da_j}{dx_i}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57.)$$

(In each of these equations the first member refers to *Hyp.* I., and the second to *Hyp.* II.; and since there is no connexion between the indices of the variables and those of the constants, the case of $i=j$ has no peculiarity.)

Theorem IV.—Let the symbol $[p, q]$ be an abbreviation for the expression

$$\Sigma_i\left(\frac{dp}{dy_i}\frac{dq}{dx_i}-\frac{dp}{dx_i}\frac{dq}{dy_i}\right)$$

(where p, q are any functions of the $2n$ variables, which may also contain any other quantities explicitly; and the differentiations are performed only so far as x_1,&c.,y_1,&c. appear explicitly in p, q); then if $a_1, \dots a_n, b_1, \dots b_n$ be expressed (*Hyp.* II.) in terms of the $2n$ variables and t, the following equations subsist identically:

$$[a_i, b_i]=-[b_i, a_i]=1, \quad [a_i, b_j]=[a_i, a_j]=[b_i, b_j]=0 \quad . \quad . \quad . \quad (58.)$$

(i being different from j); and obviously in all cases

$$[p, q]=-[q, p], \text{ and } [p, p]=0 \text{ (art. 9.)}.$$

Theorem V.—If u, v be either (1) any two functions whatever of the $2n$ constants a_1, &c., b_1, &c., or (2) any two functions whatever of the $2n$ variables x_1, &c., y_1, &c. (which may in either case also contain t explicitly)*, then

$$\Sigma_i\left\{\frac{du}{dy_i}\frac{dv}{dx_i}-\frac{du}{dx_i}\frac{dv}{dy_i}\right\}=\Sigma_i\left\{\frac{du}{da_i}\frac{dv}{db_i}-\frac{du}{db_i}\frac{dv}{da_i}\right\}. \quad \text{(59.)}$$

(When u, v represent functions of the constants, the differential coefficients in the first member of this equation refer to *Hyp.* II.; and, when functions of the variables, those in the second member refer to *Hyp.* I.) (art. 10.).

Theorem VI.—Let $x_1, \ldots x_n, y_1, \ldots y_n$ be $2n$ variables concerning which no supposition is made except that they are connected by n equations of the form

$$a_i=\phi_i(x_1, x_2, \ldots . x_n, y_1, y_2, \ldots y_n) \quad \text{(a.)}$$

(where the functions on the right are only subject to the condition that the n equations (a.) shall be algebraically sufficient to determine $y_1, \ldots y_n$ in terms of $x_1, \ldots x_n$, a_1, &c., and may contain explicitly any other quantities besides x_1, &c., y_1, &c.).

Then, if by means of the equations (a.) the n variables $y_1, y_2, \ldots y_n$ be expressed as functions of x_1, x_2, &c., a_1, &c.; in order that the $\frac{n(n-1)}{2}$ conditions

$$\frac{dy_i}{dx_j}=\frac{dy_j}{dx_i}$$

may subsist identically, it is necessary and sufficient that each of the $\frac{n(n-1)}{2}$ expressions $[a_i, a_j]$ vanish identically.

Theorem VII.—Let Z be any function whatever of $2n$ variables $x_1 \ldots x_n, y_1 \ldots y_n$, and t. If of the system of $2n$ simultaneous differential equations of the first order

$$x_i'=\frac{dZ}{dy_i}, \; y_i'=-\frac{dZ}{dx_i} \quad \text{(I.)}$$

there be given n integrals involving n arbitrary constants $a_1, a_2, \ldots a_n$, so that each of these constants may be expressed as a function of the variables x_1, &c., y_1, &c. (with or without t); then *if the* $\frac{n(n-1)}{2}$ *conditions* $[a_i, a_j]=0$ *subsist identically, the remaining* n *integrals may be found*, as follows. By means of the n given integrals let the n variables $y_1 \ldots y_n$ be expressed in terms of x_1, &c., a_1, &c.; and let (Z) be what Z becomes when $y_1 \ldots y_n$ are thus expressed. These values of $y_1, y_2 \ldots y_n$ and $-$(Z), will be the partial differential coefficients with respect to $x_1, x_2, \ldots x_n$ and t, of one and the same function; call this function X, then, since its partial differential coefficients are

* It was inadvertently stated in art. 10, that u, v must *not* contain t explicitly. But it is evident that no such limitation is implied in the demonstration of the theorem. The preceding theorem is obviously a particular case of this; namely, the case in which $u=a_j$, $v=b_j$.

all given $\left(\text{by the equations } \frac{dX}{dx_i}=y_i, \frac{dX}{dt}=-(Z)\right)$, X may be found by simple integration, and is therefore to be considered a given function of $x_1, \ldots x_n, a_1, \ldots a_n$ and t. The remaining n integrals are then given by the n equations

$$\frac{dX}{da_i}=b_i,$$

$b_1 \ldots b_n$ being n new arbitrary constants.

[On the relation between this theorem and the theories of Sir W. R. HAMILTON and JACOBI, see arts. 15–20.]

50. Other results established in the former part will be referred to as occasion may require. To the theorems enunciated in the preceding article, the following may now be added.

Returning to the equations (50.), (54.), (55.), we may observe, that if, in the *first* members of (55.), x_i, y_i be supposed expressed in terms of a_1, &c., b_1, &c. and t, then $\frac{dx_i}{dt}, \frac{dy_i}{dt}$ may be written instead of x'_i, y'_i; since on this hypothesis the total differential coefficients of x_i, y_i are obtained by differentiating with respect to t as it appears explicitly. We have therefore

$$\frac{dx_i}{dt}=\frac{dZ}{dy_i}, \quad \frac{dy_i}{dt}=-\frac{dZ}{dx_i},$$

where the first members refer to *Hyp.* I., and the second to *Hyp.* II. But since the equations (50.), (54.) involve a, b, exactly in the same way as they involve x, y, it is obvious that the same reasoning which leads to the equations just written, would lead, *mutatis mutandis*, to the following, which may be considered as an addition to the system of equations (57.) (Theorem III.):

$$\frac{da_i}{dt}=\frac{dZ}{db_i}, \quad \frac{db_i}{dt}=-\frac{dZ}{da_i}. \qquad (60.)$$

In these equations, a_i, b_i in the *first* members are supposed to be expressed in terms of the variables (*Hyp.* II.), whilst in the second members x_1, &c., y_1, &c. are supposed to be expressed in terms of the constants and t (*Hyp.* I.). As before, $Z=-\frac{dX}{dt}$, but in (60.) Z is differently expressed, being what the Z of (55.) becomes when x_1, &c., y_1, &c. are expressed according to *Hyp.* I.

It is to be remembered that all consequences deduced from the form of the system (50.), (54.) belong to the system of equations, obtained as in Theorem VII., which express the solution of the differential equations (I.). Such a solution will be called, as before, a *normal* solution; and the system of equations obtained by expressing a_1, &c., b_1, &c. in terms of the variables and t, will be called a system of *normal integrals*. (See art. 20, and the note to art. 29.)

51. Let a_1, &c., b_1, &c. be called, as before, *elements*. If then c be any function of the elements, when the latter are expressed in terms of the variables and t

(*Hyp.* II.), c becomes also a function of the same; and we have

$$\frac{dc}{dt}=\Sigma_i\left(\frac{dc}{da_i}\frac{da_i}{dt}+\frac{dc}{db_i}\frac{db_i}{dt}\right)=\Sigma_i\left(\frac{dc}{da_i}\frac{dZ}{db_i}-\frac{dc}{db_i}\frac{dZ}{da_i}\right) \quad \text{(61.)}$$

(see the last article). But, by Theorem V., this becomes

$$\frac{dc}{dt}=[c,Z]. \quad \text{(62.)}$$

It is worth observing that both this equation and (60.) might have been obtained indirectly as follows. Since c is constant, we have $c'=0$; that is, $\frac{dc}{dt}+[Z,c]=0$ (see (32.), art. 22.); this gives (62.), since $[Z,c]=-[c,Z]$, and again, by Theorem V., is changed into (61.); and if, in the latter, we put successively $c=a_j$, $c=b_j$, we obtain the system (60.).

Section V.—*On the Variation of Elements.*

52. The following general problem includes, I believe, all the cases which occur in practice. Let $P_1, \ldots P_n$, $Q_1, \ldots. Q_n$ be any functions whatever of the $2n$ variables $x_1, \ldots x_n$, $y_1, \ldots y_n$ and t. It is required to express the $2n$ integrals of the system of $2n$ simultaneous differential equations of the first order

$$x_i'=P_i,\ y_i'=Q_i \quad \text{(63.)}$$

in the same form as the integrals (supposed given) of the canonical system

$$x_i'=\frac{dZ}{dy_i},\quad y_i'=-\frac{dZ}{dx_i}, \quad \text{(I.)}$$

by substituting functions of t for the constant elements of the latter system.

Suppose a *normal solution* (see end of art. 50.) of the system (I.) to be employed. The elements a_i, b_i represent the same functions of x_1, &c., y_1, &c. and t as before, but are now *variable*; consequently we have

$$a_i'=\frac{da_i}{dt}+\Sigma_j\left\{\frac{da_i}{dx_j}x_j'+\frac{da_i}{dy_j}y_j'\right\}=\frac{da_i}{dt}+\Sigma_j\left\{P_j\frac{da_i}{dx_j}+Q_j\frac{da_i}{dy_j}\right\},$$

with a similar expression for b_i'. But, by equations (57.) and (60.), these are immediately transformed into the following:

$$\left.\begin{aligned}a_i'&=\frac{dZ}{db_i}+\Sigma_j\left\{Q_j\frac{dx_j}{db_i}-P_j\frac{dy_j}{db_i}\right\}\\ b_i'&=-\frac{dZ}{da_i}-\Sigma_j\left\{Q_j\frac{dx_j}{da_i}-P_j\frac{dy_j}{da_i}\right\}\end{aligned}\right\} \quad \text{(E.)}$$

where Z, Q_j, P_j, x_j, y_j in the second members are supposed to be expressed (*Hyp.* I.) in terms of the elements and t. Thus the system (63.) is transformed into a system involving the new variables a_i, b_i, instead of the original variables x_i, y_i.

53. If, instead of employing a set of *normal integrals* of the pattern system (I.), we take *any* complete set of integrals $c_1, c_2, \ldots c_{2n}$, then c_1, &c. may be considered as

functions of a_1, &c., and again, through them, of the variables. We have then

$$c'_i=\frac{dc_i}{da_1}a'_1+\ldots+\frac{dc_i}{db_1}b'_1+\ldots;$$

and if in this equation the values of a'_1, &c. be introduced from the formula (E.) of the last article, the following expression results:

$$c'_i=\{Z, c_i\}+\Sigma_j(Q_j\{x_j, c_i\}-P_j\{y_j, c_i\})$$

(in which the symbol $\{p, q\}$ is used to denote

$$\Sigma_k\left\{\frac{dp}{db_k}\frac{dq}{da_k}-\frac{dp}{da_k}\frac{dq}{db_k}\right\},$$

so that by (59.) (Theorem V.) we have $\{p, q\}=-[p, q]$; but in $\{p, q\}$ p and q are considered as functions of a_1, &c., b_1, &c., whilst in $[p, q]$ they are considered as functions of x_1, &c., y_1, &c.). Now, considering p, q as functions of c_1, &c., and through these, of a_1, &c., we have (by reasoning exactly similar to that employed in deducing equation (24.), art. 9.)

$$\{p, q\}=\Sigma\left(\{c_\alpha, c_\beta\}\left(\frac{dp}{dc_\alpha}\frac{dq}{dc_\beta}-\frac{dp}{dc_\beta}\frac{dq}{dc_\alpha}\right)\right)$$

(the summation referring to all binary combinations of the indices α, β). Hence we have, putting $q=c_i$,

$$\{p, c_i\}=\Sigma_\alpha\left(\{c_\alpha, c_i\}\frac{dp}{dc_\alpha}\right), \quad \ldots \quad (64.)$$

and consequently the above expression for c'_i becomes

$$c'_i=\{Z, c_i\}+\Sigma_\alpha\Sigma_j\left(\{c_\alpha, c_i\}\left(Q_j\frac{dx_j}{dc_\alpha}-P_j\frac{dy_j}{dc_\alpha}\right)\right), \quad \ldots \quad (F.)$$

an equation which is easily seen to become identical with (E.), art. 52, when $c_1\ldots c_{2n}$ represent $a_1\ldots a_n$, $b_1\ldots b_n$.

54. The simplest case is that in which the system of equations (63.), whose integrals are sought, are of the *canonical form*; that is, where

$$P_i=\frac{dW}{dy_i}, \quad Q_i=-\frac{dW}{dx_i},$$

W being a given function of the variables (with or without t). In this case the formula (E.) becomes

$$\left.\begin{aligned} a'_i&=\frac{dZ}{db_i}-\frac{dW}{db_i} \\ b'_i&=-\frac{dZ}{da_i}+\frac{dW}{da_i} \end{aligned}\right\} \quad \ldots \quad (65.)$$

whilst (F.) is easily found to be reducible, by the help of (64.), to either of the following forms:

$$c'_i=\{Z, c_i\}-\{W, c_i\} \quad \ldots \quad (66.)$$

$$c'_i=\Sigma_\alpha\left(\{c_\alpha, c_i\}\left(\frac{dZ}{dc_\alpha}-\frac{dW}{dc_\alpha}\right)\right). \quad \ldots \quad (67.)$$

If we put $W=Z+\Omega$, so that Ω may be called the "disturbing function," the above formulæ become

$$a_i'=-\frac{d\Omega}{db_i}, \quad b_i'=\frac{d\Omega}{da_i} \quad . \quad . \quad . \quad (68.)$$

$$c_i'=\Sigma_\alpha\left(\{c_i, c_\alpha\}\frac{d\Omega}{dc_\alpha}\right) \quad . \quad . \quad . \quad (69.)$$

On the first of these forms see the note to art. 38. With respect to the form (69.), if we put for $\{c_i, c_\alpha\}$ its equivalent $-[c_i, c_\alpha]$, or $[c_\alpha, c_i]$ (see Theorem V. art. 49.), we obtain the well-known expression

$$c_i'=\Sigma_\alpha\left([c_\alpha, c_i]\frac{d\Omega}{dc_\alpha}\right).$$

The difference between this last form and (69.) consists in this; that in the latter the coefficients $[c_\alpha, c_i]$ are obtained from the expressions for c_1, c_2, &c. *in terms of the variables*; whereas in (69.) the coefficients $\{c_i, c_\alpha\}$ are similarly obtained from the expressions for c_1, &c. *in terms of the normal elements* a_1, &c., b_1, &c.*; and when a *normal solution* of the undisturbed problem has been obtained, the latter process will generally be found much more convenient than the former, since the elements c_1, &c. will usually be much simpler functions of the *normal elements* than of the *variables*.

55. In illustration of this, it will be worth while to deduce the expressions for the variations of the ordinary elliptic elements of a planet's orbit from those of the normal elements given in art. 30.

Let a and e be the semiaxis major and excentricity, ι the inclination of the orbit to a fixed ecliptic, ν the longitude of the node, ϖ the longitude of the perihelion, $nt+(\varepsilon)$ the mean longitude of the planet; *longitudes* being reckoned in the plane of the ecliptic (from a fixed origin) *as far as the node*, and then *on the plane of the orbit*. As usual, n stands for $\frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}$. Also let $nt+(\varepsilon)=\int_0^t ndt+\varepsilon$, so that $\varepsilon'=(\varepsilon)'+tn'$.

If, then, we call the six normal elements α_1, α_2, α_3, β_1, β_2, β_3, we have (see art. 30.)

$$\alpha_1=\frac{m\mu}{2a}, \qquad \beta_1=\frac{\varpi-(\varepsilon)}{n},$$

$$\alpha_2=m\sqrt{\mu a(1-e^2)}, \qquad \beta_2=\varpi-\nu,$$

$$\alpha_3=m\sqrt{\mu a(1-e^2)}.\cos\iota, \qquad \beta_3=\nu;$$

from which, conversely,

$$a=\frac{m\mu}{2\alpha_1}, \qquad \varpi=\beta_2+\beta_3,$$

$$1-e^2=\frac{2\alpha_1\alpha_2^2}{m^3\mu^2}, \qquad \nu=\beta_3,$$

$$\cos\iota=\frac{\alpha_3}{\alpha_2}, \qquad (\varepsilon)=\beta_2+\beta_3-\frac{(2\alpha_1)^{\frac{3}{2}}}{m^{\frac{3}{2}}\mu}\beta_1.$$

* $\{c_i, c_\alpha\}=\Sigma_j\left(\frac{dc_i}{db_j}\frac{dc_\alpha}{da_j}-\frac{dc_i}{da_j}\frac{dc_\alpha}{db_j}\right)$.

From these expressions the values of $\{a, e\}$, $\{a, \iota\}$, &c. are found with the greatest simplicity, and the results are

$$m\mu\{a, (\varepsilon)\}=2na^2, \quad m\mu\{(\varepsilon), e\}=\frac{na\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2}),$$

$$m\mu\{\varpi, e\}=\frac{na\sqrt{1-e^2}}{e}, \quad m\mu\{(\varepsilon), \iota\}=\frac{na}{\sqrt{1-e^2}}\tan\frac{\iota}{2},$$

$$m\mu\{\varpi, \iota\}=\frac{na}{\sqrt{1-e^2}}\tan\frac{\iota}{2}, \quad m\mu\{\nu, \iota\}=\frac{na}{\sin\iota\sqrt{1-e^2}},$$

the rest all vanishing. Hence, observing that if R be taken in its usual signification we have $\Omega=-\mathrm{R}$, we obtain*

$$\mu a' =2na^2\frac{d\mathrm{R}}{d(\varepsilon)},$$

$$\mu e' =\frac{-na\sqrt{1-e^2}}{e}\left\{\frac{d\mathrm{R}}{d\varpi}+(1-\sqrt{1-e^2})\frac{d\mathrm{R}}{d(\varepsilon)}\right\},$$

$$\mu(\varepsilon)'=-2na^2\frac{d\mathrm{R}}{da}+\frac{na\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2})\frac{d\mathrm{R}}{de}+\frac{na}{\sqrt{1-e^2}}\tan\frac{\iota}{2}\frac{d\mathrm{R}}{d\iota},$$

$$\mu\varpi' =na\left\{\frac{\sqrt{1-e^2}}{e}\frac{d\mathrm{R}}{de}+\frac{1}{\sqrt{1-e^2}}\tan\frac{\iota}{2}\frac{d\mathrm{R}}{d\iota}\right\},$$

$$\mu\iota' =\frac{-na}{\sqrt{1-e^2}}\left\{\frac{1}{\sin\iota}\frac{d\mathrm{R}}{d\nu}+\tan\frac{\iota}{2}\left(\frac{d\mathrm{R}}{d(\varepsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)\right\},$$

$$\mu\nu' =\frac{na}{\sin\iota\sqrt{1-e^2}}\frac{d\mathrm{R}}{d\iota},$$

in which we may, as usual, put ε for (ε), provided that in forming the term $\frac{d\mathrm{R}}{da}$, nt be exempt from differentiation with respect to a.

56. A comparison of the above process with that by which the corresponding

* If we consider R as a function of p, q instead of ι, ν, where $p=\tan\iota\cos\nu$, $q=\tan\iota\sin\nu$, we find

$$\frac{d\mathrm{R}}{d\iota}=\sec^2\iota\left(\cos\nu\frac{d\mathrm{R}}{dp}+\sin\nu\frac{d\mathrm{R}}{dq}\right)$$

$$\frac{d\mathrm{R}}{d\nu}=\tan\iota\left(\cos\nu\frac{d\mathrm{R}}{dq}-\sin\nu\frac{d\mathrm{R}}{dp}\right),$$

and consequently

$$\mu p'=\frac{-na(\sec\iota)^2}{\sqrt{1-e^2}}\left\{\sec\iota\frac{d\mathrm{R}}{dq}+\tan\frac{\iota}{2}\cos\nu\left(\frac{d\mathrm{R}}{d(\varepsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)\right\}$$

$$\mu q'=\frac{na(\sec\iota)^2}{\sqrt{1-e^2}}\left\{\sec\iota\frac{d\mathrm{R}}{dp}-\tan\frac{\iota}{2}\sin\nu\left(\frac{d\mathrm{R}}{d(\varepsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)\right\}.$$

The formulæ will then agree with those of the Mécanique Céleste (Supplement to vol. iii. p. 360, ed. 1844), if we allow for the different mode of measuring longitudes, and neglect, as Laplace does, terms of the second order with respect to ι and $\frac{d\mathrm{R}}{d\iota}$. (Laplace uses R with the opposite sign.) Those in the text agree (allowing for notation) with the expressions given by Professor Hansen, Astr. Nachr. No. 166, art. 3, equations (2).

expressions are obtained by PONTÉCOULANT*, will show the convenience of using the coefficients $\{c_i, c_j\}$ instead of $[c_i, c_j]$ (in PONTÉCOULANT's notation (c_i, c_j)).

[It will be observed that the formulæ for $(\varepsilon)'$, ϖ', ι' at the end of the last article, do not agree with those of PONTÉCOULANT (p. 330) for the variations of the corresponding quantities ε, ω, φ. The reason of this is as follows:—In PONTÉCOULANT's notation φ expresses the same as ι in this paper, and α the same as ν. But ω (the longitude of the perihelion) is not the same as ϖ; the former being measured *entirely in the plane of the orbit* from a radius vector, *fixed in that plane*†, and assumed as the origin of longitudes. Consequently ε, in PONTÉCOULANT (which we will call $\varepsilon_{\prime}$ for distinction), is not the same as (ε) in the present paper. In fact, if we equate the expressions for the mean anomaly in the two notations, we have

$$\varepsilon_{\prime}-\omega=(\varepsilon)-\varpi;$$

also it is evident that if we put β for the angle between the node and the origin from which ω is measured, we have $d\beta=-\cos\iota d\nu$, and $\varpi=\nu+\beta+\omega$, so that

$$d\varpi=d\omega+(1-\cos\iota)d\nu.$$

If then it were allowable to consider R as capable of being expressed as a function of ω and $\varepsilon_{\prime}$ instead of ϖ and (ε), and if we represented by (R) the expression for R so transformed, we should have

$$\frac{d\mathrm{R}}{d\varpi}d\varpi+\frac{d\mathrm{R}}{d(\varepsilon)}d(\varepsilon)+\&\mathrm{c.}=\frac{d(\mathrm{R})}{d\omega}d\omega+\frac{d(\mathrm{R})}{d\varepsilon_{\prime}}d\varepsilon_{\prime}+\&\mathrm{c.};$$

and if, in the two first terms, we put for $d\varpi$ and $d(\varepsilon)$ the values $d\varpi=d\omega+(1-\cos\iota)d\nu$, $d(\varepsilon)=d\varepsilon_{\prime}+(1-\cos\iota)d\nu$, and compare the two expressions, we find

$$\frac{d\mathrm{R}}{d\varpi}=\frac{d(\mathrm{R})}{d\omega},\quad \frac{d\mathrm{R}}{d(\varepsilon)}=\frac{d(\mathrm{R})}{d\varepsilon_{\prime}},$$

$$\frac{d\mathrm{R}}{d\nu}+(1-\cos\iota)\left(\frac{d\mathrm{R}}{d(\varepsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)=\frac{d(\mathrm{R})}{d\nu}.$$

These relations, together with the equation

$$\varpi'=\omega'+(1-\cos\iota)\nu',$$

are easily seen to render the expressions at the end of art. 55 identical with those of PONTÉCOULANT; in fact, it is by an equivalent transformation that the latter are finally obtained by that author from the correct expressions in p. 328. But it is to be observed that this proceeding is founded upon a *false assumption*; for it is impossible to express R as a function of a, e, ι, ν, $\varepsilon_{\prime}$, ω, as is obvious from the consideration that R, in its original form, is not a function of $(\varepsilon)-\varpi$ merely, but also of (ε); whilst (ε) is *not expressible as a function of the new elements*, as is shown by the equation $d(\varepsilon)=d\varepsilon_{\prime}+(1-\cos\iota)d\nu$‡. It would be out of place to enter further into this sub-

* Théorie Anal. du Système du Monde, tome i. pp. 316–330.

† On the meaning of this expression, see below, art. 73.

‡ It would be a work of some trouble to trace *accurately* the process by which LAPLACE arrives at the for-

ject here, especially as some of the most important principles involved in it have been discussed elsewhere*. See also Appendix B.]

57. Returning to the expression (69.), art. 54, it may be observed that the coefficients $\{c_i, c_j\}$ are to be expressed in terms of c_1, c_2, &c., and this involves no difficulty when *each* of the two sets of elements c_1, &c., a_1, &c. can be expressed *in terms of the other explicitly,* as was the case in the example just discussed. Suppose, however, that the normal set a_1, &c., b_1, &c. are given in terms of the set c_1, &c., but that it is impracticable or inconvenient to obtain the converse equations expressing the latter in terms of the former. In this case we may proceed as follows.

Adopting the notation of art. 1†, and putting f, g for any two of the set c_1, c_2, &c., we have

$$\{f, g\}=\Sigma_i\frac{d(f,g)}{d(b_i, a_i)};$$

suppose this equation written at length, and then, after multiplying by $\frac{d(b_j, a_j)}{d(f,g)}$, let each side be summed with respect to all binary combinations f, g. The result is (see art. 1, equation (4.)),

$$\Sigma\left(\frac{d(b_j, a_j)}{d(f,g)}\cdot\{f, g\}\right)=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70.)$$

(the summation referring to the combinations f, g). Again, if the former equation be multiplied by $\frac{d(p, q)}{d(f, g)}$, where p, q represent any two of the normal elements, a_1, &c. b_1, &c., *except a conjugate pair,* and the sum be taken as before, we have

$$\Sigma\left(\frac{d(p, q)}{d(f, g)}\cdot\{f, g\}\right)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71.)$$

The two formulæ (70.), (71.) give $n(2n-1)$ linear equations for determining the $n(2n-1)$ unknown quantities $\{f, g\}$; the coefficients of the latter being all given functions of c_1, &c. But such cases will hardly occur in practice. (With respect to the form of the above system of linear equations, it is easy to show that the complete determinant of the coefficients is $=1$.)

58. The integration of the formulæ (65.), art. 53, would give the means of expressing the solution of the system

$$x'_i=\frac{dW}{dy_i}, \quad y'_i=-\frac{dW}{dx_i}$$

mulæ alluded to in a preceding note, as the various steps of it are to be found in different places, the notation is somewhat inconsistent, and the results *do not profess to be rigorous.* My impression is, however, that LAPLACE nowhere commits the fallacy of assuming (for example) that R is a function of r, v, z, or r, v, s (see vol. i. p. 295), where v is the angle described by the radius vector on the varying plane of the orbit.

* See JACOBI's two letters to Professor HANSEN in CRELLE's Journal, vol. xlii.

† *i. e.* using $\frac{d(u, v)}{d(x, y)}$ as an abbreviation for $\frac{du}{dx}\frac{dv}{dy}-\frac{du}{dy}\frac{dv}{dx}$.

in the *form* of a normal solution of any other similar system

$$x_i'=\frac{dZ}{dy_i},\quad y_i'=-\frac{dZ}{dx_i},$$

which may be chosen as the pattern.

In the most usual examples the function to be chosen for Z is naturally suggested by the circumstance, that W presents itself under the form of the sum of two functions $Z+\Omega$, of which the former, taken alone, gives an integrable system. But this is not necessarily the case; and it is worth while to observe that the formulæ (65.) take a simple and remarkable form whatever Z may be, provided that it be a function *not containing* t *explicitly.* For then, assuming the "integral of vis viva," $Z=h$, as one of the normal integrals of the pattern system*, the element conjugate to h is τ (the constant added to t); and observing that Z, in (65.), being *expressed in terms of the elements,* reduces itself simply to h, we shall have $\frac{dZ}{dh}=1$, whilst the differential coefficients of Z with respect to all the other elements vanish; so that, if we put $a_1, \ldots a_{n-1}, b_1, \ldots b_{n-1}$ for the remaining elements, the system (65.) takes the following form:—

$$\left.\begin{array}{ll} h'=-\dfrac{dW}{d\tau}, & \tau'=-1+\dfrac{dW}{dh} \\ a_i'=-\dfrac{dW}{db_i}, & b_i'=\dfrac{dW}{da_i} \end{array}\right\} \quad \ldots \ldots \ldots \quad (72.)$$

This, in dynamics, gives the process to be used in the following problem: "*To express the solution of any dynamical problem in the form of the solution of any other* (*involving the same number of variables*) *in which the principle of vis viva subsists.*"

59. As an example of the above process we may apply it to determine the motion of a simple free pendulum (not taking into account the earth's rotation).

Let l be the length of the pendulum, and let the mass of the material point m placed at its extremity be represented by unity. Also let x, y, z be the rectangular coordinates of m, the origin being at the position of rest of m, and the axis of z directed vertically upwards. The equation to the sphere described by m is

$$x^2+y^2+z^2-2lz=0,$$

and the *force-function* U is $-gz$.

Hence if we take, as the two independent coordinates, the radius vector ϱ of the projection of m on the plane of xy, and the angle θ between ϱ and the axis of x, we shall have for the differential equations of motion,

$$\left.\begin{array}{ll} \varrho'=\dfrac{dW}{du}, & \theta'=\dfrac{dW}{dv} \\ u'=-\dfrac{dW}{d\varrho}, & v'=-\dfrac{dW}{d\theta} \end{array}\right\} \quad \ldots \ldots \ldots \quad (A.)$$

* See art. 19 (where h_i in equation (29.) is a misprint for b_i).

where $u, v,$ are the variables conjugate respectively to ϱ, θ, and defined by the equations

$$u=\frac{dT}{d\varrho'},\quad v=\frac{dT}{d\theta'};$$

and W is T—U expressed in terms of ϱ, θ, u, v.

Now $x=\varrho\cos\theta, y=\varrho\sin\theta, z=l-\sqrt{l^2-\varrho^2}$; hence

$$T\left(=\frac{1}{2}(x'^2+y'^2+z'^2)\right)=\frac{1}{2}\left\{\frac{l^2}{l^2-\varrho^2}\varrho'^2+\varrho^2\theta'^2\right\},$$

from which the following expression for W is easily obtained:

$$W=\frac{1}{2}\left(\frac{l^2-\varrho^2}{l^2}u^2+\frac{v^2}{\varrho^2}\right)+g(l-\sqrt{l^2-\varrho^2}).\quad \ldots\ldots\ldots \quad \text{(W.)}$$

Now let us take as a model for the solution of the above system, a set of normal integrals (in polar coordinates) of the system

$$x''+n^2x=0,\quad y''+n^2y=0,\quad \ldots\ldots\ldots \quad \text{(B.)}$$

where $n^2=\frac{g}{l}$. In this system we have $U=-\frac{1}{2}n^2\varrho^2$; and proceeding exactly as in art. 27, we obtain the following results: the two integrals of *vis viva* and of areas are

$$\left.\begin{aligned} h&=\frac{1}{2}\left(u^2+\frac{v^2}{\varrho^2}+n^2\varrho^2\right)\\ c&=v \end{aligned}\right\}\quad \ldots\ldots\ldots \quad \text{(i.)}$$

these are to be solved for u, v; and then V is to be obtained from the equation $dV=ud\varrho+vd\theta$. This gives

$$V=c\theta+\int d\varrho\left\{2h-n^2\varrho^2-\frac{c^2}{\varrho^2}\right\}^{\frac{1}{2}};$$

and the remaining integrals are given by the equations

$$\frac{dV}{dh}=t+\tau,\quad \frac{dV}{dc}=\varpi,$$

τ and ϖ being the elements conjugate respectively to h and c. Performing the differentiations *first*, and taking the integrals in the second term so as to vanish with the expression $\left\{2h-n^2\varrho^2-\frac{c^2}{\varrho^2}\right\}^{\frac{1}{2}}$ (see Appendix A.), we find easily the final equations

$$\left.\begin{aligned} n^2\varrho^2&=h+\sqrt{h^2-n^2c^2}.\cos 2n(t+\tau)\\ c^2\varrho^{-2}&=h-\sqrt{h^2-n^2c^2}.\cos 2(\theta-\varpi) \end{aligned}\right\},\quad \ldots\ldots\ldots \quad \text{(ii.)}$$

in which ϖ is the angle between the axis of x and a (distant) apse, and $-\tau$ is the time of passage through that apse. The four equations (i.), (ii.) comprise a complete normal solution of the equations (B.). The last is the polar equation to the elliptic orbit; and if we call a, b the semiaxes of the ellipse, we have

$$c=nab,\quad h=n^2\frac{a^2+b^2}{2}.$$

60. The solution of the system (A.) of the last article will now be expressed by the same equations (i.), (ii.), if the elements h, c, τ, ϖ be variables defined by the system (see art. 58.)

$$h'=-\frac{dW}{d\tau},\quad \tau'=-1+\frac{dW}{dh}$$

$$c'=-\frac{dW}{d\varpi},\quad \varpi'=\frac{dW}{dc},$$

where W is to be obtained by substituting in the expression (W.), art. 59, the values of ϱ, θ, u, v in terms of the elements and t, derived from equations (i.), (ii.). The result of this substitution is

$$W=\frac{h}{2}+\frac{c^2}{4l^2}-\frac{h^2}{4n^2l^2}-\frac{1}{2}\sqrt{h^2-n^2c^2}.\cos 2n(t+\tau)$$

$$+\frac{h^2-n^2c^2}{4n^2l^2}\cos 4n(t+\tau)+n^2l(l-\sqrt{l^2-\varrho^2}),$$

in which the value of ϱ^2 in the last term must be understood to be substituted from the first of equations (ii.). If we call ϕ the angle between the pendulum and the vertical, we shall have evidently

$$n^2l(l-\sqrt{l^2-\varrho^2})=n^2l^2(1-\cos\phi),$$

and the differential coefficient of this term with respect to any constant k involved in the value of ϱ will be $\frac{n^2}{2\cos\phi}\cdot\frac{d(\varrho^2)}{dk}$. Observing this, we obtain the following expressions for the variations of the elements:

$$h'=-n(\sec\phi-1)\sqrt{h^2-n^2c^2}.\sin 2n(t+\tau)+\frac{h^2-n^2c^2}{nl^2}\sin 4n(t+\tau)$$

$$\tau'=-\frac{h}{2n^2l^2}+\frac{1}{2}(\sec\phi-1)\left(1+\frac{h}{\sqrt{h^2-n^2c^2}}\cos 2n(t+\tau)\right)+\frac{h}{2n^2l^2}\cos 4n(t+\tau)$$

$$c'=0$$

$$\varpi'=\frac{c}{2l^2}-\frac{1}{2}(\sec\phi-1)\frac{n^2c}{\sqrt{h^2-n^2c^2}}\cos 2n(t+\tau)-\frac{c}{2l^2}\cos 4n(t+\tau).$$

The third of these equations gives ab=*constant*; hence, by means of the equations at the end of art. 59, the following expressions are easily deduced:

$$\frac{a'}{a}=-\frac{b'}{b}=\frac{n}{2}\left\{-(\sec\phi-1)\sin 2n(t+\tau)+\frac{a^2-b^2}{l^2}\sin 4n(t+\tau)\right\}$$

$$\tau'=\frac{a^2+b^2}{4l^2}\left(-1+\cos 4n(t+\tau)\right)+\frac{1}{2}(\sec\phi-1)\left(1+\frac{a^2+b^2}{a^2-b^2}\cos 2n(t+\tau)\right)$$

$$\varpi'=nab\left\{\frac{1-\cos 4n(t+\tau)}{2l^2}-(\sec\phi-1)\frac{\cos 2n(t+\tau)}{a^2-b^2}\right\}.$$

These equations are rigorous, and in general not easier to integrate than the original system of which they are a transformation; but they may be integrated approximately on particular hypotheses. For instance, if the pendulum never deviates much from the vertical, $\frac{\varrho^2}{l^2}$ is always small, and $\sec\phi - 1 = \frac{\varrho^2}{2l^2}$ nearly; introducing this value, and substituting for ϱ^2, we have for the *non-periodic* parts of the above expressions

$$\frac{a'}{a} = -\frac{b'}{b} = 0$$

$$\tau' = -\frac{a^2+b^2}{16l^2}, \quad \varpi' = \frac{3}{8}\frac{nab}{l^2}.$$

Hence the axes of the mean ellipse are constant; also if T be the time of describing the ellipse, we shall have approximately $T = \frac{2\pi}{n} \cdot \frac{1}{1+\tau'}$, whence it follows that the motion of the apse during this period will be $\frac{3nab}{8l^2} \cdot \frac{2\pi}{n(1+\tau')}$, or $\frac{3ab}{8l^2} \cdot 2\pi$ nearly.

This agrees with the statement of the Astronomer Royal*. The above approximations would cease to be sufficiently accurate if b were very nearly equal to a, or the motion very nearly circular; but they apply to any other case, and in particular to that in which b is very small, or the motion nearly rectilinear. In the case of nearly circular motion, it would be necessary to develope $\sec\phi$ in a series of powers of $a^2 - b^2$, and the results would be applicable whether ϕ were small or not. But I shall not pursue this subject further here.

Section VI.—*On the Transformation of Variables.*

61. The method of the variation of elements, theoretically considered, consists merely in a transformation of variables of a particular kind; that kind namely, which leads to a new system of differential equations belonging to the *same general class* as the original system. But practically, the choice of variables is determined by the well-known considerations from which the method derives its name.

It is the object of the present section to consider the general class of transformations of which the method in question is a particular, and not the only useful case.

62. *Definition of Normal Transformations.*

Let $\xi_1, \xi_2, \ldots \xi_n, \eta_1, \eta_2, \ldots \eta_n$ be new variables connected with the original variables x_1, &c. y_1, &c. by $2n$ equations (which may also involve t explicitly), such that each variable of either set may be considered as a function of the variables of the other set (with or without t). Let P be any function of $\xi_1, \xi_2, \ldots \xi_n, y_1, y_2, \ldots y_n$ and t; then if the equations connecting the old and new variables can be put in the form

$$\frac{dP}{dy_i} = x_i, \quad \frac{dP}{d\xi_i} = \eta_i, \quad \ldots\ldots\ldots\ldots \quad (73.)$$

I propose to call the transformation *normal.*

* Proceedings of the Royal Astronomical Society, vol. xi. p. 160.

[It follows from Theorem I. art. 49, that the system (73.) is equivalent to each of the following:

$$\frac{dQ}{dx_i}=y_i, \quad \frac{dQ}{d\xi_i}=-\eta_i,$$

in which $Q=-P+\Sigma_i(x_iy_i)$, and is expressed in terms of $x_1, \ldots x_n, \xi_1, \ldots \xi_n$; or

$$\frac{dR}{d\eta_i}=\xi_i, \quad \frac{dR}{dy_i}=-x_i,$$

in which $R=-P+\Sigma_i(\xi_i\eta_i)$, and is expressed in terms of $y_1, \ldots y_n, \eta_1, \ldots \eta_n$; or lastly,

$$\frac{dS}{d\eta_i}=\xi_i, \quad \frac{dS}{dx_i}=y_i,$$

in which $S=-P+\Sigma_i(x_iy_i+\xi_i\eta_i)$, and is expressed in terms of $x_1, \ldots x_n, \eta_1, \ldots \eta_n$.

Any one of these forms might be used; but I shall employ the form (73.) for reasons of convenience.]

63. Inasmuch as the equations (73.) of the last article are of the same general form as the system (54.), art. 49, all the conclusions deduced from that form will subsist, *mutatis mutandis*; so that we may apply the Theorems (II.), (III.), (IV.), (V.), art. 49, by merely changing X into P, and

$x_1, \ldots x_n, y_1, \ldots y_n, a_1, \ldots a_n, b_1, \ldots b_n$, respectively into
$\xi_1, \ldots \xi_n, \eta_1, \ldots \eta_n, y_1, \ldots y_n, x_1, \ldots x_n$;

observing that instead of x'_i, y'_i we must now write $\frac{d\xi_i}{dt}, \frac{d\eta_i}{dt}$*. We thus obtain the following relations:

$$\frac{d\xi_i}{dt}=\frac{d\Psi}{d\eta_i}, \quad \frac{d\eta_i}{dt}=-\frac{d\Psi}{d\xi_i}, \qquad (74.)$$

where Ψ is a function of ξ_1, &c., η_1, &c. and t, defined by the equation

$$\Psi=-\left(\frac{dP}{dt}\right), \qquad (75.)$$

the brackets indicating that the expressions for $y_1, \ldots y_n$ in terms of the new variables ξ_1, &c., η_1, &c., are to be substituted in $\frac{dP}{dt}$ after the differentiation with respect to t; which is performed so far as t appears explicitly in the original expression for P as a function of $\xi_1 \ldots \xi_n, y_1 \ldots y_n$ and t. (See Theorem II.)

We have also the system

$$\left.\begin{array}{l} \frac{d\xi_i}{dy_j}=-\frac{dx_j}{d\eta_i}, \quad \frac{d\xi_i}{dx_j}=\frac{dy_j}{d\eta_i} \\ \frac{d\eta_i}{dy_j}=\frac{dx_j}{d\xi_i}, \quad \frac{d\eta_i}{dx_j}=-\frac{dy_j}{d\xi_i} \end{array}\right\} \qquad (76.)$$

* For in the original theorems x'_i is the same thing as the differential coefficient of x_i taken with respect to t, as t appears explicitly in the expression for x_i in terms of a_1, &c., b_1, &c. and t; the analogous quantity in the present case is therefore the differential coefficient of ξ_i taken with respect to t, as t appears explicitly in the expression for ξ_i in terms of x_1, &c., y_1, &c. and t. But this must not now be denoted by ξ'_i, inasmuch as x_1, &c., y_1, &c. are themselves afterwards to be considered as functions of t.

(see Theorem III.). And if p, q be any two functions of the variables (with or without t), then

$$\Sigma_i\left(\frac{dp}{dy_i}\frac{dq}{dx_i}-\frac{dp}{dx_i}\frac{dq}{dy_i}\right)=\Sigma_i\left(\frac{dp}{d\eta_i}\frac{dq}{d\xi_i}-\frac{dp}{d\xi_i}\frac{dq}{d\eta_i}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (77.)$$

where p and q in the first member are supposed to be expressed in terms of x_1, &c., y_1, &c., and, in the second, in terms of ξ_1, &c., η_1, &c. In other words, the value of $[p, q]$ is the same, whether it be obtained from the expressions for p, q in terms of the original variables, or by an analogous process from their expressions in terms of the new.

Particular cases of (77.) are the relations

$$[\xi_i, \eta_i]=-1, \quad [\xi_i, \xi_j]=[\eta_i, \eta_j]=[\xi_i, \xi_j]=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78.)$$

(See Theorems IV., V.)

64. The relations (74.), (76.), (77.), (78.) of the last article depend solely upon the form of the equations (73.), art. 62, which connect the new variables with the old; and are independent of any supposition as to the equations which may determine either set of variables as functions of t. Let us now, however, introduce the supposition that the original variables $x_1, \ldots x_n, y_1, \ldots y_n$ are determined as functions of t by the system of differential equations,

$$x_i'=\frac{dZ}{dy_i}, \quad y_i'=-\frac{dZ}{dx_i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

The relations just established enable us immediately to transform this system into another involving the new variables instead of the old; for we have

$$\xi_i'=\frac{d\xi_i}{dt}+\Sigma_j\left(\frac{d\xi_i}{dx_j}x_j'+\frac{d\xi_i}{dy_j}y_j'\right);$$

now
$$\frac{d\xi_i}{dt}=\frac{d\Psi}{d\eta_i} \text{ (see (74.), art. 63)};$$

and if in the remaining term we substitute for x_j', y_j' their values from (I.), and for $\frac{d\xi_i}{dx_j}$, $\frac{d\xi_i}{dy_j}$ their values $\frac{dy_j}{d\eta_i}$, $-\frac{dx_j}{d\eta_i}$, it becomes

$$\Sigma_j\left(\frac{dZ}{dx_j}\frac{dx_j}{d\eta_i}+\frac{dZ}{dy_j}\frac{dy_j}{d\eta_i}\right),$$

which is equivalent to $\frac{dZ}{d\eta_i}$, if Z be supposed expressed in terms of the new variables.

We have then
$$\xi_i'=\frac{d\Psi}{d\eta_i}+\frac{dZ}{d\eta_i},$$
and, exactly in the same way,
$$\eta_i'=-\frac{d\Psi}{d\xi_i}-\frac{dZ}{d\xi_i}.$$

This result may be stated in the form of the following *Theorem* VIII*. If the system

* This theorem, in its general form, is, to the best of my knowledge, new. But that case of it in which P does not contain t explicitly has already been proved in a different way by M. Desboves, who has, by means

of differential equations (I.) be transformed by the introduction of new variables $\xi_1, \dots \xi_n, \eta_1, \dots \eta_n$, connected with the original variables $x_1, \dots x_n, y_1, \dots y_n$, by the equations $\frac{dP}{dy_i}=x_i$, $\frac{dP}{d\xi_i}=\eta_i$, where P is any function of $\xi_1, \dots \xi_n, y_1, \dots y_n$, which may also contain t explicitly, then the transformed equations are

$$\xi'_i=\frac{d\Phi}{d\eta_i}, \quad \eta'_i=-\frac{d\Phi}{d\xi_i}, \quad \dots \dots \dots \quad (79.)$$

in which Φ is defined by the equation

$$\Phi=Z-\frac{dP}{dt},$$

and is to be expressed in terms of the new variables. (The substitution of the new variables in $\frac{dP}{dt}$ is to be made *after* the differentiation. See art. 63.).

Corollary.—If P do not contain t explicitly, $\frac{dP}{dt}=0$ and $\Phi=Z$; so that in this case the transformation is effected merely by expressing Z in terms of the new variables.

65. It follows from (77.), art 63, that if f, g be any two integrals of the system (I.), the value of $[f, g]$ is the same whether it be derived from these integrals in their original form, or similarly obtained from the same integrals after transformation by the introduction of the new variables. And consequently *if* n *integrals* $a_1, a_2, \dots a_n$ *of the original system be given, which satisfy the* $\frac{n(n-1)}{2}$ *conditions* $[a_i, a_j]=0$, *they will continue, after a normal transformation, to satisfy the analogous conditions,* so that the method of finding the remaining integrals given in Theorem VII. art. 49, will also continue to be applicable. We had an instance of this in the case of the problem of central forces (art. 27.), where the above conditions were found to subsist after the transformation from rectangular to polar coordinates. (It will be shown presently that every transformation of coordinates is a *normal transformation.*)

66. It was shown in Part I. (art. 18.), that if W be any function of $x_1, x_2, \dots x_n$, $x'_1, x'_2, \dots x'_n$ (which may also contain t explicitly), the system of n differential equations of the second order

$$\left(\frac{dW}{dx'_i}\right)'=\frac{dW}{dx_i} \quad \dots \dots \dots \quad (80.)$$

may be changed into a system of $2n$ equations of the first order of the form (I.),

of it, deduced Jacobi's form of the method of the Variation of Elements (namely, the equations (68.), art. 54), from the similar form of Lagrange, in which the elements are the initial values of the variables. It will appear in the sequel that the extension to the case in which P may contain t, is of importance. If the expression were not already appropriated, I should have proposed definitively to call P the "modulus of transformation;" and I shall use this term provisionally in the present paper, not being able to suggest a tolerable substitute. After all, as the word "modulus" itself is used without confusion in very different senses according to the subject matter, there is, perhaps, no reason why a similar liberty should not be allowed in the use of the proposed expression.

art. 64, by putting $y_i=\frac{dW}{dx'_i}$, and taking Z a function of x_1, &c., y_1, &c. and t, defined by the equation

$$Z=-W+\Sigma_i(x'_i y_i),$$

in which $x'_1, \dots x'_n$ are to be expressed in terms of y_1, &c., x_1, &c.*.

Conversely, a system of the form (I.) being given, it may be changed into a system of the form (80.) as follows: by means of the equations $x'_i=\frac{dZ}{dy_i}$, let $y_1, \dots y_n$ be expressed in terms of x'_1, &c., x_1, &c.; it follows from Theorem I. (art. 49.) that we shall have

$$y_i=\frac{dW}{dx'_i} \quad \text{. . . . . (a.)}$$

and

$$\frac{dZ}{dx_i}=-\frac{dW}{dx_i}, \quad \text{. . . . . (b.)}$$

where W is a function of x_1, &c., x'_1, &c. defined by equation

$$W=-Z+\Sigma_i(x'_i y_i),$$

in which $y_1, \dots y_n$ are to be expressed in terms of x'_1, &c., x_1, &c. The n equations $y'_i=-\frac{dZ}{dx_i}$ are then changed by (a.) and (b.) into the form (80.).

On the Transformation of Coordinates.

67. It has been seen in the preceding article, that we can always change the system of $2n$ equations of the first order of the form (I.), art. 64, into a system of n equations of the second order of the form (80.). In this latter form the equations of dynamics naturally present themselves.

Now in the case of the dynamical equations, $x_1, x_2, \dots x_n$ are the independent *coordinates* of the system (the word *coordinates* being taken in its most general sense), and when the equations are to be changed into the form (I.), the additional variables $y_1, \dots y_n$ are defined by the equations $\frac{dW}{dx'_i}=y_i$. In this case a *transformation of coordinates*, in the most general sense, consists in taking n new variables $\xi_1, \xi_2, \dots \xi_n$, connected with the original coordinates $x_1, \dots x_n$ by n equations, which may also involve t explicitly. It is a well-known theorem, that the transformation of the equations (80.) is effected merely by expressing W in terms of the new coordinates $\xi_1, \dots \xi_n$, and their differential coefficients $\xi'_1, \dots \xi'_n$, instead of the old; so that the new equations are

$$\left(\frac{dW}{d\xi'_i}\right)'=\frac{dW}{d\xi_i}: \quad \text{. . . . . (81.)}$$

the proof of this theorem does not depend upon the form of the function W; and we know also (see arts. 18 & 66.), that whatever be the form of W, these new equations may be again transformed to a system of the form (I.), by taking n additional

* This theorem is a generalization of Sir W. R. HAMILTON's transformation of the Dynamical Equations. See Part I. art. 18.

variables $\eta_1, \dots \eta_n$, defined by the equations

$$\eta_i = \frac{dW}{d\xi'_i}.$$

[It is to be observed that this last transformation is not, in general, equivalent to expressing the original Z in terms of the new variables; for the original Z is $-W+\Sigma(x'_i y_i)$ (art. 66.), and the analogous expression derived from (81.) is $-W+\Sigma(\xi'_i \eta_i)$, which is not, in general, equivalent to the former. It will be seen presently that the two expressions *are* equivalent when the equations connecting $x_1, \dots x_n$ with $\xi_1, \dots \xi_n$ do not involve t explicitly.]

68. Following the analogy of the dynamical equations, I shall adopt the following as the

Definition of a transformation of coordinates.

The original equations (I.), art. 64, having been changed into the form (80.), art. 66, let $\xi_1, \xi_2, \dots \xi_n$ be n new variables connected with the n variables $x_1, x_2, \dots x_n$ by n equations, which may also involve t explicitly; and let $\eta_1, \eta_2, \dots \eta_n$ be n other new variables defined by the equations $\frac{dW}{d\xi'_i} = \eta_i$ (where W has been expressed in terms of ξ_1, &c., ξ'_1, &c.). By means of the $2n$ assumed relations, the $2n$ original variables $x_1, \dots x_n, y_1, \dots y_n$ can be expressed as functions of the $2n$ new variables $\xi_1, \dots \xi_n, \eta_1, \dots \eta_n$. Let this substitution be called a "*transformation of coordinates.*"

It has been seen in the preceding article that the original equations (I.) are changed by a transformation of coordinates into a system of the *same form*, which however cannot in general be obtained by merely expressing Z in terms of the new variables. But we are not at liberty to assume (and it is not generally true) that a change of the system (I.) into another of the same form is a *normal transformation* (art. 62.). It has already been stated, however (end of art. 65.), that this *is true in the present case*; a proposition which I proceed to establish.

69. *Every transformation of coordinates is a normal transformation.*

To prove this theorem, we have to show that every transformation of the kind described in the last article is also of the kind defined in art. 62; in other words, that it is possible to assign a function P, of $\xi_1, \dots \xi_n, y_1, \dots y_n$ (with or without t), such that the given relations between x_1, &c., ξ_1, &c., which define the transformation of coordinates, shall be equivalent to the system of equations

$$\frac{dP}{dy_i} = x_i, \quad \frac{dP}{d\xi_i} = \eta_i.$$

Take $$P = (x_1)y_1 + (x_2)y_2 + \dots + (x_n)y_n \quad \dots \dots \quad (82.)$$

(the brackets indicating that $x_1, \dots x_n$ are to be expressed in terms of $\xi_1, \dots \xi_n$; so that P is a function of $\xi_1, \dots \xi_n, y_1, \dots y_n$, with or without t according as the equations connecting $x_1, \dots x_n$ with $\xi_1, \dots \xi_n$ do or do not contain t explicitly). Then P is the function required. For we have, at once, $\frac{dP}{dy_i}=x_i$; also

$$\frac{dP}{d\xi_i}=y_1\frac{d(x_1)}{d\xi_i}+y_2\frac{d(x_2)}{d\xi_i}+\ldots+y_n\frac{d(x_n)}{d\xi_i};$$

now the definition of η_i is $\eta_i=\frac{dW}{d\xi'_i}$, where

$$W=-(Z)+x'_1y_1+x'_2y_2+\ldots+x'_ny_n$$

(expressed in terms of $\xi_1,\ldots\xi_n$, $\xi'_1,\ldots\xi'_n$ (art. 66.)); hence, putting Z (without brackets) for the original form of Z in terms of x_1, &c., y_1, &c., and observing that (Z) becomes a function of ξ'_1, &c., only through y_1, &c., we have

$$\frac{dW}{d\xi'_i}=-\frac{dZ}{dy_1}\frac{dy_1}{d\xi'_i}-\frac{dZ}{dy_2}\frac{dy_2}{d\xi'_i}-\ldots-\frac{dZ}{dy_n}\frac{dy_n}{d\xi'_i}$$

$$+x_1\frac{dy_1}{d\xi'_i}+x_2\frac{dy_2}{d\xi'_i}+\ldots+x'_n\frac{dy_n}{d\xi'_i}$$

$$+y_1\frac{dx'_1}{d\xi'_i}+y_2\frac{dx'_2}{d\xi'_i}+\ldots+y_n\frac{dx'_n}{d\xi'_i}.$$

The two first lines of this expression vanish by virtue of the equations $x'_i=\frac{dZ}{dy_i}$; and since

$$x'_j=\frac{d(x_j)}{dt}+\frac{d(x_j)}{d\xi_1}\xi'_1+\frac{d(x_j)}{d\xi_2}\xi'_2+\ldots+\frac{d(x_j)}{d\xi_n}\xi'_n,$$

we have

$$\frac{dx'_j}{d\xi'_i}=\frac{d(x_j)}{d\xi_i};$$

so that we have, finally,

$$\eta_i=\frac{dW}{d\xi'_i}=y_1\frac{d(x_1)}{d\xi_i}+y_2\frac{d(x_2)}{d\xi_i}+\ldots+y_n\frac{d(x_n)}{d\xi_i},$$

which is evidently equivalent (see the expression (82.)) to

$$\eta_i=\frac{dP}{d\xi_i}:$$

the proposition in question is thus established, and may be enunciated as follows:—

70. *Theorem* IX.—*Every transformation of coordinates is a normal transformation, of which the modulus* * P *is a function of* $\xi_1,\ldots\xi_n$, $y_1,\ldots y_n$ (*with or without* t) *given by the equation*

$$P=(x_1)y_1+(x_2)y_2+\ldots+(x_n)y_n$$

(*the brackets indicating that* $x_1,\ldots x_n$ *are to be expressed in terms of* $\xi_1,\ldots\xi_n$). (See arts. 62, 68.)

This theorem will be made more intelligible by applying it to a very simple example.

Let it be proposed, then, to transform from rectangular to polar coordinates the differential equations of any dynamical problem referring to the motion of a single material point whose mass is m. Let x, y, z be the rectangular coordinates of m,

* See note on Theorem VIII. art. 64.

and u, v, w the variables conjugate to them; so that, putting $\mathrm{T}=\frac{1}{2}m(\mathrm{x}'^2+\mathrm{y}'^2+\mathrm{z}'^2)$, we have $\mathrm{u}=\frac{d\mathrm{T}}{d\mathrm{x}'}$, &c.; whence $\mathrm{T}=\frac{1}{2m}(\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2)$, and the equations of motion are $\mathrm{x}'=\frac{d\mathrm{Z}}{d\mathrm{u}}$, $\mathrm{u}'=-\frac{d\mathrm{Z}}{d\mathrm{x}}$, &c., where $\mathrm{Z}=\frac{1}{2m}(\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2)-\mathrm{U}$, and U is a given function of x, y, z, with or without t.

Now let r, θ, ϕ be polar coordinates of P, so that

$$\mathrm{x}=r\sin\theta\cos\phi,\quad \mathrm{y}=r\sin\theta\sin\phi,\quad \mathrm{z}=r\cos\theta.$$

Let u, v, w be the variables conjugate to r, θ, ϕ. Then the ordinary process of transformation would be as follows:—

(1) to express x′, y′, z′ in terms of $r, \theta, \phi, r', \theta', \phi'$, and thus transform T into a function of the latter quantities;

(2) to define u, v, w by the equations

$$u=\frac{d\mathrm{T}}{dr'},\quad v=\frac{d\mathrm{T}}{d\theta'},\quad w=\frac{d\mathrm{T}}{d\phi'},$$

and by means of these relations to express r', θ', ϕ' in terms of u, v, w, r, θ, ϕ, so that x′, y′, z′, and therefore, finally, T and Z, might be expressed as functions of the six new variables.

Instead of this, let us adopt the method indicated by the theorem at the beginning of this article.

We have then, for the modulus of transformation,

$$\mathrm{P}=(\mathrm{x})\mathrm{u}+(\mathrm{y})\mathrm{v}+(\mathrm{z})\mathrm{w},$$

in which x, y, z are to be expressed in terms of r, θ, ϕ; so that the proper form of P is

$$\mathrm{P}=\mathrm{u}r\sin\theta\cos\phi+\mathrm{v}r\sin\theta\sin\phi+\mathrm{w}r\cos\theta,$$

and the equations (corresponding to $\eta_i=\frac{d\mathrm{P}}{d\xi_i}$ (art. 69.)) which define the new variables u, v, w, are

$$u=\frac{d\mathrm{P}}{dr},\quad v=\frac{d\mathrm{P}}{d\theta},\quad w=\frac{d\mathrm{P}}{d\phi}.$$

These give

$$u=\sin\theta(\mathrm{u}\cos\phi+\mathrm{v}\sin\phi)+\mathrm{w}\cos\theta,$$

$$v=r\cos\theta(\mathrm{u}\cos\phi+\mathrm{v}\sin\phi)-r\mathrm{w}\sin\theta,$$

$$w=-r\sin\theta(\mathrm{u}\sin\phi-\mathrm{v}\cos\phi),$$

from which the values of u, v, w are easily obtained in terms of the six new variables. But in order to effect the transformation of T, we have only to square each side of these equations, and add them, after dividing the second by r^2 and the third by $(r\sin\theta)^2$; we thus obtain

$$\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2=u^2+\frac{v^2}{r^2}+\frac{w^2}{(r\sin\theta)^2},$$

and the transformed equations of motion are therefore

$$mr'=u,\ m\theta'=vr^{-2},\ m\phi'=w(r\sin\theta)^{-2},$$

$$mu'=r^{-3}(v^2+w^2(\sin\theta)^{-2})+m\frac{d\text{U}}{dr},$$

$$mv'=(r\sin\theta)^{-3}w^2\cos\theta+m\frac{d\text{U}}{d\theta},\ \ w'=\frac{d\text{U}}{d\phi}.$$

The preceding process is not, of course, given for the sake of the result, which may easily be verified directly, but in order to illustrate the meaning of the theorem on which it depends. It is hardly necessary to add, that if the problem involved the independent rectangular coordinates of any number of material points, the transformation to polar coordinates would be effected in the same way, by merely adding to P analogous terms for each point.

71. Before proceeding further there is an important remark to be made.

It has been hitherto assumed that the modulus of transformation P was a function of no other quantities than the $2n$ variables $\xi_1,\dots\xi_n, y_1,\dots y_n$ and t. But if every step of the demonstrations of the theorems of transformation which have been given (art. 62, &c.) be examined, it will be seen that they continue to hold good in the following more general form.

Take $$\text{P}=f(\xi_1,\xi_2,\dots\xi_n,y_1,y_2,\dots y_n,p,q,r,\dots,t),$$

where $p, q, r,\dots$ are any functions of *any or all the variables*, old and new, with or without t.

Let the equations connecting the old and new variables be, as before,

$$\frac{d\text{P}}{dy_i}=x_i,\ \frac{d\text{P}}{d\xi_i}=\eta_i, \qquad (83.)$$

with the condition that p, q, r, ... *are exempt from differentiation in forming these equations.*

Then take $\Psi=-\left(\frac{d(\text{P})}{dt}\right)$, with the following signification; (1) $\frac{d(\text{P})}{dt}$ denotes the differential coefficient of P with respect to t, so far as t is contained explicitly, and also through the variables in $p, q, r,\dots$; that is to say,

$$\frac{d(\text{P})}{dt}=\frac{d\text{P}}{dt}+\frac{d\text{P}}{dp}\cdot p'+\frac{d\text{P}}{dq}\cdot q'+\dots$$

(where $p'=\frac{dp}{dt}+\frac{dp}{dx_1}x_1'+$&c. &c., but this substitution *is not to be made* at this stage); (2) $\left(\frac{d(\text{P})}{dt}\right)$ denotes the result of substituting in the above expression the values of $y_1,\dots y_n$ in terms of $\xi_1,\dots\xi_n,\eta_1,\dots\eta_n,p,q,r,\dots$ from (83.), so far as $\frac{d(\text{P})}{dt}$ contains $y_1,\dots y_n$ explicitly (*i. e.* not involved in $p, q, r,\dots$).

Lastly, take $$\Phi=(\text{Z})+\Psi,$$

where (Z) denotes the result of substituting in Z the values of the old variables as given in terms of $\xi_1, \ldots \eta_1, \ldots p, q, r, \ldots$ from (82.). We shall then have

$$\xi'_i = \frac{d\Phi}{d\eta_i}, \quad \eta'_i = -\frac{d\Phi}{d\xi_i}, \quad \ldots \ldots \ldots \ldots \quad (84.)$$

where Φ is in general a function of

$$\xi_1, \ldots \xi_n, \eta_1, \ldots \eta_n, p, q, r, \ldots p', q', r', \ldots \text{ and } t;$$

and the differentiations with respect to ξ_i, η_i are performed only so far as those variables appear *explicitly* in Φ. But *after these differentiations*, we may introduce the actual values of $p, \ldots p', \ldots$ in terms of the variables and their differential coefficients. It is obvious that the original variables will not, *in general*, have been eliminated from the system (84.); but of course the elimination may be afterwards completed*. Similar considerations apply to that particular case of transformation which we have called a transformation of coordinates (art. 68.). We have then

$$P = (x_1)y_1 + \ldots + (x_n)y_n,$$

and the relations connecting $x_1 \ldots x_n$ with $\xi_1, \ldots \xi_n$, may contain $p, p, r, \ldots$; so that (x_1), &c. are functions of $p, q, r, \ldots$ as well as of $\xi_1, \ldots \xi_n$.

We might have deduced the preceding conclusions from the following simple consideration. Since $p, q, r, \ldots$ *are actually* functions of t, though *unknown* functions, we may imagine them to be *known*, and to be expressed explicitly in terms of t; and then the case resolves itself into that of art. 62, &c., so far as the demonstration is concerned. But as a doubt might possibly have arisen whether any fallacy was involved in the circumstance that $p, q, r, \ldots$ involve (when supposed to be expressed in terms of t) the *arbitrary constants of the problem itself*, it seemed best to refer to the original reasoning; the most important part of which is that contained (*mutatis mutandis*) in art. 6. (For the "mutanda" see the beginning of art. 63.) It is then apparent that this circumstance is perfectly immaterial with reference to the conclusions in question, though it may be important in other points of view.

72. This being premised, we will proceed to an example of transformation more interesting than the former, namely, the

Transformation from fixed to moving axes of coordinates.

Let x, y, z, u, v, w have the same signification as in art. 70, and let x, y, z, u, v, w be the new variables, where x, y, z are rectangular coordinates, referring to a system of moving axes of which the origin always coincides with that of the original fixed axes of x, y, z.

Let the direction cosines of the new (moving) axes with respect to the old be λ_0, μ_0, ν_0; λ_1, μ_1, ν_1; λ_2, μ_2, ν_2, thus†:

* The final equations in this case will not in general have the canonical form.

† I do not know who first used this convenient way of indicating the nine direction cosines by a diagram, but I first saw it in one of M. Lamé's works.

	x	y	z
x	λ_0	λ_1	λ_2
y	μ_0	μ_1	μ_2
z	ν_0	ν_1	ν_2

where λ_0, &c. are functions of t, which may be either given explicitly, or implicitly through the variables (see the last article).

The modulus of transformation P is found (art. 69.) by substituting for the variables x, y, z in the expression xu+yv+zw, their values in terms of x, y, z; we have therefore

$$\mathrm{P}=(\lambda_0 x+\lambda_1 y+\lambda_2 z)\mathrm{u}+(\mu_0 x+\mu_1 y+\mu_2 z)\mathrm{v}+(\nu_0 x+\nu_2 y+\nu_2 z)\mathrm{w};$$

and then the three equations

$$\frac{d\mathrm{P}}{dx}=u,\quad \frac{d\mathrm{P}}{dy}=v,\quad \frac{d\mathrm{P}}{dz}=w,\ \text{give}$$

$$\left.\begin{array}{l} u=\lambda_0\mathrm{u}+\mu_0\mathrm{v}+\nu_0\mathrm{w} \\ v=\lambda_1\mathrm{u}+\mu_1\mathrm{v}+\nu_1\mathrm{w} \\ w=\lambda_2\mathrm{u}+\mu_2\mathrm{v}+\nu_2\mathrm{w} \end{array}\right\},\ \text{whence}\left\{\begin{array}{l} \mathrm{u}=\lambda_0 u+\lambda_1 v+\lambda_2 w \\ \mathrm{v}=\mu_0 u+\mu_1 v+\mu_2 w \\ \mathrm{w}=\nu_0 u+\nu_1 v+\nu_2 w \end{array}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (85.)$$

Also we have (see Theorem VIII. art. 64.), since P is to be considered to contain t explicitly through λ_0, &c., only,

$$\frac{d\mathrm{P}}{dt}=(\lambda_0' x+\lambda_1' y+\lambda_2' z)\mathrm{u}+(\mu_0' x+\mu_1' y+\mu_2' z)\mathrm{v}+(\nu_0' x+\nu_1' y+\nu_2' z)\mathrm{w},$$

in which expression the values of u, v, w in terms of the new variables are to be substituted from (85.). Now if we put ω_0, ω_1, ω_2 for the angular velocities of the moving system of axes about the axes of x, y, z, respectively, so that

$$\omega_0=\lambda_2\lambda_1'+\mu_2\mu_1'+\nu_2\nu_1'=-(\lambda_1\lambda_2'+\mu_1\mu_2'+\nu_1\nu_2'),\ \text{\&c.},$$

it will be immediately seen that the usual relations between the nine direction cosines enable us to put the result of the substitution in the following form:

$$\left(\frac{d\mathrm{P}}{dt}\right)=\omega_0(yw-zv)+\omega_1(zu-xw)+\omega_2(xv-yu).$$

The original differential equations

$$\mathrm{x}'=\frac{d\mathrm{Z}}{d\mathrm{u}},\ \mathrm{u}'=-\frac{d\mathrm{Z}}{d\mathrm{x}},\ \text{\&c}$$

are then transformed into

$$x'=\frac{d\Phi}{du},\ u'=-\frac{d\Phi}{dx},\ \text{\&c. (art. 64.)},$$

where $$\Phi=(\mathrm{Z})-\left(\frac{d\mathrm{P}}{dt}\right).$$

Introducing the above value of $\left(\frac{d\text{P}}{dt}\right)$, and omitting the brackets, we obtain for the system of transformed equations,

$$\left.\begin{aligned} x'&=\frac{d\text{Z}}{du}+\omega_2 y-\omega_1 z, & u'&=-\frac{d\text{Z}}{dx}+\omega_2 v-\omega_1 w \\ y'&=\frac{d\text{Z}}{dv}+\omega_0 z-\omega_2 x, & v'&=-\frac{d\text{Z}}{dy}+\omega_0 w-\omega_2 u \\ z'&=\frac{d\text{Z}}{dw}+\omega_1 x-\omega_0 y, & w'&=-\frac{d\text{Z}}{dz}+\omega_1 u-\omega_0 v \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (86.)$$

in which Z is supposed to be expressed in terms of the new variables.

73. On the principles of the integration of this, and of transformed systems in general, I shall make some remarks hereafter. For the present, the following may be observed. If, in the transformation of the last article, we suppose the motion of the new axes *given*, then λ_0, &c., and therefore also ω_0, ω_1, ω_2, are given explicit functions of t. But if the motion of the new axes is only given by connecting it with the motion of the point m itself, then the above quantities are given functions of the variables and their differential coefficients.

The most interesting case of the latter kind is that in which the motion of the new axes is assumed to satisfy the equations

$$\frac{\omega_0}{x}=\frac{\omega_1}{y}=\frac{\omega_2}{z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\omega.)$$

which express the condition *that the instantaneous axis of rotation* (of the moving axes) *always coincides with the radius vector of the moving point* m*.

The radius vector traces, in fixed space, a certain conical surface. It also traces, with reference to the moving axes, another conical surface; and we might always assume as *one* of the conditions defining their motion, that this latter should be *any proposed surface*; that is, we might assume that the new coordinates x, y, z should always satisfy the equation $\varphi(x, y, z)=0$, φ representing any given homogeneous function. If to this last assumption we add the two conditions expressed by the formula $(\omega.)$, we further assume that *the conical surface traced by the radius vector with reference to the moving axes, rolls upon that traced in fixed space.*

Suppose, for example, we assume for the equation $\varphi(x, y, x)=0$, simply $z=0$. This, with the conditions $(\omega.)$, will express that the radius vector is always in the plane of xy, and that this plane rolls upon the conical surface traced by the radius vector in fixed space. We may then say that the plane of xy is the "plane of the orbit," and that the axes of xy, or any lines fixed with reference to them in their plane, are "fixed in the plane of the orbit†."

* See JACOBI's first letter to Professor HANSEN (CRELLE's Journal, vol. xlii. p. 21). This letter appears to refer to some unpublished (?) results of Professor HANSEN, which may possibly be similar to those of this article.

† The student of elementary treatises is, I believe, *always* left to find out for himself what this expression means, or ought to mean.

Now since we are supposing the equations to belong to a dynamical problem, we have $u=mx'$, $v=my'$, $w=mz'$, and if we substitute on each side of these equations the values derived from (85.), art. 72, we find easily

$$u=m(x'+\omega_1 z-\omega_2 y),\ v=m(y'+\omega_2 x-\omega_0 z),\ w=m(z'+\omega_0 y-\omega_1 x),$$

relations which are true *on all suppositions as to the motion of the axes*; but the assumptions (ω.) reduce them to

$$u=mx',\ v=my',\ w=mz'.$$

The further assumption $z=0$, which involves $z'=0$, gives $w=0$, and also (by the equations (ω.)) $\omega_2=0$. Thus the equations (86.) are reduced to

$$x'=\frac{dZ}{du}, \qquad u'=-\frac{dZ}{dx}$$

$$y'=\frac{dZ}{dv}, \qquad v'=-\frac{dZ}{dy}$$

$$0=\frac{dZ}{dw}, \qquad 0=-\frac{dZ}{dz}+\omega_1 u-\omega_0 v,$$

where in $\frac{dZ}{dw}$ and $\frac{dZ}{dz}$, z and w are to be put $=0$ after the differentiation. It is to be observed also, that the values of u, v, w above given reduce the three first of the equations (86.) to the form $u=m\frac{dZ}{du}$, $v=m\frac{dZ}{dv}$, $w=m\frac{dZ}{dw}$, from which it is evident that

$$Z=\frac{1}{2m}(u^2+v^2+w^2)-U,$$

where U is the original force-function, expressed in terms of the new variables. This depends only upon the conditions (ω.); but in the case now considered we have also $w=0$.

Let the origin of coordinates be the Sun; m a planet disturbed by another planet $m_{\prime}$ whose original coordinates are $x_{\prime}, y_{\prime}, z_{\prime}$, and new coordinates $x_{\prime}, y_{\prime}, z_{\prime}$; also let $x^2+y^2+z^2=r^2$, $x^2+y^2+z^2=r^2$, $x_{\prime}^2+y_{\prime}^2+z_{\prime}^2=r_{\prime}^2$, $x_{\prime}^2+y_{\prime}^2+z_{\prime}^2=r_{\prime}^2$, and $(x_{\prime}-x)^2+(y_{\prime}-y)^2+(z_{\prime}-z)^2=\delta^2$; we have evidently $r^2=r^2$, $r_{\prime}^2=r_{\prime}^2$, and

$$(x_{\prime}-x)^2+(y_{\prime}-y)^2+(z_{\prime}-z)^2=(x_{\prime}-x)^2+(y_{\prime}-y)^2+(z_{\prime}-z)^2,$$

and
$$xx_{\prime}+yy_{\prime}+zz_{\prime}=xx_{\prime}+yy_{\prime}+zz_{\prime}.$$

We have then originally

$$U=\frac{m\mu}{r}+mm_1\left(\frac{1}{\delta}-\frac{xx_{\prime}+yy_{\prime}+zz_{\prime}}{r_{\prime}^3}\right),$$

where $\mu=m+$mass of Sun; and it is evident that this expression preserves the same form when expressed in terms of the new coordinates, and also (which is essential to the validity of what follows), that $\frac{dU}{dx}$, &c. are the same whether the differentiation be performed before or after the substitution for $x_{,}$ $y_{,}$, $z_{,}$, in terms of $x_{,}, y_{,}, z_{,}$. If then we put $z=0$ and

$$R=m_{,}\left(\frac{1}{\delta}-\frac{xx_{,}+yy_{,}}{r_{,}^{3}}\right),$$

we shall have, from the fourth and fifth of the transformed differential equations*,

$$\left.\begin{aligned} x''&=-\frac{\mu x}{r^3}+\frac{dR}{dx} \\ y''&=-\frac{\mu y}{r^3}+\frac{dR}{dy} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87.)$$

from which it is evident, that, *assuming the motion of* $m_{,}$ *to be known relatively to the new axes*, the variations of the four elements of the orbit of m which determine the dimensions of the orbit, its position relatively to lines *fixed in its own plane*, and the time of perihelion passage, will be expressed in terms of the differential coefficients of R in the same way as if the plane of the orbit were fixed. But the motion of the node of the orbit upon the fixed plane of xy, and its inclination to that plane, must be determined by means of the last of the differential equations, as follows: that equation gives

$$\omega_1 u-\omega_0 v=\left(\frac{dZ}{dz}\right)(z=0)\,;$$

or if we put $-\Omega$ for the term multiplied by $mm_{,}$ in the value of U given above,

$$\omega_1 u-\omega_0 v=\frac{d}{dz}\left(-\frac{m\mu}{r}+\Omega\right),$$

and z is to be put $=0$ after the differentiation, which reduces the above to

$$\omega_1 u-\omega_0 v=\left(\frac{d\Omega}{dz}\right).$$

Let $\omega_0^2+\omega_1^2=\alpha^2$, so that α is the angular velocity of the plane of the orbit about the radius vector; then (observing that $u=mx'$, &c.) we have

$$\frac{\omega_0}{x}=\frac{\omega_1}{y}=\frac{\alpha}{r}=\frac{v\omega_0-u\omega_1}{m(xy'-x'y)},$$

whence $$v\omega_0-u\omega_1=\frac{m\alpha}{r}(xy'-x'y)$$

$$=\frac{m\alpha}{r}\sqrt{\mu a(1-e^2)}\,\dagger,$$

and therefore $$m\alpha=-\frac{r}{\sqrt{\mu a(1-e^2)}}\left(\frac{d\Omega}{dz}\right),$$

which gives α in terms of the four elements referred to above, and of t. And if we put ι for the inclination of the orbit to the plane of xy, ν for the longitude of the node

* These equations (87.) have been obtained in a different way by Mr. BRONWIN. Camb. Math. Journ. vol. iv. p. 245.

† See below, art. 80.

referred to the axis of x, and β for the angle between the axis of x and the node, the usual formulæ of rotation give

$$\left.\begin{aligned} \iota' &= \omega_0 \cos\beta - \omega_1 \sin\beta \\ \nu' \sin\iota &= \omega_0 \sin\beta + \omega_1 \cos\beta \\ \beta' &= \omega_2 - \nu' \cos\iota \end{aligned}\right\} \quad . . . . \quad (88.)$$

If in these expressions we put $\omega_0 = \frac{x}{r}\alpha$, $\omega_1 = \frac{y}{r}\alpha$, $\omega_2 = 0$, and call ϑ the angle between the radius vector and the node, so that $\omega_0 \cos\beta - \omega_1 \sin\beta = r\cos\vartheta$, $\omega_0 \sin\beta + \omega_1 \cos\beta = r\sin\vartheta$, we obtain finally

$$\iota' = \alpha\cos\vartheta, \quad \nu' = \alpha\frac{\sin\vartheta}{\sin\iota}, \quad \beta' = -\alpha\sin\vartheta\cot\iota,$$

in which the expression above given for α is to be substituted.

The actual value of $\left(\frac{d\Omega}{dz}\right)$ is $mm_{\prime}z_{\prime}\left(\frac{1}{r_{\prime}^3} - \frac{1}{\delta^3}\right)$, which (since $z_{\prime}$, $r_{\prime}$, &c. are supposed given in terms of t) may be expressed in terms of the four elements first mentioned, and t.

I propose to consider the transformation of the differential equations of the planetary theory in a more general manner in the following section. At present I shall add some remarks on normal transformations in general.

74. *Theorem.* If

$$x_i' = \frac{dZ}{dy_i}, \quad y_i' = -\frac{dZ}{dx_i} \quad . . . . \quad (89.)$$

be a system of $2n$ simultaneous differential equations, where

$$Z = f(x_1, y_1, x_2, y_2, \ldots p, q, r, \ldots, t),$$

and $p, q, r, \ldots$ are also explicit functions of x_1, &c., y_1, &c. and t, but are *exempt from differentiation* in taking the differential coefficients $\frac{dZ}{dx_i}, \frac{dZ}{dy_i}$; and if these equations be transformed by a normal substitution of new variables ξ_1, &c., η_1, &c. (art. 62, equation (73.)), then the transformed equations are, as in art. 64,

$$\xi_i' = \frac{d\Psi}{d\eta_i} + \frac{dZ}{d\eta_i}, \quad \eta_i' = -\frac{d\Psi}{d\xi_i} - \frac{dZ}{d\xi_i},$$

in which Z is expressed in terms of ξ_1, &c., η_1, &c., but the differentiations with respect to ξ_i, η_i are performed *before* the substitution of these variables in p, q, r, &c.; in other words, $p, q, r, \ldots$ are still to be exempt from differentiation in forming the differential equations.

This may be proved simply by repeating the reasoning of art. 64. The only difference is, that in the term

$$\Sigma_j \left\{ \frac{dZ}{dx_j}\frac{dx_j}{d\eta_i} + \frac{dZ}{dy_j}\frac{dy_j}{d\eta_i} \right\}$$

the differential coefficients $\frac{dZ}{dx_j}, \frac{dZ}{dy_j}$ are now taken only so far as Z contains x_j, y_j independently of p, q, r, &c.; and therefore the term represents the differential coefficient $\frac{dZ}{d\eta_i}$ taken so far as Z contains η_i independently of p, q, r, &c. The same reasoning applies to the corresponding term in the value of η_i'. The theorem is thus established.

It is evident that it may be combined with that given in art. 71, where other functions analogous to $p, q, r, \ldots$ are introduced by the modulus of transformation P.

If we call the form of the system of differential equations (89.) *canonical* when the differentiations of Z with respect to x_1, &c., y_1, &c. are *total*, we might call it *pseudo-canonical* when Z contains functions of x_1, &c., y_1, which are exempt from differentiation in forming the differential equations.

In like manner, if we call a transformation of variables *normal*, when the differentiations of the modulus P (equations (73.), art. 62.) with respect to ξ_1, &c., y_1, &c. are total (as in art. 62.), we might call the transformation *pseudo-normal* when P contains functions of the variables which are exempt from differentiation in forming the equations of transformation (as in art. 71.).

Adopting these designations, we may enunciate the following general theorem of transformation:—

Theorem X.—If a *pseudo-canonical* system be transformed by a *normal* or *pseudo-normal* substitution, the transformed equations are also *pseudo-canonical*, and may be formed by the rules applying to normal transformations of canonical systems, provided that the functions which are originally exempt from differentiation with respect to the variables, be continued exempt to the end of the process; but if such functions occur in the modulus of transformation P, they are subject to total differentiation *with respect to* t in forming the term $\frac{dP}{dt}$. (See art. 71.)

[With respect to this theorem there is one important remark to be made. If u, v be any two functions of x_1, &c., y_1, &c. (with or without $p, q, r, \ldots$ and t), the equation

$$\Sigma_i\left(\frac{du}{dy_i}\frac{dv}{dx_i}-\frac{du}{dx_i}\frac{dv}{dy_i}\right)=\Sigma_i\left(\frac{du}{d\eta_i}\frac{dv}{d\xi_i}-\frac{du}{d\xi_i}\frac{dv}{d\eta_i}\right) \quad \text{(art. 63.)}$$

is now only true on condition that the substitution of the actual values of $p, q, r, \ldots$ in terms of the variables be not performed till after all the differentiations.]

75. The theory of the variation of elements affords an interesting example of the theorem given in the last article. Consider the following system of differential equations,

$$x_i'=\frac{dZ}{dy_i}+\frac{d\Omega}{dy_i}, \quad y_i'=-\frac{dZ}{dx_i}-\frac{d\Omega}{dx_i}, \quad \ldots \ldots \ldots \quad (90.)$$

where in $\frac{dZ}{dx_i}$, $\frac{dZ}{dy_i}$ the differentiations are *total*, but Ω is supposed to contain functions of x_1, &c., y_1, &c., which are exempt from differentiation in forming the above equations. The system

$$x_i'=\frac{dZ}{dy_i}, \quad y_i'=-\frac{dZ}{dx_i}$$

is *canonical.* Let us assume then that a complete set of normal integrals $a_1, \dots a_n$, $b_1, \dots b_n$ of this latter system is known, so that we have

$$a_i = \phi_i(x_1, \&c., y_1, \&c., t), \quad b_i = \chi_i(x_1, \&c., y_1, \&c., t).$$

The assumption of these last equations to represent the solution of the complete system (90.) is simply a *transformation of variables,* (a_1, &c., b_1, &c. being the new variables); it is also a *normal transformation,* since the equations connecting the new and old variables may be put in the form (see Theorem VII. art. 49, and art. 62.)

$$\frac{dX}{dx_i} = y_i, \quad \frac{dX}{da_i} = b_i,$$

where X (the modulus of transformation) is a function of $x_1, \dots x_n, a_1, \dots a_n, t$. The function Ψ of art. 62 is now obtained by expressing $-\frac{dX}{dt}$ in terms of $a_1, \dots b_1, \dots$; but since Z is $-\frac{dX}{dt}$ expressed in terms of x_1, &c., y_1, &c., it follows that when Z is expressed in terms of the new variables a_1, &c., b_1, &c., it becomes identical with Ψ. Now if the process of art. 63 be followed, *mutatis mutandis,* it will be seen that in the present case we obtain

$$a_i' = \frac{d\Psi}{db_i} - \frac{dZ}{db_i} - \Sigma_j \left(\frac{d\Omega}{dy_j} \frac{dy_j}{db_i}\right),$$

in which expression the first two terms destroy one another, and the remaining term is evidently the differential coefficient of Ω with respect to b_i, taken so far as Ω contains b_i *independently of those functions which were exempt from differentiation* in forming the original differential equations (90.). Similar reasoning applies to the expression for b_i'.

As this result will be useful, I shall enunciate it separately as

Theorem XI.—If the original system of differential equations be formed by treating certain functions, $p, q, r, \dots$, contained in the disturbing function Ω, as exempt from differentiation with respect to x_1, &c., y_1, &c., the equations which determine the variations of any set of normal elements a_1, &c., b_1, &c.

$$a_i' = -\frac{d\Omega}{db_i}, \quad b_i' = \frac{d\Omega}{da_i}$$

on condition that $p, q, r, \dots$ be treated, in forming these equations, as exempt from differentiation with respect to a_1, &c., b_1, &c.

[It is important to recollect, that *after these equations are formed,* p, q, r, &c. are to be expressed in terms of a_1, &c., b_1, &c., and in the integration of the system a_1, &c., b_1, &c. are to be treated indiscriminately as variables, whether they originally entered through $p, q, r, \dots$ or not].

The Theorem XI. may also be immediately obtained from the general equations (E.) of art. 52 (in which it is to be remembered that Z *includes* the disturbing function). The above method of deducing it is given as an additional illustration of the general theory of transformation.

If, instead of the normal elements a_1, &c., b_1, &c., we employ any other elements, c_1, c_2, &c., which can be expressed as functions of the former, the formula (69.) of art. 54 will still be applicable, with the condition that p, q, r, &c. are exempt from differentiation with respect to c_1, c_2, &c.

Section VII.—*On the Differential Equations of the Planetary Theory.*

76. The differential equations which determine the motions of a system of mutually attracting material points *relatively to one of them,* do not, as is well known, naturally present themselves under the canonical form (I.), art. 49. It is possible indeed to reduce them, by different artifices, to that form; but it seems doubtful whether any practical advantage is gained by doing so. When the ordinary method is followed in the case of a planetary system referred to the sun, there is a distinct disturbing function for each planet; but it is easily seen that the usual expressions for the variations of the elements hold good, not merely for each planet on the hypothesis that the motions of the rest are *known,* but as a complete and rigorous set of simultaneous differential equations involving all the elements of all the orbits, and their differential coefficients with respect to t (and containing of course also t explicitly). It does not appear that we are practically farther from the attainment of the rigorous integration of this system, than we should be if it had the *canonical form,* as it might be made to have if it were derived from an original system of that form; and so far as the development of the disturbing functions is concerned, the most troublesome part of them, which is that depending on the mutual distances of the planets, is not likely to be got rid of by any conceivable artifice.

However this may be, all that I propose to do in the present section is to take the original differential equations in their ordinary form referred to rectangular axes passing through the sun and parallel to fixed directions, and then to exhibit in a general manner the effect of a transformation to new rectangular axes, still passing through the sun, but changing their directions in space according to any arbitrary law.

77. Let M be the mass of the sun, and m, $m_{\prime}$, $m_{\prime\prime}$, &c., the masses of the planets; and put $\mathrm{M}+m=\mu$, $\mathrm{M}+m_{\prime}=\mu_{\prime}$, $\mathrm{M}+m_{\prime\prime}=\mu_{\prime\prime}$, ... $\mathrm{M}+m_{(i)}=\mu_{(i)}$. And, referred to the original axes, let x, y, z be the coordinates of m, $\mathrm{x}_{\prime}$, $\mathrm{y}_{\prime}$, $\mathrm{z}_{\prime}$ of $m_{\prime}$, ..., $\mathrm{x}_{(i)}$, $\mathrm{y}_{(i)}$, $\mathrm{z}_{(i)}$ of $m_{(i)}$. Also let R, $\mathrm{R}_{\prime}$, ... $\mathrm{R}_{(i)}$ have their usual significations, so that

$$\mathrm{R}=\Sigma\left\{\frac{m_{(i)}}{\left((\mathrm{x}_{(i)}-\mathrm{x})^2+(\mathrm{y}_{(i)}-\mathrm{y})^2+(\mathrm{z}_{(i)}-\mathrm{z})^2\right)^{\frac{1}{2}}}-\frac{m_{(i)}(\mathrm{xx}_{(i)}+\mathrm{yy}_{(i)}+\mathrm{zz}_{(i)})}{(\mathrm{x}^2_{(i)}+\mathrm{y}^2_{(i)}+\mathrm{z}^2_{(i)})^{\frac{3}{2}}}\right.,$$

the summation extending to all the planets except m.

Then if we put $\mathrm{x}^2_{(i)}+\mathrm{y}^2_{(i)}+\mathrm{z}^2_{(i)}=\mathrm{r}^2_{(i)}$, the original differential equations of the second order are such as $\mathrm{x}''_{(i)}+\mu\frac{\mathrm{x}_{(i)}}{\mathrm{r}^3_{(i)}}=\frac{d\mathrm{R}_{(i)}}{d\mathrm{x}_{(i)}}$. Let $\mathrm{u}_{(i)}$, $\mathrm{v}_{(i)}$, $\mathrm{w}_{(i)}$ be the variables conjugate to

$x_{(i)}, y_{(i)}, z_{(i)}$. Then instead of the original system of differential equations of the second order, we have the following system of the first order:

$$x'_{(i)}=\frac{dZ_{(i)}}{du_{(i)}},\quad y'_{(i)}=\frac{dZ_{(i)}}{dv_{(i)}},\quad z'_{(i)}=\frac{dZ_{(i)}}{dw_{(i)}}$$

$$u'_{(i)}+\frac{dZ_{(i)}}{dx_{(i)}}=0,\quad v'_{(i)}+\frac{dZ_{(i)}}{dy_{(i)}}=0,\quad w'_{(i)}+\frac{dZ_{(i)}}{dz_{(i)}}=0,$$

in which
$$Z_{(i)}=\frac{1}{2m_{(i)}}(u^2_{(i)}+v^2_{(i)}+w^2_{(i)})-m_{(i)}\left(\frac{\mu_{(i)}}{r_{(i)}}+R_{(i)}\right).$$

These equations are not of the canonical form, because $R_{(i)}$ is not the same for all the planets. But it is easy to put them in the *pseudo-canonical* form (art. 74.), a process which is not *necessary*, but saves trouble by bringing them under the operation of the general rules of transformation established in former articles.

In fact, if we take

$$Z=\Sigma\left(\frac{u^2_{(i)}+v^2_{(i)}+w^2_{(i)}}{2m_{(i)}}-\frac{m_{(i)}\mu_{(i)}}{r_{(i)}}\right)$$

$$Q=\Sigma m_{(i)}m_{(j)}\left(\frac{x_{(i)}\bar{x}_{(j)}+y_{(i)}\bar{y}_{(j)}+z_{(i)}\bar{z}_{(j)}}{\bar{r}^2_{(j)}}+\frac{\bar{x}_{(i)}x_{(j)}+\bar{y}_{(i)}y_{(j)}+\bar{z}_{(i)}z_{(j)}}{\bar{r}^2_{(i)}}\right.$$
$$\left.-\frac{1}{((x_{(i)}-x_{(j)})^2+(y_{(i)}-y_{(j)})^2+(z_{(i)}-z_{(j)})^2)^{\frac{1}{2}}}\right),$$

where the summation in the first term extends to all the planets, and in the second to all their binary combinations, and a horizontal line placed over any letter indicates *exemption from differentiation*, we shall have

$$x'_{(i)}=\frac{dZ}{du_{(i)}}+\frac{dQ}{du_{(i)}},\quad u'_{(i)}=-\frac{dZ}{dx_{(i)}}-\frac{dQ}{dx_{(i)}}, \quad \ldots \ldots \quad (91.)$$

with similar equations for $y'_{(i)}, z'_{(i)}, v'_{(i)}, w'_{(i)}$.

[The terms $\frac{dQ}{du_{(i)}}$ &c. are only written for the sake of uniformity, being really $=0$, since Q does not involve $u_{(i)}, v_{(i)}, w_{(i)}$.]

In these equations Z and Q are *the same for the whole system*, and the differentiations of Z are *total*; but those of Q are restricted to the quantities not marked by the horizontal line, so that $\frac{dQ}{dx_{(i)}}$ is really the same thing as $-m_{(i)}\frac{dR_{(i)}}{dx_{(i)}}$.

78. Let us now refer the whole system to new (moving) rectangular axes, whose position at any instant with respect to the original axes is defined, as in art. 72, by the variable direction-cosines λ_0, &c. Let $x_{(i)}, y_{(i)}, z_{(i)}$ be the new coordinates, and $u_{(i)}, v_{(i)}, w_{(i)}$ the new variables conjugate to them. The transformation will be effected, as in art. 72, by taking for the modulus

$$P=\Sigma\Big((\lambda_0x_{(i)}+\lambda_1y_{(i)}+\lambda_2z_{(i)})u_{(i)}+(\mu_0x_{(i)}+\mu_1y_{(i)}+\mu_2z_{(i)})v_{(i)}+(\nu_0x_{(i)}+\nu_1y_{(i)}+\nu_2z_{(i)})w_{(i)}\Big),$$

and the result will be as follows: put

$$\Omega=Q+\bar{\omega}_0\Sigma(z_{(i)}v_{(i)}-y_{(i)}w_{(i)})+\bar{\omega}_1\Sigma(x_{(i)}w_{(i)}-z_{(i)}u_{(i)})+\bar{\omega}_2\Sigma(y_{(i)}u_{(i)}-x_{(i)}v_{(i)})$$

(where Q is expressed in terms of the new variables, and ω_0, ω_1, ω_2 (the angular velocities of the moving system of axes about the three moving axes themselves) are marked with the horizontal line to show that these quantities are exempt from differentiation in forming the following system of differential equations, though they may be functions of the variables); then the system (91.) is transformed into

$$x'_{(i)}=\frac{dZ}{du_{(i)}}+\frac{d\Omega}{du_{(i)}},\quad u'_{(i)}=-\frac{dZ}{dx_{(i)}}-\frac{d\Omega}{dx_{(i)}}, \quad \dots\dots \quad (92.)$$

with similar equations for y'_i, &c.

In these equations Z is to be expressed in terms of the new variables; and it is evident from the original form of Z and Q, that when so expressed, these two quantities are the same functions of the new variables that they were of the old, and involve (see art. 74.) the quantities exempt from differentiation in the same way*. Thus the transformed system (92.) contains no terms *explicitly depending upon the motion of the axes,* except those introduced by the three terms multiplied by ω_0, ω_1, ω_2 in the value of Ω given above; and the addition of these terms constitutes the only difference between the form of the old and of the new system.

79. We may now apply the method of the variation of elements to the system (92.) as follows:—

The system obtained by omitting the disturbing function Ω, namely,

$$\left.\begin{array}{lll} x'_{(i)}=\dfrac{dZ}{du_{(i)}}, & y'_{(i)}=\dfrac{dZ}{dv_{(i)}}, & z'_{(i)}=\dfrac{dZ}{dw_{(i)}} \\ u'_{(i)}+\dfrac{dZ}{dx_{(i)}}=0, & v'_{(i)}+\dfrac{dZ}{dy_{(i)}}=0, & w'_{(i)}+\dfrac{dZ}{dz_{(i)}}=0 \end{array}\right\} \quad \dots\dots \quad (93.)$$

is *canonical,* and consists simply of the aggregate of the equations representing, for each planet, undisturbed elliptic motion about the sun † (relatively to the new axes of coordinates).

The integrals of these equations may therefore be expressed in any of the usual forms. We will suppose that the elements chosen are

$$a,\ e,\ \varpi,\ (\varepsilon),\ \iota,\ \nu,$$

with significations corresponding to those given to the same symbols in art. 55. These letters unaccented will apply to the planet m, and $a_{,}$, $e_{,}$, &c., $a_{,,}$, $e_{,,}$, &c., $a_{(i)}$, $e_{(i)}$, &c., to the planets $m_{,}$, $m_{,,}$, $m_{(i)}$, &c.

The *definitions* of the elements a, e, ϖ, &c. are their expressions in terms of the six

* Since the direction cosines λ_0, &c. are exempt from differentiation in forming the equations connecting the old and new variables from the modulus P, they continue exempt throughout. (Theorem X. art. 74.) Hence we have

$$\mathrm{x}\bar{\mathrm{x}}_{,}+\mathrm{y}\bar{\mathrm{y}}_{,}+\mathrm{z}\bar{\mathrm{z}}_{,}=(\bar{\lambda}_0 x+\bar{\lambda}_1 y+\bar{\lambda}_2 z)(\overline{\lambda_0 x_{,}+\lambda_1 y_{,}+\lambda_2 z_{,}})+\&\text{c.}=x\bar{x}_{,}+y\bar{y}_{,}+z\bar{z}_{,},$$

and similarly for the rest.

† $Z=\Sigma\left(\frac{u^2+v^2+w^2}{2m}-\frac{\mu m}{r}\right)$, the summation extending to all the planets.

variables x, y, z, u, v, w *, and t, and the same expressions continue to be their definitions when they become the variable elements of the disturbed motion.

Now the general formulæ for the variations of the set of elements here chosen have already been given in art. 55; for it is evident that the process in the present case would merely be a repetition, for each planet, of the process there employed. Here however we are to use in every case the disturbing function Ω given in art. 78; but if we observe the effect of the *marks of exemption*, it will be evident that, for the planet m, the only effective part of Ω is

$$\bar{\omega}_0(zv-yw)+\bar{\omega}_1(xw-zu)+\bar{\omega}_2(yu-xv)-m\mathrm{R};$$

and similarly the effective part of Ω for any other planet will be given by suffixing the corresponding number of accents to x, y, z, u, v, w, m, R.

Now if we put A_0, A_1, A_2 for the terms multiplied respectively by ω_0, ω_1, ω_2 in the above expression, we have, by the definitions of the elements,

$$(A_0^2+A_1^2+A_2^2)^{\frac{1}{2}}=m\sqrt{\mu a(1-e^2)}$$
$$A_2=-m\sqrt{\mu a(1-e^2)}\,.\cos\iota,\quad A_1=m\sqrt{\mu a(1-e^2)}\,.\cos\nu\sin\iota$$
$$A_0=-m\sqrt{\mu a(1-e^2)}\,.\sin\nu\sin\iota,$$

so that the disturbing function, so far as m is concerned, becomes

$$-m\{\mathrm{R}+\sqrt{\mu a(1-e^2)}\,.(\bar{\omega}_0\sin\nu\sin\iota-\bar{\omega}_1\cos\nu\sin\iota+\bar{\omega}_2\cos\iota)\}. \quad . \quad . \quad . \quad . \quad (94.)$$

Consequently, since the expressions in art. 55 were obtained by taking $-m$R for the disturbing function, we have merely to add to them the additional terms derived from the part added to R in (94.). Performing the differentiations, and omitting afterwards the symbols of exemption over ω_0, ω_1, ω_2 which cease to be of any use, we obtain, after obvious reductions, the following simple results: if $\frac{\partial a}{dt}$, $\frac{\partial e}{dt}$, &c. represent those parts of the differential coefficients of a, e, &c., with respect to t, which depend upon the motion of the axes of coordinates, then

$$\left.\begin{aligned}
&\frac{\partial a}{dt}=0,\quad \frac{\partial e}{dt}=0\\
&\frac{\partial(\varepsilon)}{dt}=\frac{\partial\varpi}{dt}=\tan\frac{\iota}{2}(\omega_1\cos\nu-\omega_0\sin\nu)-\omega_2\\
&\frac{\partial\iota}{dt}=-(\omega_0\cos\nu+\omega_1\sin\nu)\\
&\frac{\partial\nu}{dt}=-\omega_2+\cot\iota\,(\omega_0\sin\nu-\omega_1\cos\nu)
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (95.)$$

where it is evident that we may write ε instead of (ε) (see art. 55.).

* *Not* their expressions in terms of x, y, z, x', y', z'; for though these are equivalent in the undisturbed equations, they are not here equivalent in the disturbed equations, and therefore the general theory, which assumes the former mode of expression, is not here applicable to the latter.

The complete variations of the elements are then found by adding to the terms just written the expressions given in art. 25.

It is easily seen that the expressions (95.) might have been deduced from geometrical considerations alone, if we had been at liberty to assume beforehand that the *mechanical* and *geometrical* parts of the variations might be calculated separately; the former as if the axes were at rest, and the latter as if there were no disturbing forces. It would not, I believe, be difficult to establish by *à priori* and simple reasoning the validity of such an assumption, and then the above results would only serve as a verification of the method which has been employed to obtain them.

80. In order however that no obscurity may rest upon the interpretation of the formulæ obtained in the last articles, it is necessary to consider the physical (or rather geometrical) meaning of the elements a, e, &c., which we have so far only defined by means of their expressions in terms of the variables x, y, z, u, &c., and to ascertain what relation they bear to the elements similarly defined by means of the original variables x, y, &c., which refer to axes whose directions are invariable.

The relations between the variables ((85.), art. 72.) give immediately

$$x^2+y^2+z^2=\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2$$

$$u^2+v^2+w^2=\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2$$

$$ux+vy+zw=\mathrm{ux}+\mathrm{vy}+\mathrm{zw}\,;$$

and if we put $\quad yw-zv=A,\quad zu-xw=B,\quad xv-yu=C$

$$\mathrm{yw}-\mathrm{zv}=\mathrm{A},\quad \mathrm{zu}-\mathrm{xw}=\mathrm{B},\quad \mathrm{xv}-\mathrm{yu}=\mathrm{C},$$

we find, by virtue of the relations $\mu_1\nu_2-\nu_1\mu_2=\lambda_0$, &c., the following equations:

$$A=\lambda_0\mathrm{A}+\mu_0\mathrm{B}+\nu_0\mathrm{C}$$

$$B=\lambda_1\mathrm{A}+\mu_1\mathrm{B}+\nu_1\mathrm{C}$$

$$C=\lambda_2\mathrm{A}+\mu_2\mathrm{B}+\nu_2\mathrm{C}$$

and $\quad A^2+B^2+C^2=\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2.$

Now $\mathrm{A}(=\mathrm{yw}-\mathrm{zv}=m(\mathrm{yz}'-\mathrm{zy}'))$ is the projection on the plane of yz of the areal velocity of m (relative to fixed space) multiplied by the mass, and B, C have analogous meanings; hence it is evident from the above equations that A, B, C are the projections on the three moving coordinate planes of yz, zx, xy of the absolute areal velocity of m relative to fixed space, multiplied by the mass. (The projections of the areal velocity *relative to the moving axes* would be $yz'-zy'$, &c., which are not proportional to $yw-zv$, &c., since u, v, w are not the same as mx', my', mz', except on a particular hypothesis as to the motion of the axes. See art. 73.).

Inasmuch as the definitions of the elements a, e, ι involve the variables only in the forms $x^2+y^2+z^2$, $u^2+v^2+w^2$, A, B, C, it follows that these three elements are respectively the *semiaxes, excentricity, and inclination to the plane of xy, of the absolute osculating ellipse of the orbit in fixed space.* Thus the instantaneous ellipse, relatively to the moving axes, is of the same dimensions and in the same plane as the true osculating ellipse; and it only remains to show that it coincides with the latter in position, for which purpose we must prove that it *touches the true orbit.* (It does *not* in general touch the relative orbit, because mx', my', mz' are not in general the same functions of the elements and t that u, v, w are.)

81. If we suppose the coordinates x, y, z of m expressed in terms of the elements a, e, &c. and t, and denote by $\frac{dx}{dt}$, &c. their differential coefficients taken so far as t appears explicitly in these expressions, then $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are proportional to the direction cosines, relatively to the axes of x, y, z, of the tangent to the (relative) instantaneous ellipse. And therefore the direction cosines of this same tangent referred to the fixed axes of x, y, z, are proportional respectively to

$$\lambda_0\frac{dx}{dt}+\lambda_1\frac{dy}{dt}+\lambda_2\frac{dz}{dt},\quad \mu_0\frac{dx}{dt}+\mu_1\frac{dy}{dt}+\mu_2\frac{dz}{dt},\quad \nu_0\frac{dx}{dt}+\nu_1\frac{dy}{dt}+\nu_2\frac{dz}{dt}.$$

On the other hand, the direction cosines of the tangent to the *absolute* instantaneous ellipse, referred to the fixed axes, are proportional to $\frac{d\mathrm{x}}{dt}, \frac{d\mathrm{y}}{dt}, \frac{d\mathrm{z}}{dt}$ (the differential coefficients of x, y, z, taken so far as t appears explicitly in the expressions for those variables in terms of the elements of the absolute ellipse). The identity of the two tangents will therefore be established, if we can show that

$$\frac{d\mathrm{x}}{dt}=\lambda_0\frac{dx}{dt}+\lambda_1\frac{dy}{dt}+\lambda_2\frac{dz}{dt}, \text{ \&c.}$$

Now $\frac{dx}{dt}$ is that part of x' which does not depend upon the disturbing function; i. e. (equations (92.), art. 78.) we have identically

$$\frac{dx}{dt}=\frac{dZ}{du},\quad \frac{dy}{dt}=\frac{dZ}{dv},\quad \frac{dz}{dt}=\frac{dZ}{dw};$$

and, in like manner,
$$\frac{d\mathrm{x}}{dt}=\frac{dZ}{d\mathrm{u}},\quad \frac{d\mathrm{y}}{dt}=\frac{dZ}{d\mathrm{v}},\quad \frac{d\mathrm{z}}{dt}=\frac{dZ}{d\mathrm{w}},$$

where Z is the same as before, but expressed in terms of x, y, z, u, v, w, instead of x, y, z, u, v, w. Now let the latter set of variables be expressed in terms of the former by the formulæ of art. 72, and we have

$$\frac{dZ}{d\mathrm{u}}=\frac{dZ}{du}\frac{du}{d\mathrm{u}}+\frac{dZ}{dv}\frac{dv}{d\mathrm{u}}+\frac{dZ}{dw}\frac{dw}{d\mathrm{u}}, \text{ \&c.,}$$

but
$$\frac{du}{d\mathrm{u}}=\lambda_0,\quad \frac{dv}{d\mathrm{u}}=\lambda_1,\quad \frac{dw}{d\mathrm{u}}=\lambda_2 \text{ ((85.), art. 72.), \&c.,}$$

and therefore
$$\frac{d\mathrm{x}}{dt}=\lambda_0\frac{dx}{dt}+\lambda_1\frac{dy}{dt}+\lambda_2\frac{dz}{dt}, \text{ \&c.,}$$

as was to be proved.

82. It follows, then, that the mode of treating the problem adopted in the preceding articles is equivalent to representing the motion of each planet by means of the true osculating ellipse of its actual orbit (relatively to the sun) in fixed space.

The definitions of all the elements (relative to the moving axes) in terms of the six new variables x, y, z, u, v, w, have the same form as those of the corresponding elements (relative to the fixed axes) in terms of x, y, z, u, v, w. The two relative elements a, e are the same as the corresponding *absolute* elements; ι is the inclination of the plane of the ellipse to the moving plane of xy, and ν the longitude of the node reckoned from the axis of x; and since the place of the body in the ellipse is evidently the same, the relations between the remaining elements ϖ and (ε) (or ε) and the corresponding *absolute elements* are purely geometrical.

Comparing these results with those of art. 79, we see that the independence of the formulæ for the mechanical and geometrical variations of the elements of the true osculating ellipse is completely established.

83. In all that precedes, the three variables ω_0, ω_1, ω_2 (the angular velocities of the system of moving axes about the axes themselves) are entirely arbitrary; they may be either explicit functions of t, involving only determinate constants, or they may depend in any way upon the relative or absolute elements of the orbits of any or all the planets, and their differential coefficients with respect to t. In the case in which the expressions for ω_0, ω_1, ω_2 involve only the *relative* elements, when these expressions are introduced in the formulæ (95.), art. 79, and these formulæ completed by the addition of the terms in art. 55, and when the corresponding sets of equations are formed for each planet, we obtain a set of simultaneous differential equations involving all the elements of all the orbits, and their differential coefficients with respect to t. The integration of these equations would determine all the elements as functions of t, and thus the motion of all the planets, relatively to the axes of coordinates, would be known. Lastly, the motion of the whole system, relatively to fixed space, would be found by integrating the system of equations

$$\left.\begin{aligned} \mathrm{L}' &= \omega_0 \cos \mathrm{X} - \omega_1 \sin \mathrm{X} \\ \mathrm{N}' \sin \mathrm{L} &= \omega_0 \sin \mathrm{X} + \omega_1 \cos \mathrm{X} \\ \mathrm{X}' &= \omega_2 - \mathrm{N}' \cos \mathrm{L} \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (96.)$$

where ω_0, ω_1, ω_2 are now given functions of t, and L is the inclination of the plane of xy to that of xy, N is the longitude of the ascending* node of the plane of xy, reckoned from the axis of x, and X is the longitude of the axis of x, reckoned upon the plane of xy, *from the node*, in the direction of positive rotation.

In the case in which ω_0, ω_1, ω_2 cannot be expressed in terms of the *relative* elements, the integrations which determine the relative motion of the system cannot be separated from those which determine the position of the axes in fixed space; but the equations (96.) must be considered simultaneously with the other differential equations of the problem.

* *Ascending* relatively to a positive rotation, *i. e.* from x to y.

Application to the Problem of Three Bodies.

84. I propose to exemplify the preceding principles by applying them to the transformation of the equations which determine the motion of three mutually attracting bodies, considered as material points. Let them be called the sun and two planets, and let the origin of coordinates be placed at the sun, and the notation be the same as in arts. 77–79, so that M, m, $m_{,}$, are the three masses, $M+m=\mu$, $M+m_{,}=\mu_{,}$, &c. Let the elements be chosen as in art. 55, and longitudes be measured, as before, along the plane of xy (from the axis of x) as far as the node, and then along the plane of the orbit.

Putting for convenience $mR=\Omega$, $m_{,}R_{,}=\Omega_{,}$, we have

$$\Omega=mm_{,}\{(r^2+r_{,}^2-2rr_{,}\cos\chi)^{-\frac{1}{2}}-rr_{,}^{-2}\cos\chi\}$$
$$\Omega_{,}=mm_{,}\{(r^2+r_{,}^2-2rr_{,}\cos\chi)^{-\frac{1}{2}}-r_{,}r^{-2}\cos\chi\},$$

where χ is the angle between the two radii vectores. Let I be the mutual inclination of the planes of the two orbits, and let the angular distances, on the planes of the orbits, between their ascending nodes on the plane of xy and their line of intersection, be respectively υ, $\upsilon_{,}$; so that $\theta-\upsilon-\nu$, $\theta_{,}-\upsilon_{,}-\nu_{,}$ are the angular distances of the two radii vectores from the line of intersection (which we may call, simply, *the line of nodes*). We shall then have

$$\left.\begin{aligned}\cos\chi&=\cos(\theta-\upsilon-\nu)\cos(\theta_{,}-\upsilon_{,}-\nu_{,})+\sin(\theta-\upsilon-\nu)\sin(\theta_{,}-\upsilon_{,}-\nu_{,})\cos I\\ \cos I&=\cos\iota\cos\iota_{,}+\sin\iota\sin\iota_{,}\cos(\nu_{,}-\nu)\,*\end{aligned}\right\} \quad . \quad (97.)$$

and υ, $\upsilon_{,}$ are functions of ι, $\iota_{,}$, $\nu_{,}-\nu$, determined by the equations

$$\left.\begin{aligned}\cot\upsilon\sin(\nu_{,}-\nu)&=-\cot\iota_{,}\sin\iota+\cos(\nu_{,}-\nu)\cos\iota\\ \cot\upsilon_{,}\sin(\nu_{,}-\nu)&=\cot\iota\sin\iota_{,}+\cos(\nu_{,}-\nu)\cos\iota_{,}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (98.)$$

Now considering Ω as expressed, on the one hand, in terms of r, $r_{,}$, θ, $\theta_{,}$, ν, $\nu_{,}$, ι, $\iota_{,}$, and on the other in terms of all the elements and t, we have rigorously, as may easily be proved in the usual way †,

$$\frac{d\Omega}{d\theta}=\frac{d\Omega}{d\varepsilon}+\frac{d\Omega}{d\varpi},\quad \frac{d\Omega}{d\theta_{,}}=\frac{d\Omega}{d\varepsilon_{,}}+\frac{d\Omega}{d\varpi_{,}},$$

---

* The arrangement referred to here and in the following articles will be made clear by the accompanying diagram.

† Since the values of r and θ in terms of the elements and t are of the forms

$$r=f(\textstyle\int ndt+\varepsilon-\varpi),\quad \theta=\int ndt+\varepsilon+\phi(\int ndt+\varepsilon-\varpi),$$

we have $$\frac{dr}{d\varepsilon}+\frac{dr}{d\varpi}=0,\quad \frac{d\theta}{d\varepsilon}+\frac{d\theta}{d\varpi}=1\,;$$

from which the equations in the text follow immediately. It may be as well to remind the reader who may happen to recollect the note to the Astronomer Royal's tract on the Planetary Theory (p. 91, ed. 1831), that that note refers to a different way of measuring longitudes.

with similar equations for $\Omega_{\prime}$. In what follows, I shall, as an abridgment, employ the symbol E to denote the operation $\frac{d}{d\epsilon}+\frac{d}{d\varpi}$, and in like manner I shall put $E_{\prime}$ for $\frac{d}{d\epsilon_{\prime}}+\frac{d}{d\varpi_{\prime}}$, so that

$$E\Omega=\frac{d\Omega}{d\epsilon}+\frac{d\Omega}{d\varpi}, \text{ \&c.} \quad \ldots\ldots\ldots \quad (99.)$$

Since r and θ, when expressed in terms of the elements and t, do not contain ι, $\iota_{\prime}$, ν, $\nu_{\prime}$, we have

$$\frac{d\Omega}{d\nu}=\frac{d\Omega}{d\cos\chi}\cdot\frac{d\cos\chi}{d\nu}, \quad \frac{d\Omega}{d\iota}=\frac{d\Omega}{d\cos\chi}\cdot\frac{d\cos\chi}{d\iota}$$

and

$$\frac{d\cos\chi}{d\nu}=-\left(1+\frac{dv}{d\nu}\right)\frac{d\cos\chi}{d\theta}-\frac{dv_{\prime}}{d\nu}\frac{d\cos\chi}{d\theta_{\prime}}$$
$$+\sin\iota\sin\iota_{\prime}\sin(\theta-v-\nu)\sin(\theta_{\prime}-v_{\prime}-\nu_{\prime})\sin(\nu_{\prime}-\nu),$$

in which expression the values of $\frac{dv}{d\nu}$, $\frac{dv_{\prime}}{d\nu_{\prime}}$ are to be obtained by differentiating the equations (98.). In like manner we find

$$\frac{d\cos\chi}{d\iota}=-\frac{dv}{d\iota}\frac{d\cos\chi}{d\theta}-\frac{dv_{\prime}}{d\iota}\frac{d\cos\chi}{d\theta_{\prime}}$$
$$+\sin(\theta-v-\nu)\sin(\theta_{\prime}-v_{\prime}-\nu_{\prime})\big(-\sin\iota\cos\iota_{\prime}+\cos\iota\sin\iota_{\prime}\cos(\nu_{\prime}-\nu)\big).$$

Analogous expressions may be found for $\frac{d\Omega_{\prime}}{d\nu_{\prime}}$, $\frac{d\Omega_{\prime}}{d\iota_{\prime}}$.

85. Hitherto we have assumed nothing concerning the motion of the axes of coordinates. Let us now however take as a first assumption that *the plane of xy shall always pass through the line of nodes.* This implies the conditions*

$$\nu_{\prime}=\nu, \quad v=0, \quad v_{\prime}=0,$$

and consequently $I=\iota_{\prime}-\iota$.

Introducing these conditions in the expressions at the end of the last article, and

* The legitimacy of the following processes will be apparent to the reader who shall have followed the general reasoning of former articles, though probably not to others. In either case it may be useful here briefly to recapitulate the principles now to be applied. The results of art. 79, and in particular the expressions (94.), (95.), were established independently of any assumption as to the values of ω_0, ω_1, ω_2, which are perfectly arbitrary. We are therefore at liberty to assume that ω_0, ω_1, ω_2 are such functions of t that any three functions of the variables shall constantly $=0$. Thus the first assumption made in the text is that $\nu_{\prime}-\nu$ shall constantly $=0$. But since such assumptions may cause (as in this case) certain elements to disappear from the expression of Ω, it is necessary to perform the differentiations of Ω with respect to such elements *first*; the differential coefficients may or may not vanish, on afterwards introducing the assumptions; if not, we find expressions for them in terms of differential coefficients with respect to *other elements which do not disappear*; so that instead of differentiations which ought to be performed *before* the assumed conditions are introduced, we have finally only such as may be performed *afterwards*. The expression "disappear" does not, it must be observed, necessarily mean the same as "vanish." Thus the condition $\nu_{\prime}-\nu=0$, causes ν and $\nu_{\prime}$ both to *disappear* from Ω, but does not of course imply that they both *vanish*.

observing that $\frac{d\Omega}{d\theta}=\text{E}\Omega$, &c. (see equation (99.)), we have evidently

$$\frac{d\Omega}{dv}=-\left(1+\left(\frac{dv}{dv_{,}}\right)\right)\text{E}\Omega-\left(\frac{dv_{,}}{dv}\right)\text{E}_{,}\Omega$$

$$\frac{d\Omega}{d\iota}=-\left(\frac{dv}{d\iota}\right)\text{E}\Omega-\left(\frac{dv_{,}}{d\iota}\right)\text{E}_{,}\Omega-\frac{d\Omega}{d\text{I}};$$

and it only remains to find the values of $\left(\frac{dv}{dv}\right)$, &c., namely, the values of $\frac{dv}{dv}$, &c., which correspond to the assumed conditions $\nu_{,}-\nu=0$, $v=v_{,}=0$.

Now in any spherical triangle of which a, b, c are the sides, and A, B, C the opposite angles, if a be considered as a function of c, A, B, by virtue of the equation

$$\cot a\sin c=\cot \text{A}\sin \text{B}+\cos c\cos \text{B},$$

we have by differentiation

$$\frac{da}{dc}=(\sin a)^2(\cot a\cot c+\cos \text{B})$$

$$=\cos a\sin a\cot c+(1-(\cos a)^2)\cos \text{B}=\cos \text{B}+\cos a\cot \text{C}\sin \text{B}$$

$$\frac{da}{d\text{A}}=\left(\frac{\sin a}{\sin \text{A}}\right)^2\frac{\sin \text{B}}{\sin c}=\frac{\sin a\sin \text{B}}{\sin \text{C}}$$

$$\frac{da}{d\text{B}}=-\frac{(\sin a)^2}{\sin c}\cot \text{A}\cos \text{B}+\cot c(\sin a)^2\sin \text{B}=\sin a\cot \text{C};$$

and if the *sides* of the triangle be indefinitely diminished, these expressions become, in the limit,

$$\frac{da}{dc}=\frac{\sin(\text{B}+\text{C})}{\sin \text{C}},\quad \frac{da}{d\text{A}}=0,\quad \frac{da}{d\text{B}}=0,$$

provided the angle C *do not vanish.*

If these results be applied to the triangle of which the sides are v, $v_{,}$, $\nu_{,}-\nu$, and the opposite angles $\pi-\iota_{,}$, ι, I, it is easily seen that the values of $\left(\frac{dv}{dv}\right)$, &c. are as follows:

$$\left(\frac{dv}{dv}\right)=-\frac{\sin \iota_{,}}{\sin \text{I}},\quad \left(\frac{dv}{dv_{,}}\right)=\frac{\sin \iota_{,}}{\sin \text{I}},\quad \left(\frac{dv_{,}}{dv}\right)=-\frac{\sin \iota}{\sin \text{I}},\quad \left(\frac{dv_{,}}{dv_{,}}\right)=\frac{\sin \iota}{\sin \text{I}}$$

$$\left(\frac{dv}{d\iota}\right)=\left(\frac{dv}{d\iota_{,}}\right)=\left(\frac{dv_{,}}{d\iota}\right)=\left(\frac{dv_{,}}{d\iota_{,}}\right)=0,$$

provided I be not $=0$. We have therefore

$$\left.\begin{aligned}\frac{d\Omega}{dv}&=-\text{E}\Omega+\frac{1}{\sin \text{I}}(\sin \iota_{,}.\text{E}\Omega+\sin \iota.\text{E}_{,}\Omega)\\ \frac{d\Omega}{d\iota}&=-\frac{d\Omega}{d\text{I}};\end{aligned}\right\}\quad \ldots\ldots\quad (100.)$$

exactly in the same way we find

$$\left.\begin{aligned}\frac{d\Omega_{,}}{dv_{,}}&=-\text{E}_{,}\Omega_{,}-\frac{1}{\sin \text{I}}(\sin \iota.\text{E}_{,}\Omega_{,}+\sin \iota_{,}.\text{E}\Omega_{,})\\ \frac{d\Omega_{,}}{d\iota_{,}}&=\frac{d\Omega_{,}}{d\text{I}}.\end{aligned}\right\}\quad \ldots\ldots\quad (101.)$$

The variations of the elements will now be found by introducing the above values of

$$\frac{d\Omega}{d\nu},\ \frac{d\Omega_{\prime}}{d\nu_{\prime}},\ \frac{d\Omega}{d\iota},\ \frac{d\Omega_{\prime}}{d\iota_{\prime}}$$

in the expressions given in art. 55, and completing these expressions by the addition of the terms (95.), art. 79. But the angular velocities ω_0, ω_1, ω_2 are no longer wholly arbitrary, since we have made *one* assumption concerning the motion of the axes, which implies *one* relation between these quantities and the elements. In order to determine ω_0, ω_1, ω_2 completely, it will be necessary to make two more assumptions; but first we will investigate the relation already implied.

86. The complete expressions for ν', $\nu'_{\prime}$, obtained in the way mentioned in the last article, from arts. 55 and 79, may be written in the following form: put for brevity

$$m\sqrt{\mu a(1-e^2)}=p,\quad m_{\prime}\sqrt{\mu_{\prime}a_{\prime}(1-e_{\prime}^2)}=p_{\prime},$$

and put $\mu^{\frac{1}{2}}a^{-\frac{3}{2}}$ for n, and $\mu_{\prime}^{\frac{1}{2}}a_{\prime}^{-\frac{3}{2}}$ for $n_{\prime}$; then

$$\left.\begin{aligned}\nu'&=\frac{1}{p\sin\iota}\frac{d\Omega}{d\iota}+\cot\iota.(\omega_0\sin\nu-\omega_1\cos\nu)-\omega_2\\ \nu'_{\prime}&=\frac{1}{p_{\prime}\sin\iota_{\prime}}\frac{d\Omega_{\prime}}{d\iota_{\prime}}+\cot\iota_{\prime}.(\omega_0\sin\nu_{\prime}-\omega_1\cos\nu_{\prime})-\omega_2\end{aligned}\right\}\quad\dots\dots\quad(102.)$$

and if the latter equation be subtracted from the former, and the conditions

$$\nu_{\prime}=\nu,\quad \frac{d\Omega}{d\iota}=-\frac{d\Omega}{d\mathrm{I}},\ \frac{d\Omega_{\prime}}{d\iota_{\prime}}=\frac{d\Omega_{\prime}}{d\mathrm{I}},\quad \iota_{\prime}-\iota=\mathrm{I}$$

be introduced, the result is easily found to be

$$(\omega_0\sin\nu-\omega_1\cos\nu)\sin\mathrm{I}=\frac{\sin\iota}{p_{\prime}}\frac{d\Omega_{\prime}}{d\mathrm{I}}+\frac{\sin\iota_{\prime}}{p}\frac{d\Omega}{d\mathrm{I}}.\quad\dots\dots\quad(103.)$$

This is the relation between ω_0, ω_1 and the elements and t, implied by the one assumed condition that the plane of xy passes through the line of nodes. The angular velocity ω_2 of the system of axes about the axis of z, is so far left, as it evidently ought to be, perfectly arbitrary.

Ω and $\Omega_{\prime}$ are now functions of the following elements* *only*:—

$$a,\ e,\ (\varepsilon),\ \varpi,\ a_{\prime},\ e_{\prime},\ (\varepsilon_{\prime}),\ \varpi_{\prime},\ \mathrm{I},\ \nu.$$

And we now have

$$\cos\chi=\cos(\theta-\nu)\cos(\theta_{\prime}-\nu)+\sin(\theta-\nu)\sin(\theta_{\prime}-\nu)\cos\mathrm{I}.$$

87. The complete expression for ι', derived from arts. 55 and 79, is easily put in the form

$$\iota'=-\frac{1}{p\sin\iota}\left\{\frac{d\Omega}{d\nu}+(1-\cos\iota)\mathrm{E}\Omega\right\}-(\omega_0\cos\nu+\omega_1\sin\nu);$$

and, on introducing the assumptions of the preceding articles, this will be found to become, after simple reductions,

$$p\sin\mathrm{I}.\iota'=-(\cos\mathrm{I}.\mathrm{E}\Omega+\mathrm{E}_{\prime}\Omega)-(\omega_0\cos\nu+\omega_1\sin\nu)\,p\sin\mathrm{I};$$

* On the difference between (ε) and ε see above, art. 55. We cannot strictly call Ω a function of a and ε.

similarly, we find

$$p_{\prime}\sin I.\iota_{\prime}'=\cos I.E_{\prime}\Omega_{\prime}+E\Omega_{\prime}-(\omega_0\cos\nu+\omega_1\sin\nu)\,p_{\prime}\sin I;$$

and since $I'=\iota_{\prime}'-\iota'$, we obtain from these

$$pp_{\prime}\sin I.I'=p(\cos I.E_{\prime}\Omega_{\prime}+E\Omega_{\prime})+p_{\prime}(\cos I.E\Omega+E_{\prime}\Omega), \quad . \quad . \quad . \quad (104.)$$

an equation which may be transformed as follows:—since

$$\frac{\partial a}{dt}=0, \quad \frac{\partial e}{dt}=0 \text{ (art. 79)},$$

we find from art. 55, $\quad p'=\frac{d\Omega}{d\varepsilon}+\frac{d\Omega}{d\varpi}=E\Omega,$

and similarly $p_{\prime}'=E_{\prime}\Omega_{\prime}$, whence it is easily seen that the above equation may be written in the form

$$(pp_{\prime}\cos I)'+pE\Omega_{\prime}+p_{\prime}E_{\prime}\Omega=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105.)$$

If we investigate, from the above expressions for ι' and $\iota_{\prime}'$, the values of $(p\sin\iota)'$ and $(p_{\prime}\sin\iota_{\prime})'$, we find

$$\sin I.(p\sin\iota)'=-(\cos\iota_{\prime}.E+\cos\iota.E_{\prime})\Omega-p\cos\iota\sin I(\omega_0\cos\nu+\omega_1\sin\nu),$$
$$\sin I.(p_{\prime}\sin\iota_{\prime})'=(\cos\iota.E_{\prime}+\cos\iota_{\prime}.E)\Omega_{\prime}-p_{\prime}\cos\iota_{\prime}\sin I(\omega_0\cos\nu+\omega_1\sin\nu),$$

and, adding these equations,

$$\sin I.(p\sin\iota+p_{\prime}\sin\iota_{\prime})'=(\cos\iota_{\prime}.E+\cos\iota.E_{\prime})(\Omega_{\prime}-\Omega)$$
$$-(p\cos\iota+p_{\prime}\cos\iota_{\prime})\sin I.(\omega_0\cos\nu+\omega_1\sin\nu). \quad . \quad . \quad . \quad (106.)$$

With respect to this equation and (103.), it may be observed that $\omega_0\cos\nu+\omega_1\sin\nu$ is evidently the *angular velocity of the plane of xy about the line of nodes*, and $\omega_1\cos\nu-\omega_0\sin\nu$ is the angular velocity of the same plane about a line in itself perpendicular to the line of nodes; or, which comes to the same thing, the *angular velocity of the line of nodes itself in fixed space, estimated perpendicularly to the plane of xy.*

88. The position of the plane of xy at any instant has been so far left arbitrary, except that it has been subjected to the condition of *passing through the line of nodes.*

As a *further assumption*, that which most naturally presents itself is, that *the plane of xy should always coincide with the principal plane.* By the principal plane I mean, of course, that on which at any instant the sum of the projections of the areal velocities (multiplied by the masses) of the two planets about the sun, is a maximum; it evidently always passes through the line of nodes, and would be the invariable plane if the disturbing functions vanished.

To determine the position of this plane we have (see art. 80.) to express the condition that ι and $\iota_{\prime}$ are always so taken, subject to the equation $\iota_{\prime}-\iota=I$, that the expression $m\sqrt{\mu a(1-e^2)}.\cos\iota+m_{\prime}\sqrt{\mu_{\prime}a_{\prime}(1-e_{\prime}^2)}.\cos\iota_{\prime}$ shall be a maximum. We will put σ for the value of this expression, so that using p and $p_{\prime}$ with the same meaning as before (see art. 86.), we have

$$\sigma=p\cos\iota+p_{\prime}\cos\iota_{\prime},$$

and the required condition of a maximum will evidently be

$$p\sin\iota+p_{\prime}\sin\iota_{\prime}=0\ ^{*}; \qquad (107.)$$

adding the squares of these expressions, we obtain

$$\sigma^{2}=p^{2}+p_{\prime}^{2}+2pp_{\prime}\cos\mathrm{I}, \qquad (108.)$$

which determines the actual value of σ; moreover we have

$$-\frac{\sin\iota}{p_{\prime}}=\frac{\sin\iota_{\prime}}{p}=\frac{\sin\mathrm{I}}{\sigma}; \qquad (109.)$$

and it is easy to find

$$\left.\begin{aligned}\sigma\cos\iota&=p+p_{\prime}\cos\mathrm{I}\\ \sigma\cos\iota_{\prime}&=p_{\prime}+p\cos\mathrm{I}\end{aligned}\right\}. \qquad (110.)$$

so that σ, $\sin\iota$, $\sin\iota_{\prime}$, $\cos\iota$, $\cos\iota_{\prime}$ are all simply expressible in terms of p, $p_{\prime}$ and I. The variation of σ is easily found by means of the equation (105.), which gives

$$\sigma\sigma'=(p_{\prime}\mathrm{E}_{\prime}-p\mathrm{E})(\Omega_{\prime}-\Omega). \qquad (111.)$$

The equation (103.) now gives (see (109.))

$$\sigma(\omega_{1}\cos\nu-\omega_{0}\sin\nu)=\frac{d}{d\mathrm{I}}(\Omega_{\prime}-\Omega); \qquad (112.)$$

and from (106.) we obtain

$$\sigma^{2}\sin\mathrm{I}(\omega_{0}\cos\nu+\omega_{1}\sin\nu)=\{(p+p_{\prime}\cos\mathrm{I})\mathrm{E}_{\prime}+(p_{\prime}+p\cos\mathrm{I})\mathrm{E}\}(\Omega_{\prime}-\Omega). \quad (113.)$$

The two last equations determine the motion of the principal plane in space, irrespectively of any arbitrary *sliding* motion which we may attribute to it in its own plane. For they give the angular velocities with which it is at any instant moving about two lines at right angles to one another in its own plane (see the end of art. 87.). They may be put in another form as follows:—the actual value of $\Omega_{\prime}-\Omega$ is

$$mm_{\prime}\left(\frac{r}{r_{\prime}^{2}}-\frac{r_{\prime}}{r^{2}}\right)\cos\chi,$$

where $\cos\chi$ has the value given above (end of art. 86.); and if the operations indicated be actually performed, observing that $\mathrm{E}r=0$, $\mathrm{E}_{\prime}r=0$, &c., and that

$$\mathrm{E}\cos\chi=\frac{d\cos\chi}{d\theta}, \ \&c.,$$

the results will be found to be

$$\sigma(\omega_{1}\cos\nu-\omega_{0}\sin\nu)=mm_{\prime}\sin\mathrm{I}.\left(\frac{r_{\prime}}{r^{2}}-\frac{r}{r_{\prime}^{2}}\right)\sin(\theta-\nu)\sin(\theta_{\prime}-\nu)$$

$$\sigma^{2}(\omega_{0}\cos\nu+\omega_{1}\sin\nu)=mm_{\prime}\sin\mathrm{I}.\left(\frac{r_{\prime}}{r^{2}}-\frac{r}{r_{\prime}^{2}}\right)\times(p\cos(\theta-\nu)\sin(\theta_{\prime}-\nu)+p_{\prime}\sin(\theta-\nu)\cos(\theta_{\prime}-\nu)).$$

Here $\theta-\nu$, $\theta_{\prime}-\nu$ represent, it will be remembered, the angular distances of the planets from the line of nodes.

We will assume for the present the condition $\omega_{2}=0$, so that the plane of xy may have no sliding motion, but *roll* upon the conical surface to which it is always a tan-

* Referring to the arrangement supposed in the diagram, it will be seen that ι becomes *negative* in the case now considered.

gent plane. If, then, we call ω the actual angular velocity of the principal plane about its instantaneous axis of rotation, so that $\omega=\sqrt{\omega_0^2+\omega_1^2}$, and put j for the angle between the line of nodes and the instantaneous axis (which is the line in which the principal plane is intersected by its consecutive), we shall have

$$\omega_0 \cos\nu+\omega_1 \sin\nu=\omega\cos j \text{ and } \omega_1\cos\nu-\omega_0\sin\nu=\omega\sin j;$$

and the above equations give

$$\sigma\cot j=p\cot(\theta-\nu)+p_{\prime}\cot(\theta_{\prime}-\nu),$$

a result which may also be put in the form (see (109.))

$$\sin \mathrm{I}\cot j=\sin\iota_{\prime}\cot(\theta-\nu)-\sin\iota\cot(\theta_{\prime}-\nu).$$

It is very easy to show by spherical trigonometry that this equation signifies that the instantaneous axis coincides with the line in which the principal plane is intersected by the plane of the two radii vectores*. In other words, we have this theorem :

The principal plane always turns about the line in which it is intersected by the plane of the two radii vectores.

It follows of course that the principal plane always touches the conical surface described in fixed space (relatively to the sun) by the said line. I think it probable that most persons would expect, at first sight, that the principal plane would always touch the conical surface described by the *line of nodes*, which, as has been just shown, is not the case. It is perhaps worth while to verify this result by independent reasoning.

89. Let Roman letters refer, as in former articles, to axes of coordinates having any arbitrary *fixed* directions. Then, putting $\mathrm{A}=m(\mathrm{yz}'-\mathrm{zy}')$, &c. (as in art. 80.), and using ξ, η, ζ for current coordinates, the equation to the principal plane is

$$(\mathrm{A}+\mathrm{A}_{\prime})\xi+(\mathrm{B}+\mathrm{B}_{\prime})\eta+(\mathrm{C}+\mathrm{C}_{\prime})\zeta=0; \quad \ldots\ldots \quad (114.)$$

and the line in which this plane is cut by its consecutive is determined by combining the equation (114.) with that obtained from it by differentiating the coefficients of ξ, η, ζ with respect to t; namely,

$$(\mathrm{A}'+\mathrm{A}'_{\prime})\xi+(\mathrm{B}'+\mathrm{B}'_{\prime})\eta+(\mathrm{C}'+\mathrm{C}'_{\prime})\zeta=0. \quad \ldots\ldots \quad (115.)$$

Now from the fundamental equations

$$m\mathrm{x}''+m\mu\frac{\mathrm{x}}{\mathrm{r}^3}=\frac{d\Omega}{d\mathrm{x}}, \text{ \&c.,}$$

we obtain $\quad \mathrm{A}'=m(\mathrm{yz}''-\mathrm{zy}'')=\mathrm{y}\frac{d\Omega}{d\mathrm{z}}-\mathrm{z}\frac{d\Omega}{d\mathrm{y}}=mm_{\prime}(\mathrm{yz}_{\prime}-\mathrm{zy}_{\prime})(\delta^{-3}-\mathrm{r}^{-3})$

* Let ψ be the angle between the principal plane and the plane of the radii vectores; the former plane divides the angle I into two parts, of which one is $\iota_{\prime}$ and the other is $-\iota$; and we get two spherical triangles which have a common side j, with adjacent angles $\iota_{\prime}$ and ψ in one, $-\iota$ and $\pi-\psi$ in the other; hence

$$\cot(\theta-\nu)\sin j=\cot\psi\sin\iota+\cos j\cos\iota$$
$$\cot(\theta_{\prime}-\nu)\sin j=\cot\psi\sin\iota_{\prime}+\cos j\cos\iota_{\prime};$$

and if cot ψ be eliminated between these, the result is the equation in the text.

(where δ is the distance between m and $m_{\prime}$); and in like manner

$$A'=-mm_{\prime}(yz_{\prime}-zy_{\prime})(\delta^{-3}-r_{\prime}^{-3});$$

we have therefore $$A'+A'_{\prime}=mm_{\prime}(yz_{\prime}-zy_{\prime})(r_{\prime}^{-3}-r^{-3}),$$

with similar expressions for $B'+B'_{\prime}$, $C'+C'_{\prime}$; so that, when the common factor $mm_{\prime}(r_{\prime}^{-3}-r^{-3})$ is omitted, the equation (115.) becomes

$$(yz_{\prime}-zy_{\prime})\xi+(zx_{\prime}-xz_{\prime})\eta+(xy_{\prime}-yx_{\prime})\zeta=0,$$

which is evidently the equation to the plane containing the two radii vectores. Thus the theorem in question is verified.

90. To return from this digression: the motion of the line of nodes *in the principal plane* will be given by putting $\omega_2=0$ in either of the equations (102.), and introducing the value of $\omega_0\sin\nu-\omega_1\cos\nu$ from (103.). In this way we find, after slight reductions,

$$pp_{\prime}\sin I.\nu'=p_{\prime}\cos\iota_{\prime}\frac{d\Omega}{dI}+p\cos\iota\frac{d\Omega_{\prime}}{dI},$$

in which we may substitute for $\cos\iota$, $\cos\iota_{\prime}$, the values given by equations (110.); this gives

$$\sigma pp_{\prime}\sin I.\nu'=(p_{\prime}^2+pp_{\prime}\cos I)\frac{d\Omega}{dI}+(p^2+pp_{\prime}\cos I)\frac{d\Omega_{\prime}}{dI};$$

or, if we introduce the actual values of $\frac{d\Omega}{dI}$, $\frac{d\Omega_{\prime}}{dI}$, we find

$$pp_{\prime}\nu'=-mm_{\prime}rr_{\prime}\sin(\theta-\nu)\sin(\theta_{\prime}-\nu)\times\{p\cos\iota.(\delta^{-3}-r^{-3})+p_{\prime}\cos\iota_{\prime}(\delta^{-3}-r_{\prime}^{-3})\}.$$

It is not my purpose however to enter further into details; and I shall conclude this subject by briefly examining the consequences of a slightly different assumption as to the motion of the axes of coordinates. I shall suppose, namely, that the plane of xy still always coincides with the principal plane, but has a *sliding* motion such that the axis of x *always coincides with the line of nodes.*

91. The assumption made at the end of the last article implies the condition $\nu=0$; and ω_2 will no longer be 0, but must be determined by equations (102.); either of these gives (putting $\nu'=\nu'_{\prime}=0$, and reducing by means of (103.), (109.), &c.)

$$pp_{\prime}\sin I.\omega_2=p_{\prime}\cos\iota_{\prime}\frac{d\Omega}{dI}+p\cos\iota\frac{d\Omega_{\prime}}{dI}, \quad \ldots\ldots \quad (116.)$$

which coincides, as of course it ought, with the expression given for ν' on the former hypothesis (art. 90.). The difference is that Ω, $\Omega_{\prime}$ now no longer contain ν.

The values of ω_0, ω_1 are obtained at once by putting $\nu=0$ in the equations (112.), (113.); and all the conclusions which were derived independently of any supposition as to the value of ω_2, subsist as before, when modified by putting $\nu=0$.

We may add one more equation, which is required in forming some of the expressions for the variations of the elements; namely,

$$\tan\frac{\iota}{2}=\frac{-p_{\prime}\sin I}{\sigma+p+p_{\prime}\cos I}. \quad \ldots\ldots \quad (117.)$$

This is easily obtained from (109.) and (110.); and in like manner

$$\tan\frac{\iota_{,}}{2}=\frac{p\sin \mathrm{I}}{\sigma+p_{,}+p\cos \mathrm{I}}.$$

I shall now recapitulate the results of the last supposition, so as to exhibit in one view the transformed differential equations of the problem of three bodies. It will be as well to repeat also the explanation of the symbols.

92. *Signification of the Symbols.*

M, m, $m_{,}$ are the masses of the sun, and of the two planets, and $\mathrm{M}+m=\mu$, $\mathrm{M}+m_{,}=\mu_{,}$.

a and e are the semiaxis and excentricity of the instantaneous ellipse described by m about the sun.

$a_{,}$ and $e_{,}$ have the same meanings with reference to $m_{,}$.

I is the inclination of the plane of the former ellipse to that of the latter.

θ, $\theta_{,}$ are the longitudes of the two planets, measured in the planes of their orbits *from the common line of nodes.*

r, $r_{,}$ their radii vectores.

ϖ, $\varpi_{,}$ the longitudes of the perihelia, measured likewise in the planes of the orbits *from the line of nodes.*

ε, $\varepsilon_{,}$ two elements such that $\int_0^t ndt+\varepsilon$, $\int_0^t n_{,}dt+\varepsilon_{,}$ are the *mean longitudes*, where n, $n_{,}$ are defined as usual by the equations $n^2=\mu a^{-3}$, $n_{,}^2=\mu_{,}a_{,}^{-3}$.

Then $\int_0^t ndt+\varepsilon-\varpi$ is the mean anomaly of m, and r, θ are functions of the mean anomaly and mean longitude given by the laws of elliptic motion. The same is true for $m_{,}$, *mutatis mutandis.*

χ is the angle between the radii vectores, so that

$$\cos\chi=\cos\theta\cos\theta_{,}+\sin\theta\sin\theta_{,}\cos \mathrm{I}.$$

Let δ be the distance between the planets, so that

$$\delta^2=r^2+r_{,}^2-2rr_{,}\cos\chi.$$

Ω, $\Omega_{,}$ are the disturbing functions, defined by the equations

$$\Omega=mm_{,}\left(\frac{1}{\delta}-\frac{r\cos\chi}{r_{,}^2}\right),\quad \Omega_{,}=mm_{,}\left(\frac{1}{\delta}-\frac{r_{,}\cos\chi}{r^2}\right),$$

and, when expressed in terms of the elements and t, are functions of

$$a,\ a_{,},\ e,\ e_{,},\ \mathrm{I},\quad \int_0^t ndt+\varepsilon,\quad \int_0^t ndt+\varepsilon-\varpi,\quad \int_0^t n_{,}dt+\varepsilon_{,},\quad \int_0^t n_{,}dt+\varepsilon_{,}-\varpi_{,}.$$

When Ω, $\Omega_{,}$ are considered on the one hand as expressed in this way, and on the other, in their original form as functions of r, $r_{,}$, θ, $\theta_{,}$, and I, we have, as applied to either of them,

$$\frac{d}{d\theta}=\frac{d}{d\varepsilon}+\frac{d}{d\varpi},\quad \frac{d}{d\theta_{,}}=\frac{d}{d\varepsilon_{,}}+\frac{d}{d\varpi_{,}}.$$

$p, p_{,}, \sigma$ are defined by the equations

$$p=m\sqrt{\mu a(1-e^2)},\quad p_{,}=m_{,}\sqrt{\mu_{,}a_{,}(1-e_{,}^2)},\quad \sigma^2=p^2+p_{,}^2+2pp_{,}\cos \mathrm{I}.$$

$\iota, \iota_{,}$ are the angles between the *principal plane* and the planes of the two orbits, and are given functions of the elements $a, a_{,}, e, e_{,}, \mathrm{I}$, by virtue of the equations

$$-p\sigma \sin\iota=p_{,}\sigma\sin\iota_{,}=pp_{,}\sin \mathrm{I},$$

from which we have also

$$\sigma\cos\iota=p+p_{,}\cos \mathrm{I},\quad \sigma\cos\iota_{,}=p_{,}+p\cos \mathrm{I}$$

$\left(\text{for the values of } \tan\frac{\iota}{2}, \tan\frac{\iota_{,}}{2}, \text{ see art. 91.}\right)$.

ω_0 is the angular velocity of the principal plane about the line of nodes;

ω_1 the angular velocity of the principal plane about a line in itself perpendicular to the line of nodes;

ω_2 the angular velocity of the line of nodes estimated in the direction of the principal plane.

Differential Equations of the Problem.

93. The nine *intrinsic elements,* as we may perhaps appropriately call them, namely,

$$a,\ a_{,},\ e,\ e_{,},\ \varepsilon,\ \varepsilon_{,},\ \varpi,\ \varpi_{,},\ \mathrm{I},$$

are determined as functions of t by the following system of nine simultaneous differential equations of the first order:

$$m\mu a'=2na^2\frac{d\Omega}{d\varepsilon},\quad m_{,}\mu_{,}a_{,}'=2n_{,}a_{,}^2\frac{d\Omega_{,}}{d\varepsilon_{,}}$$

$$m\mu e'=-\frac{na\sqrt{1-e^2}}{e}\left\{\frac{d\Omega}{d\varpi}+(1-\sqrt{1-e^2})\frac{d\Omega}{d\varepsilon}\right\}$$

$$m_{,}\mu_{,}e_{,}'=-\frac{n_{,}a_{,}\sqrt{1-e_{,}^2}}{e_{,}}\left\{\frac{d\Omega_{,}}{d\varpi_{,}}+(1-\sqrt{1-e_{,}^2})\frac{d\Omega_{,}}{d\varepsilon_{,}}\right\}$$

$$m\mu\varepsilon'=-2na^2\frac{d\Omega}{da}+\frac{na\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2})\frac{d\Omega}{de}$$

$$-\frac{m\mu}{\sin \mathrm{I}}\left\{\frac{\cos \mathrm{I}}{p}\frac{d\Omega}{d\mathrm{I}}+\frac{1}{p_{,}}\frac{d\Omega_{,}}{d\mathrm{I}}\right\}$$

$$m_{,}\mu_{,}\varepsilon_{,}'=-2n_{,}a_{,}^2\frac{d\Omega_{,}}{da_{,}}+\frac{n_{,}a_{,}\sqrt{1-e_{,}^2}}{e_{,}}(1-\sqrt{1-e_{,}^2})\frac{d\Omega_{,}}{de_{,}}$$

$$-\frac{m_{,}\mu_{,}}{\sin \mathrm{I}}\left\{\frac{\cos \mathrm{I}}{p_{,}}\frac{d\Omega_{,}}{d\mathrm{I}}+\frac{1}{p}\frac{d\Omega}{d\mathrm{I}}\right\}$$

$$m\mu\varpi'=\frac{na\sqrt{1-e^2}}{e}\frac{d\Omega}{de}-\frac{m\mu}{\sin \mathrm{I}}\left\{\frac{\cos \mathrm{I}}{p}\frac{d\Omega}{d\mathrm{I}}+\frac{1}{p_{,}}\frac{d\Omega_{,}}{d\mathrm{I}}\right\}$$

$$m_{,}\mu_{,}\varpi_{,}'=\frac{n_{,}a_{,}\sqrt{1-e_{,}^2}}{e_{,}}\frac{d\Omega_{,}}{de_{,}}-\frac{m_{,}\mu_{,}}{\sin \mathrm{I}}\left\{\frac{\cos \mathrm{I}}{p_{,}}\frac{d\Omega_{,}}{d\mathrm{I}}+\frac{1}{p}\frac{d\Omega}{d\mathrm{I}}\right\}$$

$$pp_{,}\sin \mathrm{I}.\mathrm{I}'=p_{,}\left\{\cos \mathrm{I}\left(\frac{d\Omega}{d\varepsilon}+\frac{d\Omega}{d\varpi}\right)+\frac{d\Omega}{d\varepsilon_{,}}+\frac{d\Omega}{d\varpi_{,}}\right\}$$

$$+p\left\{\cos \mathrm{I}\left(\frac{d\Omega_{,}}{d\varepsilon_{,}}+\frac{d\Omega_{,}}{d\varpi_{,}}\right)+\frac{d\Omega_{,}}{d\varepsilon}+\frac{d\Omega_{,}}{d\varpi}\right\}.$$

94. The only parts of the preceding expressions of which the deduction is not perfectly obvious, are the terms involving I in the values of ε', $\varepsilon'_{,}$, ϖ', $\varpi'_{,}$. They are obtained, as has been sufficiently explained, from the expressions in art. 55, to which are to be added the values of $\frac{\partial\varepsilon}{dt}$, &c. ((95.), art. 79.); on putting $\nu=0$, ω_0 disappears from the latter; and the values of ω_1, ω_2, $\cos\iota$, $\cos\iota_{,}$, $\tan\frac{\iota}{2}$, $\tan\frac{\iota_{,}}{2}$ are to be introduced (equations (110.), (112.), (116.), (117.)). After some rather troublesome reductions, the expressions above given will be found.

In these equations it will be recollected that the mean longitude of m is represented by $\int_0^t ndt+\varepsilon$, and the differentiation with respect to a in $\frac{d\Omega}{da}$ is only performed so far as a appears explicitly. If we wished that the mean longitude should be expressed by $nt+\varepsilon$, the only change in the equations would be that the differentiation with respect to a must be total; i.e. must extend to a as contained in n. A similar remark applies of course to $\varepsilon_{,}$.

In actual use it would be more convenient to introduce R, $\mathrm{R}_{,}$ instead of Ω, $\Omega_{,}$; the latter functions give a rather more symmetrical form to the equations, and are more convenient in general investigations. (The relation between them here is merely $\Omega=m\mathrm{R}$, $\Omega_{,}=m_{,}\mathrm{R}_{,}$; in another part of the paper the symbol Ω was used for $-m\mathrm{R}$ (art. 55.)).

95. If the equations of art. 93. were completely integrated, the *intrinsic* motion of the system would be completely determined; that is, we should know at any instant the dimensions of the two orbits, the mutual inclination of their planes, the position of their axes with respect to the line of nodes, the place of each planet in its orbit, and (by (110.)) the inclination of each orbit to the principal plane.

The position of the system relatively to fixed space would then have to be separately determined as follows:—

The three quantities ω_0, ω_1, ω_2 (see end of art. 92.), of which the values are ((112.), (113.), (116.)) given by the equations

$$\sigma^2\sin \mathrm{I}.\omega_0=\left\{(p+p_{,}\cos \mathrm{I})\left(\frac{d}{d\varepsilon_{,}}+\frac{d}{d\varpi_{,}}\right)+(p_{,}+p\cos \mathrm{I})\left(\frac{d}{d\varepsilon}+\frac{d}{d\varpi}\right)\right\}(\Omega_{,}-\Omega)$$

$$\sigma\omega_1=\frac{d}{d\mathrm{I}}(\Omega_{,}-\Omega)$$

$$\sigma\sin \mathrm{I}.\omega_2=\left(\frac{p_{,}}{p}+\cos \mathrm{I}\right)\frac{d\Omega}{d\mathrm{I}}+\left(\frac{p}{p_{,}}+\cos \mathrm{I}\right)\frac{d\Omega_{,}}{d\mathrm{I}},$$

would be given functions of t. Then if we call

J the inclination of the principal plane to an arbitrary fixed plane;

Ω the longitude of the line of intersection of these two planes, reckoned in the fixed plane from a fixed line;

N the angle between this line of intersection and the *line of nodes*;

we should have (as in art. 83. with a different notation)

$$\left.\begin{aligned} J' &= \omega_0 \cos N - \omega_1 \sin N \\ \Omega' \sin J &= \omega_0 \sin N + \omega_1 \cos N \\ N' &= \omega_2 - \cot J(\omega_0 \sin N + \omega_1 \cos N) \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (118.)$$

and the integration of this system would give J, Ω, N as functions of t, and so determine at any instant the position of the principal plane and of the line of nodes, relatively to fixed space.

With respect to the motion of the principal plane, the following may be added. It has already been shown (art. 88, 89.) that the line about which it is at any instant turning, coincides with that in which it is intersected by the plane of the radii vectores; and the values of ω_0, ω_1 (see art. 88, putting $\nu=0$ in the expressions there given) may be put in the form

$$\sigma^2\omega_0 = mm_, \sin I.\left(\frac{r_,}{r^2} - \frac{r}{r_,^2}\right)(p \cos\theta \sin\theta_, + p_, \sin\theta \cos\theta_,)$$

$$\sigma\omega_1 = mm_, \sin I\left(\frac{r_,}{r^2} - \frac{r}{r_,^2}\right) \sin\theta \sin\theta_,.$$

If the latter of these be multiplied by σ, and then both sides of each squared, and the results added (after putting for σ^2 on the right its value $p^2+p_,^2+2pp_, \cos I$), we find, observing that $\cos\theta \cos\theta_, + \sin\theta \sin\theta_, \cos I = \cos\chi$,

$$\sigma^2\sqrt{\omega_0^2+\omega_1^2} = mm_, \sin I\left(\frac{r_,}{r^2} - \frac{r}{r_,^2}\right)(p^2 \sin^2\theta_, + p_,^2 \sin^2\theta + 2pp_, \sin\theta \sin\theta_, \cos\chi)^{\frac{1}{2}},$$

an expression which may be further transformed as follows. Let λ, $\lambda_,$ be the *latitudes* of m, $m_,$ (with reference to the principal plane); then $\sin\lambda = \sin\theta \sin\iota$, $\sin\lambda_, = \sin\theta_, \sin\iota_,$; hence, since $p = \frac{\sigma \sin\iota_,}{\sin I}$, $p_, = \frac{-\sigma \sin\iota}{\sin I}$, we obtain

$$\sigma\sqrt{\omega_0^2+\omega_1^2} = mm_,\left(\frac{r_,}{r^2} - \frac{r}{r_,^2}\right)(\sin^2\lambda + \sin^2\lambda_, - 2\sin\lambda \sin\lambda_, \cos\chi)^{\frac{1}{2}}.$$

This gives the absolute angular velocity with which the principal plane is at any instant changing its direction in space; it is evident that (if the supposition $r_,=r$ be excluded) it can never vanish except when both planets are in the line of nodes. The *direction* of the rotation is determined by the signs of ω_0 and ω_1.

96. The system of differential equations given in art. 93. affords an example of the so-called "elimination of the nodes" in the problem of three bodies. Jacobi, by a very remarkable and ingenious transformation, has effected the elimination in a quite

different manner*. The equations of art. 93. are *merely* transformations of the original differential equations of the problem, without any integrations; they are however in a form which might perhaps be used advantageously in certain cases for the purposes of physical astronomy. Those of JACOBI are obtained by employing all the four usual integrals of the problem, and are shown to include an *additional integration.* They have however the disadvantage of substituting the coordinates of two fictitious bodies for those of the actual planets, and would probably be inconvenient for ordinary practical use; though in a theoretical point of view they seem to deserve more attention than they have hitherto received. It would be wrong to take leave of this celebrated problem without referring to another transformation by M. BERTRAND†, which, as has been remarked by a recent writer in the same journal, effects *six* integrations, and therefore represents the furthest advance which has yet been made towards a rigorous solution.

APPENDIX A.

When the method described in Theorem VII. (art. 49.) is applied to the solution of a system of equations of the form (I.), of which n integrals, $a_1 \dots a_n$, satisfying the conditions $[a_i, a_j]=0$, are given, the first step is to express the $n+1$ partial differential coefficients $\frac{dX}{dx_1}$, &c., and $\frac{dX}{dt}$; namely, the values of $y_1, \dots y_n$ and $-Z$ in terms of $x_1, \dots x_n$, $a_1, \dots a_n$ and t. The *direct* process is then to find X by integrating the expression $y_1dx_1+y_2dx_2+\dots+y_ndx_n-Zdt$, and afterwards to form the remaining integral equations $\frac{dX}{da_1}=b_1$, &c.: when this process is adopted, the inferior limits in the integrations are perfectly arbitrary; in other words, we may add to X an arbitrary function of $a_1, a_2, \dots a_n$, without altering any of the general properties of the final system of integrals.

But it is generally much more convenient to perform the differentiations with respect to $a_1, \dots a_n$ *first*, and integrate afterwards; thus we obtain the remaining equations in the form

$$b_i=\int\left(\frac{dy_1}{da_i}dx_1+\frac{dy_2}{da_i}dx_2+\dots+\frac{dy_n}{da_i}dx_n-\frac{dZ}{da_i}dt\right).$$

When this plan is followed, the limits are still arbitrary if it be only required that the equations thus obtained shall be *true*; but if it be required that they shall form, with the given integrals, a *normal solution*, it is necessary to take the limits in such a manner that the functions equated to $b_1, b_2, \dots b_n$ shall be the partial differential coefficients with respect to $a_1, a_2, \dots a_n$, of *one and the same function*; which will not generally be the case unless care be taken that it should be so.

In practice, the expression for dX usually consists of several terms, of which each

* Comptes Rendus, 1842, part 2. p. 236, &c.

† LIOUVILLE's Journal, 1852.

contains one of the variables *only*. Suppose one of these terms is

$$\phi(x, a_1, a_2, \dots a_n)dx,$$

so that, so far as this term is concerned, we have

$$X=\int_A^x \phi(x, a_1, a_2, \dots a_n)dx,$$

where A is an arbitrary function of $a_1, \dots a_n$. Consequently

$$\frac{dX}{da_i}=\int_A^x \frac{d\phi}{da_i}dx-\phi(A, a_1, \dots a_n)\frac{dA}{da_i},$$

and we see that we should not *in general* obtain the differential coefficients with respect to $a_1, \dots a_n$ of one and the same function X, by merely integrating $\frac{d\phi}{da_1}$, $\frac{d\phi}{da_2}$, &c., with respect to x, from the same inferior limit A, chosen at hazard.

But it is evident that we shall attain this end if we adopt the following simple rule:—

Integrate $\frac{d\phi}{da_1}$, &c. with respect to x, taking the same inferior limit in each case, namely, either

(1) a value A of x which satisfies the equation $\phi(x, a_1, \dots a_n)=0$, or

(2) any *determinate* constant (i.e. not a function of $a_1, \dots a_n$).

For example, in the problem of central forces (Part I., art. 28, &c.), we had (see art. 19.)

$$dX=-hdt+cd\theta+\left(2m(h+\phi(r))-k^2r^{-2}\right)^{\frac{1}{2}}dr+(k^2-c^2\sec^2\lambda)^{\frac{1}{2}}d\lambda$$

(where r, θ, λ are the three variables).

The very troublesome process of differentiating X with respect to h, k and c *after* first finding X by integrating the above expression, is avoided by the method adopted in art. 29; namely, by differentiating first, and integrating afterwards. In the integrations with respect to r, the inferior limit is one of the roots of the equation

$$2m(h+\phi(r))-k^2r^{-2}=0,$$

namely (in the case of elliptic motion), the perihelion distance; and in those with respect to λ, the inferior limit is 0; so that the rule above given is observed.

At the time of writing the article referred to, neither the rule itself, nor the necessity of attending to the limits, had occurred to me; it was therefore, strictly speaking, accidental that the final integrals were obtained in a *normal form*.

In treating the problem of rotation (Section III.), I perceived the necessity of caution as to the limits, if the former order of proceeding were adopted; but preferred avoiding the risk of error altogether, by performing the integrations first, so as to obtain the actual expression for V. The final equations (R.), art. 44, might however be obtained in a more simple way by differentiating *first*; thus we should have (see equations (45.), (46.)), observing that $\frac{dk}{dh}=\frac{k}{2h}$, &c. (art. 44.), and putting

$$(1-\cos^2 i-\cos^2 j+2\cos i\cos j\cos\theta-\cos^2\theta)^{\frac{1}{2}}=\mathrm{Q},$$

$$\frac{d\mathrm{V}}{dh}=\frac{k}{2h}\left\{\psi\cos i+\varphi\cos j+\int\frac{\mathrm{Q}}{\sin\theta}d\theta\right\}=t+\tau$$

$$\frac{d\mathrm{V}}{d\cos i}=k\left\{\psi+\int\frac{\cos j\cos\theta-\cos i}{\mathrm{Q}\sin\theta}d\theta\right\}=\alpha \quad \text{(i.)}$$

$$\frac{d\mathrm{V}}{d\cos j}=\frac{(\mathrm{C}-\mathrm{A})k^3\cos j}{2\mathrm{AC}h}\left\{\psi\cos i+\varphi\cos j+\int\frac{\mathrm{Q}}{\sin\theta}d\theta\right\}$$
$$+k\left\{\varphi+\int\frac{\cos i\cos\theta-\cos j}{\mathrm{Q}\sin\theta}d\theta\right\}=\beta.$$

In order to get rid of the troublesome integration involved in the term $\int\frac{\mathrm{Q}}{\sin\theta}d\theta$, we may (1) eliminate this term between the first and last of these equations, and (2) eliminate ψ between the first and second. We thus find the two following equations,

$$\varphi+\int\frac{\cos i\cos\theta-\cos j}{\mathrm{Q}\sin\theta}d\theta=\frac{\beta}{k}-\frac{\mathrm{C}-\mathrm{A}}{\mathrm{AC}}k\cos j.(t+\tau) \quad \text{(ii.)}$$

$$\cos j\left\{\varphi+\int\frac{\cos i\cos\theta-\cos j}{\mathrm{Q}\sin\theta}d\theta\right\}+\int\frac{\sin\theta d\theta}{\mathrm{Q}}=\frac{2h}{k}(t+\tau)-\frac{\alpha\cos i}{k},$$

which last, combined with the preceding, gives

$$\int\frac{\sin\theta d\theta}{\mathrm{Q}}=\frac{k}{\mathrm{A}}(t+\tau)-\frac{\alpha\cos i+\beta\cos j}{k}; \quad \text{(iii.)}$$

and we may take (i.), (ii.) and (iii.) as expressing the solution of the problem.

Now we have $\cos\mathrm{I}=\frac{\cos i-\cos j\cos\theta}{\sin j\sin\theta}$, from which it is easy to find (observing the conditions which determine the sign of Q)

$$-d\mathrm{I}=\frac{\cos i\cos\theta-\cos j}{\mathrm{Q}\sin\theta}d\theta, \text{ and similarly,}$$

$$-d\mathrm{J}=\frac{\cos j\cos\theta-\cos i}{\mathrm{Q}\sin\theta}d\theta;$$

we have moreover $\qquad d\Theta=\frac{\sin\theta d\theta}{\mathrm{Q}}$.

All the integrations may therefore now be performed immediately; and we may take for the inferior limit of θ any value which satisfies the equation $\mathrm{Q}=0$, or

$$(\cos\theta-\cos i\cos j)^2-\sin^2 i\sin^2 j=0;$$

this is satisfied by $\theta=i+j$, which evidently corresponds to $\mathrm{I}=0$, $\mathrm{J}=0$, $\Theta=0$, and it is manifest that equations (i.), (ii.), (iii.) will thus become identical with equations (R) of art. 44.

I do not regret however having introduced the rather prolix investigation of arts. 39 and 43, because it is interesting to know the actual value of V (equation (48.)), which the method just given leaves undetermined.

Appendix B.

On the subject of the transformation of elements, the following additional remarks will hardly be superfluous. Suppose Ω is originally a function of the elements a, b, c, &c. with t; and let α, β, γ, &c. be other quantities connected with a, b, c, &c., by equations such as

$$da = \mathrm{A}d\alpha + \mathrm{B}d\beta + \mathrm{C}d\gamma + \ldots + \mathrm{K}dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a.)$$

where A, B, C, ... K are given functions of $\alpha, \beta, \gamma, \ldots\, t$; or by equations such as

$$d\alpha = \mathrm{A}_{,}da + \mathrm{B}_{,}db + \ldots + \mathrm{K}_{,}dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha.)$$

where $\mathrm{A}_{,}$, &c. are given functions of $a, b, \ldots, t$. In either case, if each of the equations be integrable *per se*, we may consider $a, b, c, \ldots$ as functions of $\alpha, \beta, \gamma, \ldots, t$; and such equations as

$$\frac{d\Omega}{d\alpha} = \mathrm{A}\frac{d\Omega}{da} + \mathrm{B}\frac{d\Omega}{db} + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\Omega.)$$

are both significant and true.

But if the expressions on the right of the equations $(a.)$ be *not* differentials *per se*, the equations $(\Omega.)$ are either unmeaning or untrue. For the symbol $\frac{d\Omega}{d\alpha}$ implies one of two things; either, that Ω is expressed in terms of $\alpha, \beta, \ldots\, t$ *without* arbitrary constants (i. e. that the transformation of Ω can be actually effected *without* integrating the differential equations of the problem), which is manifestly impossible, unless $(a.)$, &c. be integrable *per se*; or else, that the differential equations are to be conceived to have been completely integrated, so that a, b, &c., and consequently $\mathrm{A}_{,}, \mathrm{B}_{,}$, &c., are known as *functions of* t *and arbitrary constants*, whereby the right-hand side of $(\alpha.)$ becomes an *explicit function of* t (and arbitrary constants), so that α, β, &c. may by integration be expressed in the same way, and, by means of $(a.)$, a, b, &c. may be similarly expressed, and finally, by algebraical elimination, a, b, &c. become functions of α, β, &c., t, and arbitrary constants. On this supposition, $\frac{d\Omega}{d\alpha}$ has a meaning, but the equation $(\Omega.)$ is *untrue*; for we must have

$$\frac{d\Omega}{d\alpha} = \frac{d\Omega}{da}\frac{da}{d\alpha} + \frac{d\Omega}{db}\frac{db}{d\alpha} + \ldots ;$$

and it is manifestly not true that $\frac{da}{d\alpha} = \mathrm{A}$, &c. in this case, because the equation

$$da = \mathrm{A}d\alpha + \mathrm{B}d\beta + \ldots + \mathrm{K}dt,$$

not being integrable *per se*, only subsists for those variations of α, β, &c. which *actually take place* during the instant dt; whereas the equation

$$da = \frac{da}{d\alpha}d\alpha + \frac{da}{d\beta}d\beta + \ldots + \frac{da}{dt}dt$$

subsists for *arbitrary variations* of all the variables. This view of the subject entirely coincides, in substance, with that taken by JACOBI; but the above mode of stating it may tend to make it clearer, and to call attention to a matter which, so far as I know, is not so much as mentioned in any of the elementary works usually in the hands of students of physical astronomy.

[*Addition to* APPENDIX B.]

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The remark made above, that the symbol $\frac{d\Omega}{d\alpha}$ is *unmeaning* in the case considered, is not of course intended to imply that a meaning *may not be given to it*; but then such meaning is different from the ordinary signification of the symbol, which is a *partial derived function.*

The whole matter may be strikingly illustrated by a simple example.

Consider the movement of a rigid body about a fixed point. Adopting the notation of art. 40 (Part I.), we have

$$pdt = -\cos\phi d\theta - \sin\theta\sin\phi d\psi$$
$$qdt = \sin\phi d\theta - \sin\theta\cos\phi d\psi$$
$$rdt = d\phi + \cos\theta d\psi.$$

Let α, β, γ be three new variables, defined by the equations

$$\alpha = \int_0^t pdt, \quad \beta = \int_0^t qdt, \quad \gamma = \int_0^t rdt;$$

so that $d\alpha = pdt$, &c. Then the above equations give

$$d\theta = -\cos\phi d\alpha + \sin\phi d\beta, \quad d\phi = d\gamma + \cot\theta(\sin\phi d\alpha + \cos\phi d\beta)$$
$$d\psi = -\operatorname{cosec}\theta(\sin\phi d\alpha + \cos\phi d\beta).$$

Here α, β, γ are the sums of the elementary angles described about the axes in the course of the motion; and no one would maintain that θ, ϕ, ψ are *functions* of α, β, γ, for the values of the latter variables at any time *do not determine* the values of the former. If therefore we choose to write such equations as

$$\frac{d\theta}{d\alpha} = -\cos\phi, \ \frac{d\theta}{d\beta} = \sin\phi, \text{ \&c.},$$

we must admit that $\frac{d\theta}{d\alpha}, \frac{d\theta}{d\beta}$, &c. are not partial derived functions in the ordinary sense. At most, $\frac{d\theta}{d\alpha}$ is the derived function of *that function of* α *which* θ *would become* if β and γ were maintained invariable, *i. e.* if the motion were restricted to a rotation about the A-axis. Again, if we admit such symbols as $\frac{d^2\theta}{d\beta d\alpha}, \frac{d^2\theta}{d\alpha d\beta}$, we must interpret them as follows:

$$\frac{d^2\theta}{d\beta d\alpha} = \frac{d}{d\beta}\frac{d\theta}{d\alpha} = -\frac{d\cos\phi}{d\beta} = \sin\phi\frac{d\phi}{d\beta},$$

but $\frac{d\phi}{d\beta}=\cot\theta\cos\phi$, and therefore

$$\frac{d^2\theta}{d\beta d\alpha}=\cot\theta\sin\phi\cos\phi\,;$$

and in like manner we find the same value for $\frac{d^2\theta}{d\alpha d\beta}$, so that in *this particular case* the condition $\frac{d^2\theta}{d\alpha d\beta}=\frac{d^2\theta}{d\beta d\alpha}$ is verified.

But if we take $\frac{d^2\theta}{d\gamma d\alpha}$ and $\frac{d^2\theta}{d\alpha d\gamma}$ in the same way, we find *the former* $=\sin\phi$ *and the latter* $=0$, so that in this case the condition *is not verified.* The geometrical meaning of this is obvious; analytically it is merely an instance of a general fact, pointed out by JACOBI; namely, that the effect of two successive *pseudo-differentiations* with respect to two independent variables, is not generally independent of the order of operations.

If V be the potential of another body, given in position, upon the body considered, then V is a function of θ, ϕ, ψ, and

$$dV=\frac{dV}{d\theta}d\theta+\frac{dV}{d\phi}d\phi+\frac{dV}{d\psi}d\psi\,;$$

and if we substitute for $d\theta$, $d\phi$, $d\psi$ their values in terms of $d\alpha$, $d\beta$, $d\gamma$, we obtain an expression which we may call $Ld\alpha+Md\beta+Nd\gamma$, L, M, N being functions of θ, ϕ, ψ, $\frac{dV}{d\theta}$, $\frac{dV}{d\phi}$, $\frac{dV}{d\psi}$, of which, as is well known, the mechanical meanings are the moments of the attraction of the second body about the three axes. Here again no one would maintain that V is a *function* of α, β, γ; and if, as is often done, we say $\frac{dV}{d\alpha}=L$, &c., the above remarks apply in all respects to these equations.

I should have thought it superfluous to dwell so much on these points if it had not appeared that writers on physical astronomy have in some instances either overlooked the distinction between *real* and *pseudo*-differentiation, or at least have failed to point it out to their readers. The only discussion of the subject which I have met with is that given by JACOBI, in the correspondence referred to.

It may be added, that in general investigations, where symbols such as $\frac{dV}{d\alpha}$, &c. may be used without defining the nature of V, or the precise meaning of α, β, &c., serious errors might be committed if it were assumed that the condition $\frac{d^2V}{d\alpha d\beta}=\frac{d^2V}{d\beta d\alpha}$ always subsisted.

APPENDIX C.

The theorems relating to the transformation of coordinates, given in Section VI., may be made more general, and in many cases more useful, as follows:—

If x_1, x_2, ... x_n be the coordinates employed in the first statement of any dynamical problem, the differential equations are comprehended in the formula

$$\Sigma_i\left\{\left(\frac{dW}{dx'_i}\right)'-\frac{dW}{dx_i}\right\}\delta x_i=0. \qquad \text{(D.)}$$

[If there be any forces, such as those arising from a resisting medium, which do not satisfy the natural conditions of integrability, then on the right-hand side of the formula (D.), instead of 0 we shall have an expression such as $\Sigma_i(X_i\delta x_i)$; but such terms are easily introduced and allowed for separately, and do not affect the following investigation. I shall therefore here assume that they do not exist.]

In the above formula, W is a given function of x_1, ... x_n, x'_1, ... x'_n, which may also explicitly contain t.

In Section VI. the only case contemplated was that in which x_1, ... x_n are *independent* coordinates; in which case the formula (D.) is equivalent to n separate equations, since δx_1, &c. are wholly arbitrary and independent.

In practice, however, it is often more convenient to use, at first, a set of coordinates more in number than the independent variables of the problem, and therefore subject to equations of condition.

Let us assume then that the n coordinates x_1, ... x_n are subject to r equations of condition,

$$L_1=0,\ L_2=0,\ \dots\ L_r=0,$$

where L_1, &c. may explicitly contain t, besides the n variables x_1, &c.

If we introduce the n conjugate variables y_1, ... y_n defined by the equations $y_i=\frac{dW}{dx'_i}$, and take Z a function of x_1, &c., y_1, &c. (with or without t), defined by the equation

$$Z=-[W]+[x'_1]y_1+\dots+[x'_n]y_n$$

(the brackets indicating that x'_1, &c. are expressed in terms of y_1, &c.), then it follows exactly as in art. 18 (Part I.), that the formula (D.) will be changed into the system

$$\left.\begin{aligned} x'_i&=\frac{dZ}{dy_i} \\ \Sigma_i\left(y'_i+\frac{dZ}{dx_i}\right)\delta x_i&=0 \end{aligned}\right\} \qquad \text{(E.)}$$

[In the most usual problems W is of the form T+U, where T is homogeneous and of the second degree in x'_1, &c., and U does not contain x'_1, &c. at all. In this case Z is only T−U expressed in terms of y_1, &c., instead of x'_1, &c. But T is *not necessarily* homogeneous; in fact it is not so in problems relating to motion *relative to the earth,* as affected by the earth's rotation.]

Let us now suppose that the system (E.) is to be transformed by the introduction of the m *independent* coordinates ξ_1, ξ_2, ... ξ_m, and of the new conjugate variables η_1, η_2, ... η_m; where $m=n-r$. And let it be required to investigate a theorem by means of which the transformation may be effected *without recurring to the original formula* (D.).

The definitions of the new coordinates ξ_1, &c. will furnish m equations (which may explicitly contain t) by means of which $\xi_1, \dots \xi_m$ may be expressed as functions of $x_1, \dots x_n$ (with or without t); and conversely, by means of these m equations, together with the r equations of condition $L_1=0$, &c., the n variables $x_1, x_2, \dots x_n$ may be expressed as functions of $\xi_1, \dots \xi_m$, with or without t. When $x_1, \dots x_n$ are so expressed, let them be represented by $(x_1), \dots (x_n)$. We shall have then

$$x'_i=\frac{d(x_i)}{dt}+\frac{d(x_i)}{d\xi_1}\xi'_1+\dots+\frac{d(x_i)}{d\xi_m}\xi'_m, \quad \dots \dots \quad (x'.)$$

so that x'_1, &c. are expressible (and in only one way) as functions of ξ_1, &c., ξ'_1, &c.

If then the formula (D.) be transformed by expressing x_1, &c., x'_1, &c. in this manner, it becomes, as is well known,

$$\Sigma_i\left\{\left(\frac{d(\mathrm{W})}{d\xi'_i}\right)'-\frac{d(\mathrm{W})}{d\xi_i}\right\}\delta\xi_i=0,$$

where (W) represents the result of transforming W as above; and since $\delta\xi_1$, &c. are now independent, this formula breaks up into the m separate equations

$$\left(\frac{d(\mathrm{W})}{d\xi'_i}\right)'=\frac{d(\mathrm{W})}{d\xi_i}. \quad \dots \dots \quad (\mathrm{F.})$$

Moreover, if we now define η_i by the equation $\frac{d(\mathrm{W})}{d\xi'_i}=\eta_i$, and put

$$\Psi=-(\mathrm{W})+(\xi'_1)\eta_1+\dots+(\xi'_m)\eta_m,$$

where (ξ'_1), &c. are expressed in terms of η_1, &c., we know already (art. 18.) that the system (F.) becomes

$$\xi'_i=\frac{d\Psi}{d\eta_i}, \quad \eta'_i=-\frac{d\Psi}{d\xi_i}. \quad \dots \dots \quad (\mathrm{G.})$$

Now let P be a function of the m new variables $\xi_1, \dots \xi_m$, and of the n old variables $y_1, \dots y_n$ (with or without t), defined by the equation

$$\mathrm{P}=(x_1)y_1+(x_2)y_2+\dots+(x_n)y_n.$$

Since $\eta_i=\frac{d(\mathrm{W}}{d\xi'_i}$, and since (W) contains ξ'_i, &c., only through x'_i, &c., we have, observing that $\frac{d\mathrm{W}}{dx'_i}=y_i$,

$$\eta_i=y_1\frac{dx'_1}{d\xi'_i}+y_2\frac{dx'_2}{d\xi'_i}+\dots+y_n\frac{dx'_n}{d\xi'_i};$$

but from the equation $(x'.)$ we have $\quad \frac{dx'_j}{d\xi'_i}=\frac{d(x_j)}{d\xi_i},$

consequently $\quad \eta_i=y_1\frac{d(x_1)}{d\xi_i}+y_2\frac{d(x_2)}{d\xi_i}+\dots+y_n\frac{d(x_n)}{d\xi_i},$

an expression evidently equivalent to $\frac{d\mathrm{P}}{d\xi_i}$. Thus η_i *may be defined by the equation*

$$\frac{d\mathrm{P}}{d\xi_i}=\eta_i. \quad \dots \dots \quad (\eta.)$$

without recurring to the formula (D.). And since each equation of condition gives, if differentiated totally with respect to t, with the substitution of $\frac{dZ}{dy_i}$ for x'_i, an equation such as

$$0=\frac{dL}{dt}+\frac{dL}{dx_1}\frac{dZ}{dy_1}+\ldots+\frac{dL}{dx_n}\frac{dZ}{dy_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (L.)$$

the m equations (η.) with the r equations L (in which last x_1, &c. must be expressed, after the differentiation, in terms of ξ, &c.), give n equations by means of which $y_1, \ldots y_n$ may be expressed in terms of ξ_1, &c., η_1, &c., and can be so expressed *only in one way.*

Lastly, the value of Ψ (see equations (G.)), may be obtained as follows:—

Since $$\Psi=-(W)+(\xi'_1)\eta_1+\ldots+(\xi'_m)\eta_m$$

and $$Z=-[W]+[x'_1]y_1+\ldots+[x'_n]y_n,$$

and (W) is only [W] differently expressed, we have, without reference to modes of expression,

$$\Psi-Z=\Sigma_i(\xi'_i\eta_i)-\Sigma_i(x'_iy_i).$$

On the other hand, since P can contain t explicitly only through (x_1), &c., we have

$$\frac{dP}{dt}=\Sigma_i\left(y_i\frac{d(x_i)}{dt}\right),$$

and also $$\Sigma_i(x'_iy_i)=\Sigma_i\left(y_i\frac{d(x_i)}{dt}\right)+\Sigma_i\left(\frac{dP}{d\xi_i}\xi'_i\right);$$

hence, observing that $\frac{dP}{d\xi_i}=\eta_i$, we obtain

$$\Sigma_i(\xi'_i\eta_i)-\Sigma_i(x'_iy_i)=-\frac{dP}{dt};$$

and therefore, finally, $$\Psi=Z-\frac{dP}{dt},$$

so that $Z-\frac{dP}{dt}$ must become identical with Ψ, when expressed entirely in terms of the new variables.

These results may be stated in the form of the following *theorem.* The system (E.) is transformed into the system (G.) by the following substitutions:—

(1) $x_1, \ldots x_n$ are expressed in terms of $\xi_1, \ldots \xi_m$ by means of the m equations which define the latter variables, together with the $n-m$ equations of condition

$$L_1=0, \ldots L_r=0 \ (r=n-m).$$

(2) $\eta_1, \ldots \eta_m$ are defined by the m equations $\frac{dP}{d\xi_i}=\eta_i$; where the modulus of transformation P is given by the equation

$$P=(x_1)y_1+(x_2)y_2+\ldots+(x_n)y_n,$$

(x_1), &c. being expressed in terms of ξ_1, &c., so that P is explicitly a function of $\xi_1, \ldots \xi_m$, $y_1 \ldots y_n$, with or without t.

(3) Ψ is defined by the equation $\Psi = Z - \frac{dP}{dt}$, in which (after the explicit differentiation of P with respect to t), x_1, &c., y_1, &c. are to be expressed in terms of the new variables. y_1, &c. are thus expressible by the help of the m equations $\frac{dP}{d\xi} = \eta_i$ and the $n-m$ equations $\frac{dL}{dt} + \Sigma_i \left(\frac{dL}{dx_i} \frac{dZ}{dy_i} \right) = 0$.

If (x_1), &c., do not contain t explicitly, then $\frac{dP}{dt} = 0$, and Ψ is obtained merely by expressing Z in terms of the new variables.

It may be observed that the whole of the above reasoning would apply to the case in which the new variables $\xi_1, \ldots \xi_m$ are more in number than the independent variables of the problem (or $m > n - r$), *with this exception*; that the m equations $\frac{dP}{d\xi_i} = \eta_i$, together with the r equations obtained by differentiating the equations of condition totally with respect to t, would be *more than sufficient* to express $y_1, \ldots y_n$ in terms of the new variables; consequently y_1, &c. might be so expressed in *different ways*, and therefore, although the *value* of Ψ obtained by the above rule would certainly be the same as that obtained by recurring to the original formula (D.), the *form* of Ψ might be different, and therefore the resulting formula erroneous.

There must doubtless exist some rule for choosing $n-m$ combinations of the equations of condition in such a way as to lead to the correct *forms* of $y_1, \ldots y_n$ as functions of the new variables; but I have not at present attempted to investigate it, and perhaps it would be hardly worth while. The theorem in the case in which the new coordinates are independent, may, I believe, be practically useful.

ERRATA IN PART I.

Art. 1. equation (4.), for dx read dx_i.

Art. 10. In paragraph preceding equation (26.) *omit* the words "not containing t explicitly."

Art. 18. equation (β), for y_i read y_i'.

Art. 19. equation (29.), for h_i read b_i.

Art. 24. second line after equation (L.), *for* "such as h, k" *read* "such as f, g."

Art. 30. The expressions equated to h, k, c, and the three terms in the left-hand column of the table of elements, should each be multiplied by m.

Art. 42. near the end, *for* "according as Θ is between o and π, or not" *read* "according as Θ is between π and 2π, or between o and π."

XVI. *On the* Megatherium (Megatherium Americanum, Cuvier *and* Blumenbach). Part II.—*Vertebræ of the Trunk*. *By Professor* Owen, *F.R.S. &c.*

Received November 8, 1850—Read May 8, 1851.

In commencing the description of the skeleton of the Megatherium, now in London, Plate XVII., which is the most complete that has yet reached Europe, a brief statement may be premised of the chief steps which have led to the restoration of the species to which it belongs.

Cuvier, in communicating to the 'Annales du Muséum' (t. v. 1804) a translation of the first memoir on this subject—that, viz. by Garriga and Bru, published at Madrid in 1796,—gives all the requisite details respecting the discovery of the skeleton therein described, and adds his own more important deductions as to its affinities from an examination and comparison of the plates of the Spanish work.

It appears that proofs of these plates were transmitted in 1795 to the Institute of France, and that Cuvier, having been called upon by the 'Class of Sciences' at that period to give a report upon them, developed his views of the affinity of the animal to the *Sloths* and other *Edentates**, and proposed for it the name 'Megatherium†.'

M. Roume, the correspondent of the French Institute to whom that distinguished scientific body were indebted for the proof impressions of Garriga's work, and who had an opportunity of examining the skeleton itself at Madrid (which Cuvier never enjoyed), inserted a brief notice of it in the 'Bulletin de la Société Philomathique' of the Republican year IV. (1795); in which, after particularly noticing that the pelvis was open towards the abdomen, the pubis being absent, without any indication of its having ever existed‡, concludes that the animal was intermediate, as to form, between the Cape Anteater (*Orycteropus*) and the Great Anteater of America (*Myrmecophaga jubata*). But he adds, that M. Cuvier, from an examination of the engravings of the skeleton, had recognized that the species was much more nearly allied to the Sloths than to the Anteaters.

* "Je développai dès-lors l'affinité de cet animal avec les *paresseux* et les autres *édentés*."—Annales du Muséum, t. v. p. 377.

† *μέγας great, θηρίον beast.*

‡ "Son bassin est composé des os sacrum, iléum et ischium, mais il n'y a point de pubis ni d'indication qu'il ait existé. Ce bassin est ouvert du côté de l'abdomen."—p. 97. It will be shown in this memoir that the supposed want of pubic bones was due to accidental mutilation of the pelvis of the skeleton at Madrid: the true profile of the pelvis is given at 62, 63, 64, Plate XVII.

M. Abildgaard, a professor at Copenhagen, having had the opportunity of studying the skeleton of the Megatherium at Madrid in 1793, published a short notice of it in Danish, illustrated by a rude figure of the skull and of the hind limbs, and referred the species to the *Bruta* of Linnæus, an order afterwards modified to form the *Edentés* of Cuvier: this notice, though published the year after Cuvier's Report, appears to have been independent of it, and the conclusions to be the result of the author's own observations and reflections. It is, therefore, to be regarded as an additional testimony to the true affinities of the species.

Cuvier's comments on the figures in the engravings for Garriga's memoir are accompanied by reduced copies of them, given in the above-cited volume of the 'Annales du Muséum,' and afterwards in the fourth volume of the first edition of the 'Recherches sur les Ossemens Fossiles,' 4to, 1812. In both works Cuvier sums up his conclusions as to the habits and food of the Megatherium, as follows:—"Its teeth prove that it lived on vegetables, and its robust fore-feet armed with sharp claws, make us believe that it was principally their roots which it attacked. Its magnitude and its talons must have given it sufficient means of defence. It was not of swift course, nor was this requisite, the animal needing neither to pursue nor to escape*."

In the year 1821 Drs. Pander and D'Alton published their beautiful Monograph on the Megatherium, the result of personal examination and depiction of the then unique skeleton at Madrid; they represent the bones more artistically and in more natural juxtaposition than in the plates of Bru's memoir, but the subject being the same, the same deficiencies, to be presently specified, were unavoidable.

As the accomplished and learned authors of the German work reasoned, like Abildgaard, from actual inspection of the fossil skeleton, their conclusions as to the nature, affinities and habits of the animal to which it belonged merit a respectful consideration. In it they recognize, with Cuvier, all the important points of resemblance to the skeletons of the existing species of Sloth, of which they append excellent figures. But, imbued with the principles of the transcendental and transmutative hypotheses, then prevalent in the schools of Germany, they regard the great Megatherium and Megalonyx as being not merely predecessors but progenitors of those still lingering remnants of the tardigrade race, into which such ancestral giants are supposed to have dwindled down by gradual degeneration and alteration of characters. But they deem the living habits of the Megatherium to have been far different from those of its puny scansorial progeny: it was, in their opinion, a fossorial animal; and not merely an occasional digger of the soil, as Cuvier concluded, but altogether

* "Ses dents prouvent qu'il vivoit de végétaux, et ses pieds de devant, robustes et armées d'ongles tranchans, nous fait croire que c'étoit principalement leurs racines qu'il attaquoit. Sa grandeur et ses griffes devoient lui fournir assez de moyens de défense. Il n'étoit pas prompt à la course, mais cela ne lui étoit pas nécessaire, n'ayant besoin ni de poursuivre ni de fuir." *Tom. cit.* 'Sur le Mégatherium,' p. 29. See also the posthumous edition of the 'Ossemens Fossiles,' 8vo, 1836, tom. viii. p. 363.

a creature of subterranean habits; some earth-whale, as it were, or colossal mole. PANDER and D'ALTON nevertheless give to this animal, which they truly characterized as one of the most extraordinary of its class, the name of 'Riesen-faulthier,' *Bradypus giganteus*, or Gigantic Sloth*.

CUVIER, in preparing his new and enlarged edition of the famous 'Recherches sur les Ossemens Fossiles,' availed himself, in the fifth volume, published in 1823, of the labours of the German anatomists and draughtsmen, just cited, and substituted copies of their figures for those which he had previously borrowed from BRU.

The teeth of the Megatherium are still described as being implanted by two roots, and as being sixteen in number, formulized as $m, \frac{4-4}{4-4}$: there is the same deficiency of the sternal ribs, pubic bones and tail; the manubrium sterni continues to be represented in a reversed position: but, with regard to the bones of the fore-foot, the organization of which was involved in obscurity, owing to the faulty manner in which CUVIER believed them to have been articulated, he endeavours to throw some new light on their arrangement. After a comparison of the figures given by PANDER and D'ALTON with the bones of the fore-foot in existing Edentata, CUVIER concludes that the fore-feet in the Madrid skeleton are transposed, the right being on the left, and the left on the right side; that the index, medius and annular digits were the only ones provided with claws; that the thumb was clawless, and the little finger rudimental and concealed, in the living Megatherium, under the skin; the hand being thereby specially formed for cleaving the soil and digging, like that of the *Dasypus gigas*. On this hypothesis, names are applied by CUVIER to certain bones of the carpus, none of which had before been determined†. CUVIER, however, adds, that "in order to verify his conjectures it must be necessary to have access to the skeleton itself, and to compare separately each bone of the fore-foot with their homologues in that species of Armadillo‡." His ideas of the affinities of the Megatherium have now undergone some modification: the following paragraph is added to the summary on this head given in the earlier edition of the great work:—"Its analogies approximate it to different genera of the Edentate family. It has the head and the shoulder of a Sloth, whilst the legs and the feet offer a singular mixture of characters peculiar to the Anteaters and Armadillos§." CUVIER concludes his account of the Megatherium in the second edition of the 'Ossemens Fossiles,' by appending a note communicated to him by

* Das Riesen-Faulthier, &c. fol. 1821, p. 16.

† Recherches sur les Ossemens Fossiles, 4to. t. v. pt. 1. 1823, p. 185. pl. 16. fig. 13. The letters indicative of the carpal bones have been omitted, by oversight, in the plates, but there is no difficulty in adding them according to the description given by CUVIER in the text.

‡ "Mais on sent que, pour vérifier ces conjectures, il faudroit être auprès du squelette, et en comparer séparément tous les os avec leurs analogues dans ce tatou, ce que j'espère que quelque anatomiste espagnol ne tardera pas à faire."—*Ib.* p. 185.

§ "Ses analogies le rapprochent des divers genres de la famille des édentés. Il a la tête et l'épaule d'un paresseux, et ses jambes et ses pieds offrent un singulier mélange de caractères propres aux fourmilliers et aux tatous."—*Ib.* p. 189.

M. Auguste St. Hilaire, "which announces," he says, "that the Megatherium had pushed its affinity to the Armadillos so far as to be covered like them with a scaly cuirass."

This opinion derived apparent confirmation from the description by Professor Weiss of portions of an osseous tessellated dermal armour of some gigantic quadruped, sent to Berlin by the traveller Sellow, which armour he figures, and attributes to the Megatherium, in a "Geological memoir on the Provinces of S. Pedro do Sal and the Banda Oriental," published in 1827*.

In the year 1832, a highly valuable and important collection of the bones of the Megatherium, discovered in the Rio Salado, with a portion of a bony tessellated dermal covering of an animal, found in Lake Averias, province of Buenos Ayres, indicative of a frame as great as that of the skeleton from the Rio Salado, was transmitted from Buenos Ayres by Sir Woodbine Parish, K.H., and presented by him to the Royal College of Surgeons. These specimens formed the subject of a memoir communicated by William Clift, Esq., F.R.S., to the Geological Society, June 13, 1832†, in which, although, with the characteristic caution of the author, the armour is not directly affirmed to belong to the Megatherium, nothing is stated to prevent the inference that it formed part of the 'Remains' of that animal which it is the object of the memoir to describe: and, in the description of the map engraved in pl. 43, the specimen figured in pl. 46, with other portions of the bony armour, are comprehended amongst "those Remains of the Megatherium which have hitherto been sent to Europe‡." Further countenance to the later opinion of Cuvier as to the affinities of the Megatherium to the Armadillo, was afforded by a few remarks in the text of Mr. Clift's memoir: thus, in noticing the "bony or pseudo-cartilaginous pieces which unite the true ribs to the sternum," Mr. Clift adds, "as is also the case in the Armadillo §." And in the description of the caudal vertebræ, he remarks, "they have the inferior spines (*i. e.* the chevron or V-shaped bones), manifesting in this their relation to other Edentata, as the *Myrmecophagæ* and *Dasypodæ*||."

The additional parts of the Megatherium, supplied by Sir Woodbine Parish, and deficient in the skeleton at Madrid, were two of the ossified cartilages of the ribs, two of the smaller bones of the sternum, twelve caudal vertebræ, and ten of the separate 'chevron bones,' partly belonging to them and partly indicating other caudal vertebræ: they also included a part of the os hyoides. Mr. Clift, with his accustomed ingenuity and artistic ability, gives at one view an idea of "all the parts hitherto known, or supposed to be known," of the Megatherium, by taking as his basis the outline of the view of the skeleton given by Pander and D'Alton in the first plate of their work; leaving in simple outline those parts which are present in

* Abhandlungen der Kön. Akad. der Wissenschaften zu Berlin, 1827.

† Geological Transactions, 2nd series, vol. iii. p. 437.

‡ *Ibid.* Description of the Plates. § *Ibid.* p. 439. || *Ibid.* p. 444.

the Madrid skeleton, but not in Sir Woodbine Parish's collection; expressing by a pale tint the parts in that collection which also exist in the Madrid skeleton; and indicating by a dark tint the additional parts which are deficient in the Madrid skeleton, and had not before been figured.

Besides the important elements thus added towards the completion of our knowledge of the skeleton, Mr. Clift was enabled to correct an error into which Cuvier had been led by a figure of a mutilated tooth in tab. 4. fig. 5, f, of Garriga's memoir, which seemed to show that it had been implanted, as Cuvier describes it to have been, by two fangs. Pander and D'Alton give a similar figure of one of the teeth in their tab. 2. fig. 15. The figure of the natural size of one of the teeth of the Megatherium transmitted by Sir Woodbine Parish, given in the third plate (pl. 45. fig. 2.) of Mr. Clift's memoir, is the first accurate representation of these characteristic parts, and shows that the implanted base is widely excavated for a persistent matrix, as in the Sloths and Armadillos.

The prevalent belief among Comparative Anatomists and Naturalists at this period, founded upon the additional observations by Weiss and Clift to those contained in the second edition of the 'Ossemens Fossiles' of Cuvier, may be gathered from such notices as were then published of the opinions expressed by the eminent professors of those sciences on the subject. Thus Dr. Robert Grant, treating of the Armadillos, in his Lectures on Comparative Anatomy, says, "The Megatherium itself appears to have been such a digging loricated animal, and in many of its bones resembles the Armadillos *."

The Very Rev. Dr. Buckland, in his Bridgewater Treatise published in 1836, admitting the probability, from the evidence at that time adduced, that the Megatherium had been defended by a bony tessellated armour, argues that—"A covering of such enormous weight would have been consistent with the general structure of the Megatherium: its columnar hind legs and colossal tail were calculated to give it due support; and the strength of the loins and ribs, being very much greater than in the Elephant, seems to have been necessary for carrying so ponderous a cuirass as that which we suppose to have covered the body." He next calls attention to the broad and rough flattened surface of a part of the crest of the ileum, to the broad summits of the spines of many vertebræ, and also to the superior convex portion of certain ribs, on which the armour could rest, as affording "evidence of pressure, similar to that we find on the analogous parts of the skeleton of the Armadillo, from which,"

* Lecture, reported in the 'Lancet,' March 29, 1834: and again, in 1839, the Megatherium is described by Dr. Grant as being "allied in structure to the *Bradypus*, and shielded with cutaneous plates like the *Dasypus*." —Thomson's 'British Annual' for 1839, p. 274. M. Desmarest, in the art. *Megathere* of the 'Dictionnaire des Sciences Naturelles,' 1823, writes as follows:—"Leurs membres étaient robustes et terminés par cinq gros doigts. Des observations récentes paroissent prouver que sa peau, épaissée et comme ossifiée, était partagée en une foule d'écussons polygones et rapprochés les uns des autres comme les pièces qui entrent dans la composition d'une mosaïque."—"La forme des molaires et la taille de ces animaux semblent indiquer qu'ils se nourrissoient de végétaux et sans doute de racines."

he remarks, "we might have inferred that the Megatherium also was covered with heavy armour, even had no such armour been discovered near bones of this animal in other parts of the same level district of Paraguay*."

The estimable and justly celebrated author of the 'Bridgewater Treatise,' notwithstanding his bias for the hypothesis of the affinities of the Megatherium to the Armadillos, enunciates his conclusion with philosophic caution, and affirms that the other "remarkable character of the Megatherium, in which it approaches most nearly to the Armadillo and Chlamyphorus, consists in its hide having probably been covered with a bony coat of armour, varying from three-fourths of an inch to an inch and a half in thickness†." In the same work is given an original figure of the pelvis and hind limb of the Megatherium, from a front view of those specimens in the Museum of the College of Surgeons.

M. Laurillard, in the posthumous edition of the 'Ossemens Fossiles' of Cuvier, published in 1836, whilst admitting it to be very possible for the Megatherium to have been covered by a cuirass, appends a note of warning against too hastily attributing to that animal the fragments of the gigantic osseous armour that had been found in the same formations of South America; because, in the casts of some of the bones which were transmitted with that armour by Sir Woodbine Parish, M. Laurillard had recognized a calcaneum, an astragalus and a scaphoid, differing from those of the living Armadillos only by their size and by some merely specific modifications‡.

But that which Baron Cuvier and M. Laurillard had ventured to regard as very possible, and Dr. Buckland as probable, M. de Blainville a few years later announced to be a positive fact. He communicated, in 1839, to the Academy of Sciences of the French Institute, a statement that bones of the Megatherium had recently been discovered, accompanied with fragments of a carapace belonging indubitably to the same animal; and he adds that the association of a bony armour with the internal skeleton of the Megatherium can be demonstrated as surely by *à priori* reasoning as by the *à posteriori* fact; but he adduces no observations or arguments from the skeleton in addition to those of which Dr. Buckland had previously availed himself, simply affirming that "the Megatherium is proved to have been certainly covered by an osteo-dermal carapace, by the disposition of the spinous processes of the vertebræ, by the angles of the ribs, by the articulation of the pelvis with the vertebral column," &c.; and he concludes by announcing "that the Megatherium was a gigantic species of Armadillo, most nearly allied to the diminutive *Chlamyphorus* §."

* Geology and Mineralogy considered with reference to Natural Theology, vol. i. pp. 160 and 161.

† *Ibid.* p. 159.

‡ Recherches sur les Ossemens Fossiles, 8vo, 1836, tom. viii. p. 354.

§ "Recherches sur l'ancienneté des Edentés terrestres à la surface de la terre," Comptes Rendus de l'Acad. des Sciences, 1839, p. 65.

With regard to the fossils from South America, unequivocally referable to the Armadillo family, I had myself pointed out the generic distinction of that large quadruped, some bones of which had been transmitted along with the gigantic dermal armour by Sir Woodbine Parish, and proposed for it the name of 'Glyptodon' in Sir Woodbine Parish's work on Buenos Ayres *; and afterwards, stimulated by the general tendency of anatomists and palæontologists to regard the Megatherium as being, likewise, a gigantic Armadillo, I entered upon a critical review of all the facts of the case which at that time had been obtained, and communicated the result in a memoir to the Geological Society, read March 23, 1839 †. The general conclusions from this memoir were:—

1. The opinions of Cuvier and Weiss, in favour of the Megatherium being so armed, rest on no better ground than the mere fact of bony armour of some gigantic quadruped and the skeleton of the Megatherium having been discovered in the same continent.

2. The skeleton, or its parts, which have been actually associated with the bony armour above mentioned, belongs to a quadruped distinct from and less than the Megatherium.

3. No part of the skeleton of the Megatherium presents those modifications which are related to the support of a dermal covering.

4. The proportions of the component tesseræ of the bony armour in question to the skeleton of the Glyptodon are the same as those between the dermal tesseræ and skeleton of existing Armadillos, but are much smaller as compared with the bones of the Megatherium.

5. No bony armour composed of tesseræ having the same relative size to the bones of the Megatherium as in the Glyptodon and existing Armadillos, has yet been discovered.

In 1837 I had been put in possession of an additional test of the affinities of the Megatherium, by portions of teeth, obtained by Mr. Charles Darwin at Punta Alta in Northern Patagonia, from which specimens I was kindly permitted to take the requisite sections for microscopical examination. Previous researches by Professor Retzius and myself into the structure of the teeth of the Mammalia generally, had made me acquainted with the marked difference between the teeth of the Armadillos and those of the Sloths in internal structure, and I now found that the Megatherium presented the same remarkable compound structure of the teeth as in the Sloths, but with additional complexity, by which they still further departed from the comparatively simple structure of the teeth of the Armadillos: the examination at the same time proved that there was no true enamel in the teeth of the Megatherium ‡.

* P. 178 *b*, frontispiece, 8vo, 1838.

† "On the *Glyptodon clavipes*," Transactions of the Geological Society, second series, vol. vi. p. 98.

‡ Zoology of the Voyage of Her Majesty's Ship 'Beagle,' Fossil Mammalia, 4to, 1838–40, p. 103, pls. 31 and 32. fig. 1.

he remarks, "we might have inferred that the Megatherium also was covered with heavy armour, even had no such armour been discovered near bones of this animal in other parts of the same level district of Paraguay*."

The estimable and justly celebrated author of the 'Bridgewater Treatise,' notwithstanding his bias for the hypothesis of the affinities of the Megatherium to the Armadillos, enunciates his conclusion with philosophic caution, and affirms that the other "remarkable character of the Megatherium, in which it approaches most nearly to the Armadillo and Chlamyphorus, consists in its hide having probably been covered with a bony coat of armour, varying from three-fourths of an inch to an inch and a half in thickness†." In the same work is given an original figure of the pelvis and hind limb of the Megatherium, from a front view of those specimens in the Museum of the College of Surgeons.

M. Laurillard, in the posthumous edition of the 'Ossemens Fossiles' of Cuvier, published in 1836, whilst admitting it to be very possible for the Megatherium to have been covered by a cuirass, appends a note of warning against too hastily attributing to that animal the fragments of the gigantic osseous armour that had been found in the same formations of South America; because, in the casts of some of the bones which were transmitted with that armour by Sir Woodbine Parish, M. Laurillard had recognized a calcaneum, an astragalus and a scaphoid, differing from those of the living Armadillos only by their size and by some merely specific modifications‡.

But that which Baron Cuvier and M. Laurillard had ventured to regard as very possible, and Dr. Buckland as probable, M. de Blainville a few years later announced to be a positive fact. He communicated, in 1839, to the Academy of Sciences of the French Institute, a statement that bones of the Megatherium had recently been discovered, accompanied with fragments of a carapace belonging indubitably to the same animal; and he adds that the association of a bony armour with the internal skeleton of the Megatherium can be demonstrated as surely by *à priori* reasoning as by the *à posteriori* fact; but he adduces no observations or arguments from the skeleton in addition to those of which Dr. Buckland had previously availed himself, simply affirming that "the Megatherium is proved to have been certainly covered by an osteo-dermal carapace, by the disposition of the spinous processes of the vertebræ, by the angles of the ribs, by the articulation of the pelvis with the vertebral column," &c.; and he concludes by announcing "that the Megatherium was a gigantic species of Armadillo, most nearly allied to the diminutive *Chlamyphorus* §."

* Geology and Mineralogy considered with reference to Natural Theology, vol. i. pp. 160 and 161.

† *Ibid.* p. 159.

‡ Recherches sur les Ossemens Fossiles, 8vo, 1836, tom. viii. p. 354.

§ "Recherches sur l'ancienneté des Edentés terrestres à la surface de la terre," Comptes Rendus de l'Acad. des Sciences, 1839, p. 65.

With regard to the fossils from South America, unequivocally referable to the Armadillo family, I had myself pointed out the generic distinction of that large quadruped, some bones of which had been transmitted along with the gigantic dermal armour by Sir WOODBINE PARISH, and proposed for it the name of 'Glyptodon' in Sir WOODBINE PARISH's work on Buenos Ayres *; and afterwards, stimulated by the general tendency of anatomists and palæontologists to regard the Megatherium as being, likewise, a gigantic Armadillo, I entered upon a critical review of all the facts of the case which at that time had been obtained, and communicated the result in a memoir to the Geological Society, read March 23, 1839 †. The general conclusions from this memoir were:—

1. The opinions of CUVIER and WEISS, in favour of the Megatherium being so armed, rest on no better ground than the mere fact of bony armour of some gigantic quadruped and the skeleton of the Megatherium having been discovered in the same continent.

2. The skeleton, or its parts, which have been actually associated with the bony armour above mentioned, belongs to a quadruped distinct from and less than the Megatherium.

3. No part of the skeleton of the Megatherium presents those modifications which are related to the support of a dermal covering.

4. The proportions of the component tesseræ of the bony armour in question to the skeleton of the Glyptodon are the same as those between the dermal tesseræ and skeleton of existing Armadillos, but are much smaller as compared with the bones of the Megatherium.

5. No bony armour composed of tesseræ having the same relative size to the bones of the Megatherium as in the Glyptodon and existing Armadillos, has yet been discovered.

In 1837 I had been put in possession of an additional test of the affinities of the Megatherium, by portions of teeth, obtained by Mr. CHARLES DARWIN at Punta Alta in Northern Patagonia, from which specimens I was kindly permitted to take the requisite sections for microscopical examination. Previous researches by Professor RETZIUS and myself into the structure of the teeth of the Mammalia generally, had made me acquainted with the marked difference between the teeth of the Armadillos and those of the Sloths in internal structure, and I now found that the Megatherium presented the same remarkable compound structure of the teeth as in the Sloths, but with additional complexity, by which they still further departed from the comparatively simple structure of the teeth of the Armadillos: the examination at the same time proved that there was no true enamel in the teeth of the Megatherium ‡.

* P. 178 *b*, frontispiece, 8vo, 1838.

† "On the *Glyptodon clavipes*," Transactions of the Geological Society, second series, vol. vi. p. 98.

‡ Zoology of the Voyage of Her Majesty's Ship 'Beagle,' Fossil Mammalia, 4to, 1838–40, p. 103, pls. 31 and 32. fig. 1.

Another important evidence of the affinity of the Megatherium to the Sloths was brought to light by a fragment of the skull from Punta Alta, which demonstrated a fifth small molar tooth on each side of the upper jaw, thus showing that in the number as well as in the structure and the kind of teeth the Megatherium agreed with the *Bradypodidæ*, and especially with the Ai or Three-toed Sloth; the anterior pair of molars not manifesting the excess of size and laniary form which characterize them in the Unau or Two-toed species*.

These additional evidences of the concordance of structure between the Megatherium and the Sloths, manifested by the hard and enduring parts which are most intimately related to the food of the animal, induced me to reconsider the conclusions of Cuvier, Pander and D'Alton, and Dr. Buckland relative to the sources of its nutriment and its habits of life; and ultimately to arrive at a conviction of the correspondence of the food of the Megatherium with that of the Sloths, and of the relation of the modified form of the Megatherium to its peculiar mode of obtaining such food, the grounds for which conviction are given in my memoir on the *Mylodon robustus*, published in 1842.

By the analogy of this smaller species of the great extinct terrestrial Sloths of South America, I endeavoured to dissipate some of the doubt and obscurity which shrouded the true structure of the fore and hind feet of the Megatherium†; but the light so obtained served rather to increase the desire to inspect the skeleton itself at Madrid, and obtain, *ex visu*, a conviction of the accuracy of my views; for I participated entirely in the doubts expressed by my experienced colleague Mr. Clift‡ relative to that skeleton, then unique in Europe, viz. as to "whether it had been properly or improperly mounted, *i. e.* whether all the parts were of one or more individuals, whether they belong to the situation or position in which they are placed, whether all the parts are genuine or partly modelled, or whether parts are eked out by bones that do not belong to the part or situation in which they are collected:", concurring at the same time with Mr. Clift, that "no blame was attributable to the articulator, who, probably, had little or no guide in such a difficult task."

Year after year, however, passed away without bringing with it the requisite liberty from official duties for a visit to Madrid. I availed myself of the rare opportunities afforded by journeys of scientific friends to that city to endeavour to obtain information on some of the discrepant or doubtful points, and more especially relative to the exact number of teeth or sockets of teeth in the skull; but usually without any satisfactory result, owing chiefly to the difficulty which the mode of preserving the famous skeleton presents to any close inspection. Dr. Daubeny, the accomplished Professor of Botany at Oxford, in reply to one of my requests, wrote to me from Madrid:—"I have examined the Megatherium and can discern only four teeth in

* Zoology of the Voyage of Her Majesty's Ship 'Beagle,' Fossil Mammalia, 4to, 1838–40, p. 102.

† Description of the Skeleton of the *Mylodon robustus*, 4to, pp. 102, 131–136.

‡ "Notice on the Megatherium," Geol. Trans. Second Series, vol. iii., description of plate 44.

either jaw, which are all perfect and double, but whether or not there be the rudiment of a tooth behind cannot be distinctly ascertained, unless the glass case which covers the specimen be removed; for there is no door or any way of getting close to the skeleton. I should advise a memorial to be drawn up stating the reasons for wishing the point to be determined, in which case, perhaps, the authorities might consent to allow a pane of glass to be removed."

Fortunately the necessity of the endeavour to overcome these obstacles was in great measure obviated by the arrival in 1845, in this country, of a very important and remarkable accession of remains of the Megatherium, discovered in 1837, near Luxan, Buenos Ayres, which, with other fossils of large extinct South American animals, were purchased by the Trustees of the British Museum. This collection included an entire cranium and lower jaw of the Megatherium; the entire tail; complete series of the bones of both fore- and hind-feet:—in short, parts which, in combination with those previously deposited in the Museum of the Royal College of Surgeons, by Sir Woodbine Parish and Mr. Darwin, completed the entire skeleton of the animal, including the hyoid and sesamoid bones, which are too often wanting in the skeletons of our recent and common quadrupeds.

Accurate plaster casts of the huge pelvis and most of the other bones of the Megatherium in the College Museum had been prepared at the expense of the College and presented to the British Museum: and, after an examination and comparison of the whole series of the remains of the Megatherium in both collections, and a consultation with the ingenious articulator of the Mylodon, Mr. Flower, I suggested to the able keeper of the Mineralogical Department in the British Museum, Charles König, K.H., F.R.S., the advantage which would arise, if similar plaster casts should be taken of all the other bones of the skeleton, and if such casts, coloured to match the original bones, should be mounted; the originals being preserved separate, for the greater facility of their comparison and for the advantage of examining their articular surfaces.

The Trustees of the British Museum having taken the proposition under mature consideration, the Council of the Royal College of Surgeons having liberally sanctioned the moulding of all the requisite bones in their Museum, and my own consent to superintend the co-adjustment and attitude of the skeleton having been given, the models and plaster casts were ordered by the Trustees to be made, and their articulation was confided to Mr. Flower.

The result is the exhibition in our National Museum of the entire skeleton of the Megatherium, Plate XVII., in a much more complete state, and, I believe I may add, more natural attitude, than that of the same extraordinary quadruped, which previously had been unique and the glory of the Royal Museum of Natural History at Madrid.

For the full fruition by comparative anatomists and palæontologists of so rich an accession to our evidences of one of the strangest animals of a former world, there still remained one condition,—viz. the power to employ a competent artist to depict the skeleton and its several parts. There could be no question that the opportunity of supplying the omissions, correcting the errors, and clearing up the doubts, in the descriptions founded on the Madrid skeleton, ought to be embraced without loss of time. The enlightened and liberal grant of £1000, placed by Lord John Russell, then Prime Minister, at the disposal of the Council of the Royal Society, in aid of the labours of men of science, seemed to me to afford the means of removing the only difficulty that stood in the way of completing the object of my wishes. I therefore submitted the case to the "Committee of Recommendations for the application of the Government Grant," and the Council of the Royal Society has been pleased to approve the recommendation of the Committee, viz. "that £100 be granted to Professor Owen, to be applied to the procurement of drawings of the undescribed and unfigured or inaccurately figured parts of the skeleton of the Megatherium, on the understanding that Professor Owen undertakes to select the subjects, direct the artists, and communicate his descriptions to the Royal Society; and that Mr. Darwin, Mr. Bell, and Dr. Sharpey, be a Committee to ascertain the proper application of the funds."

The memoir, and its illustrations from the accurate pencil of Mr. Joseph Dinkel, herewith communicated to the Society, are the result of that recommendation; and I have only to add, that the descriptions and figures of the several parts of the skeleton of the Megatherium, now in London, have been taken from the actual bones; and the views of the entire skeleton from the articulated casts, which are so beautifully exact, as, for all the essential purposes of science, to be of the same value and utility as the bones themselves would be if so articulated together.

Of the Spinal Column.

The skeleton of the Megatherium, like that of all other vertebrate animals, being composed of a series of segments, similar in their composition, and referable under all their modifications to a common type, answering to that which is figured as the 'typical vertebra' in my work on the Vertebrate Skeleton*, I shall commence its description by one of those segments which deviate least from the archetypal character; and such segments we find to constitute the major part of the trunk, where they form what, in human anatomy, would be termed 'dorsal vertebræ,' 'ribs,' 'cartilages of ribs' and 'sternal bones.'

Plate XVIII. fig. 1, gives a front view of the fifth segment of the dorsal or thoracic region of the trunk: it deviates from the archetype inasmuch as the neurapophyses, *n*, *n*, have coalesced, as in other mammals, with the centrum, *c*, below, and are connate with the neural spine, *ns*, above: the hæmal arch is also vastly expanded in relation to the greatly developed vascular centres which it was destined to encompass; the

* On the Archetype and Homologies of the Vertebrate Skeleton, 8vo, 1848, p. 81, fig. 16, and p. 82, fig. 15.

pleurapophyses, *pl*, *pl*, being elongated and bent down, and the hæmapophyses, *h.h*, removed from the centrum and articulated to the ends of the pleurapophyses, and, by a double synovial joint, *s's'*, to the hæmal spine or sternal bone *h.s*.

The coalesced centrum and neural arch constitute the so-called 'dorsal vertebra,' and the one selected is the fifth of that series counting backwards.

The centrum, *c*, or body of the vertebra, is wedge-shaped, with its base upwards, forming the floor of the capacious neural canal, and the sides—slightly concave lengthwise, almost flattened vertically—converge to the inferior surface, which is formed by an obtuse ridge: the centrum expands slightly at its articular ends, and so that the contour of the anterior one is rather oval than trihedral; this surface is slightly depressed at its middle, slightly convex in the rest of its extent; the posterior articular surface is larger than the anterior one and flatter, but is also a little depressed at the middle: the two upper angles of the hinder end, probably contributed by the neurapophyses in the development of the vertebra, are slightly produced and truncate, offering each a smooth, flat, small subcircular surface, *c'*, for the head of the rib of the preceding vertebra; the corresponding part of the rib of the present segment is marked *c''*. The neurapophyses, *n*, *n*, rise each by a slender base which has coalesced with the anterior half of the upper and outer angle of the centrum: they diverge from each other and expand as they rise; then, developing some articular surfaces and exogenous processes from their outer surface, arch towards each other, increasing rapidly in antero-posterior extent, and coalesce above the neural canal; where they support the zygapophyses, *z*, *z*, and the thick and strong neural spine *ns*. The roof of the neural arch, thus formed, projects some way beyond the anterior surface of the centrum, and extends almost to the posterior surface. The inner surface of the neural arch is as remarkable for its even smoothness, as the outer surface is for its various prominences and depressions. The outer side of the basal half of the neurapophysis supports a large elliptical articular surface (*n'*), concave from above downwards and backwards: the overhanging fore-part of the arch supports the two flat oval anterior zygapophyses, *z*, *z*, the articular surfaces of which look almost directly upwards; on the under surface of the back part of the arch are the two posterior zygapophyses looking almost downwards, and between these is a rough longitudinal prominent ridge. The neural spine is moderately long, subcompressed and subtrihedral, with a sharp anterior margin, smooth sides, and a rough thick posterior surface, developing a median longitudinal ridge: the summit expands into a rough triangular almost flattened surface.

For the convenience of describing and comparing the different exogenous processes developed from the neural arch in the class *Mammalia*, I have indicated them by single-worded names; having found that, although they varied much in size and a little in relative position, when traced through the series, they could be identified from species to species.

The most common and constant of these processes is that which usually stands out, or transversely, from the base of the neural arch, and affords a joint or a surface of confluence for the rib; this I have proposed to call '*diapophysis**': the second process has a range of variety in its position from the upper part of the diapophysis to that of the anterior zygapophysis, but, as it is commonly somewhere between these two, I have called it '*metapophysis*†'; the third process projects most commonly more or less backwards, from the base of the diapophysis, and I have termed it '*anapophysis*‡'. As each of the above processes developes in some species an articular surface, and as each is usually more or less oblique in position, I have called those processes which more constantly support such surfaces '*zygapophyses*§'. The comparative anatomy of these and other 'exogenous' processes has been the subject of a previous part of this memoir, because their extreme degree of variety in the Order *Edentata*, and extraordinary development in the Armadillos and true Anteaters, render them of unusual importance in the question of the affinities of the Megatherium.

The narrow compressed base of the neurapophysis of the fifth dorsal vertebra in that animal having risen above the centrum, developes on its outer surface a large vertically oval articular surface, n', concave in the direction of its long axis, almost flat transversely; to which surface a corresponding convex articular surface, n'', on the upper part of the neck of the rib near the head, is adapted. Above the surface, n', the diapophysis, d, stands out, short, thick, subdepressed, expanded at its extremity, which is slightly produced backwards and supports on its outer surface an elliptical articular concavity, d', with the long axis directed from above downwards and backwards, and articulated to a corresponding convexity, d'', upon the tubercle of the rib. A rugged tuberosity, m, on the upper and fore-part of the diapophysis represents in rudiment the metapophysis. A smaller tuberosity is interposed between the anterior zygapophysis, z, and the neurapophysial surface, n', for the rib. Thus the neural arch of the vertebra in question presents ten distinct articulations besides the two sutural ones now obliterated by its anchylosis to the centrum, which part has its two large terminal articulations distinct from those for the head of the rib, c', which I reckon among the neurapophysial ones. The rib, which term I confine to the pleurapophysis, or 'vertebral rib' of comparative anatomists, 'pars ossea costæ' of anthropotomy, presents a small flat subcircular surface, c'', for articulation with that on the base of the neurapophysis forming the upper angle of the body of the vertebra in advance of the segment to which the rib belongs. The neck of the rib rapidly expands as it quits the head, developes the convex oval surface, n'', on its upper part

* Lectures on the Comparative Anatomy of the Vertebrate Animals, 8vo, Longmans, 1846, vol. i. p. 42; from διὰ *trans*, and ἀπόφυσις *processus*.

† On the Anatomy of the Male Aurochs, Proceedings of the Zoological Society, November 14, 1848, p. 131, from μετὰ *inter*, and ἀπόφυσις.

‡ On the Anatomy of the Aurochs, *ut supra*, p. 131; from ἀνὰ *retro*, and ἀπόφυσις.

§ Lectures on the Comparative Anatomy of the Vertebrate Animals, p. 43; from ζυγὸν *junctura*, and ἀπόφυσις; these are called the "oblique or articular processes" in Human Anatomy.

for the concavity on the neurapophysis of its own vertebra; and is indented behind where it joins the tubercle. The oval convex articular surface, d'', for that on the diapophysis, is situated on the upper and towards the back part of the tubercle. The body of the rib is moderately convex on its outer side, more convex transversely on its inner side, where the convexity is bounded by a groove on each side, extending half-way down the rib, near its rather sharp margins: in its lower half the rib becomes a little broader and less thick. The outer surface of the rib is well marked by grooves and ridges for muscular attachment; the best-developed eminence being at a short distance from the tubercle; and the largest and deepest groove being behind the tubercle.

The hæmapophysis, or 'sternal rib,' *h*, is a straight subcompressed bone, with a very irregular surface, which is somewhat convex on the inner side, but is traversed by strong oblique ridges with intervening deep and wide channels on part of the outer side. The surface of junction with the pleurapophysis is a very rough and irregular one for ligamentous union: the opposite end of the hæmapophysis divides into two convex condyles, $s' s'$, separated by an oblique, deep and rather narrow groove; the outer condyle projects further than the other; on the shorter one the articular surface passes continuously from one side to the other, describing a semicircle; on the longer condyle the articulation is divided by a median constriction into two oval convex surfaces. One half of each of the condyles articulates with a corresponding concavity on the 'sternal bone' (*h s*) of its own segment, the other half of each condyle with the contiguous sternal bone.

The sterneber or sternal bone, completing, as 'hæmal spine,' *h s*, the typical segment in question, is a cuboid piece, divided into an outer or peripheral, and an inner or central portion. The outer portion is subpentagonal, having its four corners excavated by as many concavities for the hæmapophysis; the two upper concavities, $s'' s''$, being divided by a flat rough tract, the two lower concavities by a rough tuberosity. The outer surface is flat and rough. The inner portion, or that next the cavity of the chest, is larger than the other and has a hexagonal contour; the four angular concave articular surfaces, s'', s'', for the hæmapophyses being separated, at the sides of the bone, by rough tracts; and, above and below, by a flat articular surface, *hs*, by which the bone articulates with contiguous sternal bones. The inner surface of this portion is flat and rough, having been apparently covered by a strong aponeurosis in the living animal. Thus the whole bone presents not fewer than ten articular surfaces, viz. a flat semicircular one above or in front of, and a similar one behind, the posterior division; and two concave articulations, s'', s'', on each side of both divisions for the double condyles of two pairs of hæmapophyses.

The chief modification in the sixth segment of the chest is the development of a third articular surface at the back part of the base of the spine, between the two posterior zygapophyses; and the somewhat greater production of the ridge which stands out from the fore-part of the base of the spine between the anterior zygapophyses. The inferior or hæmal arch is also augmented by increased length of the ribs. The key-bone of that arch, sterneber or hæmal spine, Plate XXVII. figs. 4–7, repeats the characters of that of the previous segment, save that the sternal articulation, *s*, and the hæmapophysial ones, *hp*, *hp*, of the central division of the bone are more continuous, as shown in fig. 6. Fig. 5 shows the surface which was presented towards the integument; fig. 4 that which was turned towards the cavity of the chest; fig. 7 is a side view showing the four articular cavities for the double condyles of the sixth and seventh hæmapophyses; fig. 6 shows the under surface of this remarkable type of sternal bone.

In the seventh dorsal vertebra (Plate XIX. figs. 1, 2, 3), a third articular surface, *mz*, fig. 1, is developed between the two anterior zygapophyses, *z*, *z*, to join that upon the back part of the sixth vertebra; so that there are three zygapophyses, a median and two lateral, on both the fore and the back part (mz', fig. 2) of the arch of this vertebra, making, with the three articular surfaces on each side (fig. 3, c', n', d') for the ribs, and with the anterior and posterior surfaces of the centrum, not fewer than fourteen joints. In this and the two following segments of the back (D_8 and $_9$, Plate XVII.), the ribs attain their greatest length. In the tenth segment the hæmapophyses cease to articulate below directly with a hæmal spine. The median zygapophyses continue to be developed both before and behind to the twelfth dorsal vertebra inclusive. In the thirteenth (Plate XXVI. fig. 4) this supplementary articulation is suppressed behind; and the costal articulations have disappeared from the diapophyses *d*. Those on the neurapophyses are almost circular, n', and those on the upper and posterior angles of the centrum, c', have increased in size. The metapophysis (*m*), which was indicated by a protuberance above the diapophysis in the preceding dorsals, begins to assume the form of a rugged thick vertical ridge. The fourteenth dorsal vertebra shows the progressive increase of size of the centrum, and the absence of the median zygapophysis before as well as behind; and in it the costal articulation on the centrum for the penultimate rib is lost, as well as that on the diapophysis for the antepenultimate one, and only the subcircular concave neural surface for the rib remains. In the fifteenth dorsal vertebra the posterior zygapophyses are convex transversely at their inner border, slightly concave in the rest of their extent; the back part of the neural spine between these processes is deeply grooved; the metapophysial ridge increases in height and length; a short and thick anapophysis is developed from the back part of the base of the diapophysis, and on the under part of the anapophysis there is a distinct, nearly flat, articular surface. The sixteenth dorsal vertebra (Plate XIX. figs. 4 and 5, Plate XXVI. fig. 5) offers a corresponding modification at the fore-part of each neurapophysis, in the development of a short, strong, wedge-shaped process, *p*, fig. 5, answering to the parapophysis in *Myrmecophaga** and *Dasypus*, with an articular surface, *pa* (Plate XIX. fig. 4), on its upper part for junction with the anapophysis of the preceding vertebra.

* Philosophical Transactions, 1851, Plate L. fig. 22, *p*, *pa*.

The metapophysis, *m*, projects forward above the zygapophysis, the articular surface of which (*z*, fig. 5) is continued upward upon the metapophysis (*mz*, fig. 4). The anapophysis, *a a*, fig. 5, Plate XXVI., forms a strong thick square plate of bone projecting upward, outward and backward from the diapophysis. The anapophysial articular surface, *a'*, *a'*, fig. 5, Plate XIX., is on the under and back part of this plate, nearly parallel with the posterior zygapophysis, *z'*, the convex inner border of which has increased in thickness. The body of the vertebra assumes, with its larger size, a more decided trihedral or wedge-shaped figure at this part of the spine.

The rib of the thirteenth segment loses the convex articular surface on the tubercle, which is attached by ligaments to the diapophysis. The rib of the fourteenth segment loses the small flat surface at the extremity of the head; and only the large convex surface on the upper part of that end of the neck remains, which surface, extending to the free end of the neck, reduces that part to an edge: the tubercle exists in this rib, and is rough for ligamentous insertions as in the preceding. In the fifteenth (Plate XIX. figs. 1 and 3) and sixteenth (*ib.* fig. 2) ribs the tubercle subsides; the neck of the rib is defined by the rugosity of its whole upper surface, save that part where the articular convexity, *n''*, remains for articulating with the neural arch, as shown at *n'*, fig. 5, Plate XIX.

The hæmal spine ('sterneber') of the eighth segment, Plate XXVII. figs. 8–12, may be the last of the so-called bones of the sternum. It is divided, like those in advance, into a peripheral and a central portion. The peripheral portion (fig. 9) is of a subquadrate form, the four corresponding articular surfaces (*ha*, *ha'*) for the hæmapophyses almost touching each other at their margins; the outer roughened surface is convex; the anterior hæmapophysial articular surface is suppressed on the left side of this division of the bone, fig. 10, to which the corresponding hæmapophysis seems to have been united by ligament. The central division of the bone (fig. 8) presents the median flat surface (*s*, fig. 10) on its upper or fore-part for the antecedent sterneber, and the concave hæmapophysial surface, *hp*, *hp*, on each side of its anterior half; but posteriorly the hæmapophysial surface on each side is confluent with that of the same side belonging to the peripheral division of the bone, which thus presents at its lower or hinder part only two long oval concave articular surfaces (*hp a*, figs. 11 & 12) for the pair of hæmapophyses of the ninth segment of the chest, and the concavities almost meet at the under surface of the sterneber, which there presents no articular surface for a succeeding one. The number of articular surfaces, therefore, of this bone is reduced to six; one, *s*, for the antecedent sterneber, two on the anterior half of the right side, *hp*, *ha*, fig. 12, for the bifid condyle of the eighth hæmapophysis, one on the anterior half of the left side, *ha*, fig. 10, for one of the condyles of the opposite hæmapophysis; and a pair of surfaces on the posterior and lateral parts, *hp a*, for the ninth pair of hæmapophyses which terminate each by a single convex condyle. It is possible that a more simplified sterneber may have intervened between the hæmapophyses of the tenth segment.

In the three segments of the trunk, Plate XVII. *L* 1, 2, 3, succeeding the last of the dorsal series, both pleurapophyses and hæmapophyses are wanting as distinct ossified parts, and those segments are reduced to the coalesced elements, constituting the 'lumbar vertebræ' of Human Anatomy. The accessory articulations between the parapophyses and anapophyses are continued in these vertebræ, which do not become anchylosed together in the Megatherium as in the Mylodon.

I next proceed to trace the modifications of the segments as they recede from the typical one in the opposite direction or towards the head.

The fourth dorsal segment much resembles the fifth, which has been taken as the type; the pleurapophyses are shorter, especially at their cervix; but the complex articulations of these and of the hæmapophyses are repeated.

In the third segment the pleurapophysis, *pl*, fig. 3, Plate XXV., and hæmapophysis, ib. *h*, are anchylosed together: both are shortened, but the pleurapophysis in a greater degree; this retains its three articular surfaces, *c''*, *n''*, *d''*, on the head, neck and tubercle; and the hæmapophysis, *h*, has its double condyle, *s'*, *s''*, at the sternal end, fig. 3 *b*, the inner one being single, the outer one divided by a narrow groove into an anterior and a posterior convexity. The concave border of the rib is less produced than in the fifth segment.

In the second dorsal segment (Plate XVII. *D* 2) the neural spine is increased in height, and the metapophysial tubercle is diminished in size. The vertebral, *pl*, and sternal, *h*, parts of the rib (Plate XXV. fig. 2) are anchylosed, and both are shortened: the former retains its three articular surfaces on the head, *c''*, neck, *n''*, and tubercle, *d''*, that on the tubercle being the largest, and being partially divided into two convexities (fig. 2*a*, *d*). The pleurapophysis (*pl*, fig. 2) is diminished in length, but increases in breadth to its place of coalescence with the hæmapophysis (*h*); the convex articular surface (fig. 2*b*, *s''*) on the outer condyle of the hæmapophysis is not divided: the inner condyle, *s'*, is much reduced and has only a small articular surface.

The first dorsal segment (Plate XVII. *D* 1) is remarkable for the superior height and antero-posterior extent of the neural spine, the summit of which expands into a broad flat subtriangular rough surface, Plate XX. fig. 5. *D* 1, *ns*. The anterior margin of the spine is sharp and produced. The anterior zygapophyses are not so near each other as in the succeeding vertebræ; and they are continued outwardly upon the base of the metapophysis, which is here more distinct from the diapophysis than in the succeeding vertebræ; and the articular surfaces of the zygapophyses are slightly concave transversely. The costal concavity below the diapophysis is continuous with the smaller articular surface upon the side of the neurapophysis.

The pleurapophysis (Plate XXV. *pl*, fig. 1) is much reduced in length, and is confluent below with a short, thick, broad, subquadrate hæmapophysis, *h*. The short neck of the rib terminates in a small obtuse end without any distinct articular surface: this end seems to have been imbedded in the ligamentous mass between the seventh cervical and first dorsal vertebræ: the short neck quickly expands into the shaft of the rib: a small elliptical surface, fig. 1 *a*, n'', on the upper part of the neck, is continued at its outer end into the larger convex surface upon the upper part of the tubercle, d'': from the middle of the anterior border of this surface a strong ridge is continued down the outer surface of the rib to its hinder border. The sternal end of the hæmapophysis, fig. 1 *b*, presents a large subtriangular surface, slightly concave in one direction, slightly convex in the other, adapted to a similar single concavo-convex surface on the side of the much developed and modified hæmal spine called 'manubrium sterni.'

This bone (Plate XXVII. figs. 1, 2, 3) is of an oval or cordiform figure, with a prominence on each side near its inferior truncated apex, below the lateral articulations, *hp*, for the first pair of sternal ribs. The outer surface is principally concave lengthwise, and is sculptured by the impressions of the coarse aponeurotic structures which have been adherent to it in the living animal: a short median longitudinal ridge projects from its lower part. The inner surface is chiefly convex, but is very irregular. At its upper half a strong median prominence divides the shallow rough depressions, *cl*, for the attachment of the clavicular ligaments: these depressions are deepest above the costal articulations which are supported on well-marked triangular prominences. Between these prominences the bone is rather concave. A strong rough tuberosity projects below the lower angle of the costal surface. The contracted inferior end of the manubrium terminates in an oval convex articular surface, s', for the second sternal bone. There are no articular surfaces for the hæmapophyses of the second dorsal segment.

The skeletal segment (Plates XVII. C_7; XX. fig. 7) in advance of the first of the dorsal series is reduced to its centrum and neural arch; both pleurapophysis, hæmapophysis and hæmal spine are absent; and its remaining anchylosed elements constitute the 'seventh cervical vertebra' of Descriptive Anatomy. This is most remarkable for the great development of the neural spine (Plate XX. fig. 7, *ns*), which exceeds that of the first dorsal vertebra: the summit is similarly expanded and flattened above (Plate XX. fig. 5, *ns* 7); but the anterior margin is still more produced, forming a low angle about half-way down the spine (fig. 7, *ns*). The anterior zygapophyses are more remote from each other than in the first dorsal, and their articular surfaces are chiefly supported by the inner side of the base of the metapophyses, figs. 5, 7 *m*, which are here well developed and more distinct from the diapophyses, *d*, fig. 7, than in the dorsal region. The diapophyses, figs. 5 & 7, *d*, are strong, rugged, stand out from the sides of the neural arch, and terminate in rough truncate ends. The posterior zygapophyses (figs. 6 & 7, z') are slightly convex.

In the sixth cervical vertebra (Plates XVII. C_6; XXI. figs. 5 & 6) the spine, *ns*, is much shortened, though still long and large in proportion to the neural arch. The metapophyses, *m*, *m*, stand out from the upper part of the side of the neural arch behind the anterior zygapophyses, *z*. The diapophyses, *d*, *d*, are developed from the base of the neural arch, which has descended lower upon the sides of the centrum; and now we find another element—the pleurapophysis, *pl*—restored to the segment, but reduced to rudimental proportions and anchylosed at two points. Its vertebral end is bifid; one portion, answering to the head of the rib, has coalesced with the side of the centrum (at *p*, fig. 5); the other, answering to the tubercle, has united with the under part of the diapophysis, *d*: what may be termed the body of the rib is a short but broad rhomboidal plate (fig. 6, *pl*), projecting outward, downward and a little backward. The space intercepted between the pleurapophysis and diapophysis forms the canal, *v*, for the vertebral artery.

The fifth cervical (Plate XVII. C_5) differs from the sixth by its smaller dimensions, especially by its shorter spine, and by the diminution in the breadth of the pleurapophysis, which terminates by a thick obtuse end: it sends out, however, a thin plate forwards from its vertebral end. The metapophysis is a large obtuse tubercle, Plate XX. fig. 5, m_5.

In the fourth and third cervicals (Plate XXI. figs. 3 & 4) the neural spine is still more reduced, and, contracting from its base, assumes a triangular shape, fig. 4, *ns*. The anterior zygapophyses, fig. 3, *z*, are concave transversely, and look upward and inward; the posterior ones, fig. 4, z', are convex, with the reverse aspect: the metapophysis, *m*, continues to be developed as a distinct tuberosity, external and posterior to the prozygapophyses; and the pleurapophysis continues to send forward the pointed plate from its fore-part, fig. 4, pl', its outer end, *pl*, being thick and tuberous, like the diapophysis, *d*, above.

The dentata (Plate XXI. figs. 1 & 2) has its spine extended in the antero-posterior direction, and of great strength, though low; with a thick angular ridge projecting from its fore-part and overhanging the neural canal; it is broad, flattened and almost vertical behind, and has a subbifid summit (Plate XX. fig. 5, $_2$, and Plate XXI. fig. 1, *ns*). There are no metapophyses and no anterior zygapophyses, but the analogous articular surfaces (figs. 41 & 42, *zn*) have descended upon the antero-lateral parts of the coalesced centrum of the atlas or 'odontoid process,' and are adapted to corresponding surfaces of the bases of the neural arch of the atlas. The posterior zygapophyses, z', are wide apart, and are convex. The diapophysis, *d*, is short and obtuse; the pleurapophysis, *pl*, still developes its anterior angle, Plate XXI. fig. 2, pl'. The fore-part of the odontoid process, *o*, is a rounded tuberosity, on the under surface of which is the oval, slightly convex surface for articulating with the hypapophysis (Plate XX. fig. 3, *o*, *hy*), which has coalesced with the neural arch, *ns*, of the atlas, and is commonly called the 'body of the atlas.' The under surface of the centrum of the dentata developes a hypapophysial ridge.

The atlas, viewed from behind, as in Plate XX. fig. 3, is a large, transversely oblong, subdepressed, shuttle-shaped bone, perforated by a large aperture, quadrate below for its detached centrum the 'odontoid process,' arched above for the spinal cord.

The thinnest and smallest part of the ring of the atlas is formed by the hypapophysis, *hy*, which has coalesced with part of the bases of the neural arch, *nn*, and has supplanted, as it were, the proper centrum, *o*, Plate XXI. figs. 1 & 2, which has remained anchylosed to that of the axis. The upper surface of the hypapophysis presents a shallow articular surface, *o*, Plate XX. fig. 3, for that centrum to rest and turn upon. The hinder half of the base of the neurapophysis developes, on each side, a slightly concave, subcircular, articular surface, z', for a moveable articulation with that on the side of the odontoid, *zn*, fig. 1, Plate XXI. The atlas is perforated anterior and external to this by a foramen, Plate XX. fig. 1, *s*, answering to that called 'foramen alare posterius' in the Horse, in which it gives passage to the posterior branch of the occipital artery; in the Megatherium the foramen or canal is bridged over by a narrow oblique bar of bone, dividing its external outlet into two, *r* & *s*, and through the hinder, *s*, of the divisions it is probable that a branch of the second spinal nerve may have passed.

The diapophysis, *d*, is a broad, depressed, rounded aliform process, with a protuberance from its under and back part, like the rudiment of a pleurapophysis, *pl*. Anterior to this process the under surface of the diapophysis is deeply and widely excavated and perforated by the vertebral artery, the canal for which, opening upon the upper surface of the diapophysis, is then continued obliquely inward, perforating at *q*, fig. 4, Plate XX. the upper part of the neural arch, just within the upper part of the condyloid concavities. A large part of the canal for the first spinal nerve, fig. 1, *v*, opens into the outer commencement of the vertebral canal, and answers to that called 'foramen alare anterius' in the Horse, which transmits the inferior branch of the first spinal nerve as well as the anterior branch of the occipital artery. The condyloid concavities, fig. 2, *nz*, are semioval, large and deep, and occupy nearly the whole of the anterior surface of the neural arch, being separated above by a rough tract of three inches' extent, upon which the vertebral canals open. There is a triangular rough surface at the back and inner part of each condyloid concavity.

Such are the modifications of the different cervical vertebræ of the Megatherium. With regard to the dorsal vertebræ, their chief characteristics may be briefly recapitulated as follows:—

The first dorsal vertebra is distinguished by the confluence of the neuro-costal and dia-costal surfaces, and by the superior height of the spine.

The second to the fifth dorsals inclusive, like the first, have only the ordinary pair of zygapophyses before and behind, but have the neural and diapophysial surfaces for the rib distinct.

The sixth dorsal is recognizable by having a third median zygapophysis behind, but not in front. The seventh to the twelfth dorsals inclusive have the three zygapophyses both before and behind. The thirteenth dorsal has the median zygapophysis in front but not behind: the costal surface has disappeared from the diapophysis. The fourteenth dorsal has only the ordinary pair of zygapophyses before and behind, but may be distinguished from the second, third, fourth and fifth dorsals by the absence of the costal articulation on the diapophysis, and of that on the upper and hinder angle of the centrum. The fifteenth dorsal has an anapophysis on each side with an articular surface, and has only the costal articulation on the neurapophysis. The sixteenth dorsal has on each side, at the fore part of the neural arch, a parapophysis with a superior articular surface, and behind, an anapophysis with an inferior articular surface. But it differs from the lumbar vertebræ by the costal surface on the neurapophysis.

Having now described the principal characters of those segments of the skeleton, the centrums and neural arches of which are comprehended in Anthropotomy under the term of 'true vertebræ,' on account of their freedom of motion on each other, I next proceed to the description of the 'false vertebræ;' and first, of those that, being anchylosed together, form the 'sacrum.'

This part of the skeleton includes five vertebræ (Plate XXIII. $_{1-5}$), which are not only anchylosed to each other, but to both the iliac and ischial bones: the length of the sacrum is 22 inches, its extreme breadth across the fifth vertebra, fig. 1, d_5, is 20 inches. The centrum of the first vertebra (Plate XXII. *c*) presents a transversely oblong, subquadrate, flattened, articular surface for that of the last lumbar vertebra, with its margin a little produced forwards, and developed below into a pair of rough ridges, * *. The neural arch overhangs this surface, and developes a metapophysis from the fore part of each side of its base, with a broad articular surface on its under part, and a similar surface above (Plate XXIII. fig. 1, *z*), representing the anterior zygapophysis; the two surfaces meeting at a right angle at their inner borders. The broad diapophysis of the first sacral, $_1$, is perforated by a small subvertical canal, *d'*, at its confluence with the ilium, and is separated from the corresponding part of the next diapophysis by a larger orifice, o_1, which is the first of the four superior or posterior sacral outlets. The neural arch of the first sacral vertebra, n_1, is separated from that of the second, n_2, by a narrow transversely elongated elliptical vacuity. The neural arches of the three succeeding vertebræ are completely confluent: a pair of triangular closely approximated apertures, n_4, divides the base of the neural spine of the fourth, ns_4, from that, ns_5, of the fifth sacral vertebra. The neural spines of the first four sacral vertebræ have coalesced into a strong vertical ridge, *ns*, ns_4, increasing in thickness as it extends backwards, and being there from two-thirds of an inch to one inch and a half thick across the broken summit. The second posterior sacral canal, o_2, intervenes between the diapophysis of the second and that of the third vertebra. The metapophysis, m_4, of the fourth appears as a low angular tubercle above and a little behind the diapophysis of the third sacral, $_3$. The diapophysis of the fourth sacral, d_4, extends outwards beyond the ilium, as a subdepressed broad process with a rough free extremity; the back part of the process coalesces with a similar but stronger and longer diapophysis of the fifth sacral, d_5, from the fore-part of the base of which a tuberous metapophysis, m_5, projects upward and forward. The third, o_3, and fourth, o_4, upper sacral outlets are wider apart than the second and first; the sacrum expanding posteriorly. The back and under part of the diapophysis of both the fourth and fifth vertebræ coalesce with the ischium and with the thick and strong parapophysis extended from the side of the centrums. The neural arch of the fifth sacral developes a pair of posterior zygapophyses, Plate XXIII. z', z', with a flat surface looking outward and a little downward, and with the lower angle continued upon a small rough subarticular surface. The posterior surface of the last sacral vertebra, Plate XXIII. fig. 2, is on the same vertical parallel as the posterior zygapophyses; it is nearly flat and transversely elliptic. The neural canal of the sacrum, the anterior aperture of which is 3 inches in vertical and 4 inches in transverse diameter, expands in the sacrum, and opens below by three wide foramina on each side: of these the first and second are of great size: into the second foramen the third upper sacral canal leads: the third lower sacral foramen, which is the smallest, corresponds with the fourth upper one: the fifth canal for the fifth pair of sacral nerves broadly grooves the back part of the parapophysis and side of the centrum. The posterior aperture of the neural canal is 2 inches in vertical and 4 inches 3 lines in transverse diameter. Both diapophyses and parapophyses of the first three sacral vertebræ coalesce with the ilia. The sacrum is concave below both transversely and lengthwise.

The tail of the Megatherium was of great strength: it is so long as to touch the ground when the trunk is raised at an angle of forty-five degrees from the horizontal position: it includes eighteen vertebræ, which progressively diminish in size from the first to the last, Plate XVII. Cd_{1-18}.

The first vertebra, Plate XVIII. fig. 2, is remarkable for the length and strength of its diapophyses, *d*, which are expanded at both ends, and, like those of the sacral vertebræ, are probably lengthened out by connate or coalesced pleurapophyses, *pl*. The base of the process, *dp*, extends from the side of the centrum to the base of the neural arch, is widely excavated behind for the passage of the first pair of caudal nerves, and is subcompressed before it expands into its rugged free termination. These processes are shorter than those of the last sacral vertebra. The neural canal, *n*, is triangular, 3 inches in vertical and 3 inches 9 lines in transverse diameter. The neural arch developes two posterior zygapophyses, z', with their articular surfaces looking downwards and outwards: two anterior zygapophyses, *z*, with their articular surfaces looking upwards and inwards, these being strengthened by a strong tuberous metapophysis, *m*, on their outer side: the spine, *ns*, is of moderate length, carinate behind, obtuse and slightly expanded above. From the under part of the transversely elliptical centrum are developed two hypapophyses, *hy*, each with an oblong articular surface, Plate XXVI. fig. 6, *hy*, to which is joined a hæmapophysis, Plate XVIII. *h*. Each hæmapophysis is a long slender conical bone, with an articular surface at each angle of the base, hy', hy'', and an obtuse slightly inflected apex: the inner side of the bone is slightly concave, the outer one convex transversely: a rough tuberosity dividing it from the inflected apex. The essential differences between the first caudal segment and the dorsal one delineated on the same Plate are, that the pleurapophysis, *pl*, is short and anchylosed to the diapophysis, the hæmopophyses, *h*, articulate with the centrum, and the hæmal spine is absent, in fig. 2.

The second caudal vertebra (Plate XXIV. figs. 1 and 2) differs from the first in having an anterior, fig. 2, *hy*, as well as a posterior, ib. *hy'*, pair of hypapophyses; and in the confluence of the hæmapophyses, fig. 1, *h*, at their apices forming the so-called 'chevron bone' (*os en chevron*, Cuvier). This vertebra is smaller than the first caudal in all its parts except the hæmapophyses, and in all its dimensions except the vertical diameter, which is due to the development of the coalesced parts of those elements into a long and strong hæmal spine, *hs*. The anterior hypapophyses, fig. 2, *hy*, which are the smallest, articulate with the surface on the back part of the base of the hæmapophyses of the first caudal vertebra: the posterior hypapophyses, *hy'*, which are more oblong and closer together, articulate with the anterior and larger pair of surfaces, *hy'*, of their own hæmapophyses, fig. 1, *h*. There is a strong rough tuberosity projecting backwards external to each of the anterior hypapophyses. The posterior articular surface is, in the present instance, developed only on the right hæmapophysis, fig. 1, *hy''*; on the left it is represented by a rough tubercle.

From the third to the fifth caudal vertebræ inclusive, the proximal end of each hæmapophysis has both the large anterior transversely oblong surface for its own centrum, and the smaller subcircular posterior surface for the next centrum: the spine, or coalesced portions, of the third pair is the longest in the caudal series; beyond this it progressively diminishes. In the sixth caudal vertebra the hæmapophyses have a rough protuberance instead of the posterior articular surface. After the eighth the protuberance subsides to a rough ridge. In the eleventh caudal (Plate XXIV. figs. 3–6) the distal end of the coalesced and shortened hæmapophyses, *hy*, is truncate, and as broad as the divided bases. The under surface of the corresponding centrums of the sixth to the eleventh caudal offers the articular surface on the anterior pair of hypapophyses, fig. 6, *hy'*; the posterior pair, ib. *hy*, are rough tuberosities. The posterior zygapophyses (figs. 3, 4, 5, *z' z'*) retain their articular surfaces to the tenth caudal: in the eleventh they are mere angular projections. The metapophyses, *m m*, are continued to the fourteenth caudal. The neural spine is reduced to a low tuberosity on the thirteenth caudal (fig. 7, *ns*): the neural arch continues complete to the sixteenth, fig. 8, *n*: in the seventeenth, fig. 10, the neurapophyses, *n*, are mere exogenous ridges, bounding the sides of an open neural groove. The hæmapophyses are continued to the fourteenth vertebra: two pair of rough low hypapophysial tubercles, *hy*, *hy*, fig. 9, continue to be developed to the sixteenth: they subside on the penultimate caudal, fig. 11, in which the diapophyses are represented by an obtuse ridge on each side of the centrum. In the last centrum, figs. 12, 13, all the processes have disappeared, and it presents the form of a low rounded cone, with a smooth concave pentangular base, fig. 13, and a rough tuberous summit, fig. 12. This simplified modification of the central element terminates the vertebral series.

Comparison of the Vertebral Column.

In the number of the true vertebræ, as well as of their kinds, the *Myrmecophaga jubata,* amongst the *Edentata,* agrees with the Megatherium. The Ai, or Three-toed Sloth, *Bradypus tridactylus,* has the same number of dorsal and lumbar vertebræ, but has two more in the cervical region; the Unau, or Two-toed Sloth, *Cholœpus didactylus,* has the same number of cervical and lumbar, but has eight additional dorsal vertebræ, being the greatest number known in any mammalian quadruped. The Short-tailed Manis (*Manis brevicaudata*) has seven cervical and sixteen dorsal vertebræ, but it differs from the Megatherium in having five lumbar vertebræ. The Armadillo tribe (*Dasypodidæ*) differ most from the Megatherium in the inferior number of the dorsal vertebræ, which do not exceed eleven in some species, nor twelve in any. The Orycterope, *Orycteropus capensis,* shows its affinity to the Armadillos in having but thirteen dorsal vertebræ: and, like them, it has five lumbar vertebræ. With regard to the structure of the vertebræ, the Anteaters, both hairy (*Myrmecophaga*) and scaled (*Manis*) most resemble the Megatherium in the length and the uniform backward inclination of the spinous processes; but these processes are not so long in proportion to their antero-posterior extent. The spinous processes of the dorsal vertebræ are short and, in the hinder ones, obsolete in the Sloths: the Unau shows the nearest affinity to the Megatherium by having a few of the anterior dorsal spines better developed than in the Ai. In the Orycterope the last dorsal spine is vertical, indicating a centre of motion in the trunk, those behind and those before slightly converging towards this centre.

In the development of the accessory articular surfaces upon both anapophysis and parapophysis of the last dorsal and lumbar vertebræ, the Megatherium manifests a more direct departure from the Sloths and a proportionate affinity to the Anteaters. The Armadillos, which likewise possess these accessory joints, have superadded peculiarities of the posterior dorsal and lumbar vertebræ, in relation to the support of their peculiar bony armour, of which the Megatherium offers as little trace as do the *Myrmecophagæ*: I allude to the progressively and rapidly increasing length of the metapophysis*. These, in the lumbar region, equal in length the spinous process itself; to which the metapophyses bear the same relation in the support of the overarched carapace that the tie-bearers do to the king-post in the architecture of a roof. From the fact of the metapophyses in the dorsal and lumbar vertebræ of the Megatherium, Plate XVII. and Plate XIX. figs. 4 and 5, *m*, not being developed beyond the state of a tubercle, I long ago drew the inference that, like the Sloths and Anteaters, it was not covered by a bony armour†.

* See Plate XLIX. figs. 18 & 19, *m*, of Part I. of this memoir, Philosophical Transactions, 1851.

† Geological Transactions, 2nd Series, vol. vi. p. 101 (1839).

With regard to the cervical vertebræ, the fact of the Megatherium having the normal number in the Mammalian class, seven—if it were not sufficiently established by the well-adjusted articulations of those in the skeleton here described, rendering any supplemental vertebræ inadmissible,—would have been made most probable by the same number being present in the skeleton of the Megatherium at Madrid, and in the more complete skeleton of the *Mylodon* in the Museum of the Royal College of Surgeons in London. Moreover, that one of the Megatherioids had seven cervical vertebræ and no more is certain: the skeleton of the *Scelidotherium*, discovered and deposited by Mr. Darwin in the Museum of the Royal College of Surgeons, having been imbedded, without disturbance of the true vertebræ, and those of the neck being exposed in the ordinary number, and in their natural juxtaposition, on the removal of the stony matrix*.

The atlas in both the Ai and Unau presents but two perforations on each side upon the upper surface; one in front, the other behind the base of the transverse process, and this is less produced and is of a quadrate rather than a triangular form.

The dentata of the Unau resembles more that of the Megatherium in the size of the spinous process than that of the Ai does; but the spine in the Unau is pointed behind, not bifurcate. In the forms and proportions of the spines of the succeeding cervical vertebræ the Unau approaches nearer to the Megatherium than does any other existing Edentate species; but the spine of the seventh cervical is by no means proportionally so developed, and metapophyses are not present in any. The Armadillos are distinguished from all other *Bruta* by the great breadth, the shortness and the anchylosis of the middle cervical vertebræ. In the Anteaters (*Myrmecophaga*) the spine of the dentata is low and is extended more forwards than backwards; the spines of the other cervical vertebræ are still less elevated. In the long-tailed Manis very similar proportions of the cervical spines prevail.

The closest correspondence with the Megatherium in the form and structure of the cervical vertebræ is presented as might be expected by its extinct congeners, the *Mylodon* and *Scelidotherium*.

A resemblance of the Armadillos to the Megatherium has been pointed out in the ossification of the sternal ribs, but this is a character common to the order Edentata, and is consequently equally manifested by the Sloths. The Anteaters most resemble the Megatherium in the double joints by which the sternal ribs articulate with the sternum. There is, however, a character by which the Sloths peculiarly resemble the Megatherium, viz. in the anchylosis of the sternal with the vertebral portion of the rib in those of the first three dorsal segments. The Unau most resembles the Megatherium in the form of the manubrium sterni, having the same prolongation of that bone in advance of the expanded part giving articulation to the first rib. This prolongation, which is not present in the *Bradypus tridactylus*, relates to the complete

* Description of the Fossil Mammalia collected during the Voyage of the Beagle, 4to. 1838–40. pl. 20. p. 84.

development of the clavicles in the *Cholœpus didactylus,* and serves, as in the Megatherium, for their ligamentous attachment. In other existing *Bruta* the manubrium sterni has a broader and shorter figure, and is generally emarginate anteriorly. The succeeding sternal bones present the nearest resemblance to those of the Megatherium in the genus *Myrmecophaga,* in which they are divided into two parts, each having articular surfaces for those on the bifurcate ends of the ossified sternal ribs; but here the broad depressed portion of the sternal bone is external, the stumpy cylindrical part internal or toward the thoracic cavity. The small subcubical sternebers of the Sloths represent the peripheral divisions only of the more complex bones in the Megatherium. Only the Sloths and Anteaters resemble the Megatherium in the small number of the lumbar vertebræ, and the Megatherium most resembles the *Myrmecophaga* in the number and complexity of the articulations between these. The genera *Manis, Dasypus* and *Orycteropus* have five lumbar vertebræ.

Both the *Mylodon* and *Scelidotherium* have three lumbar vertebræ; but these had coalesced with each other and with the sacrum in the skeleton of the *Mylodon robustus* described by me*.

The *Myrmecophagæ* have five sacral vertebræ as in the Megatherium: the Orycteropus has six: so likewise has the *Bradypus tridactylus*: the *Bradypus didactylus* has seven sacral vertebræ; the Armadillos depart furthest from the Megatherium in the unusual number of vertebræ which coalesce to form the sacrum, these ranging from eight to ten in the different species.

It would be unsafe, however, to infer that the Megatherioids were more nearly allied to the Anteaters than to the Sloths in respect of the structure of the sacrum of the Megatherium, for the *Mylodon robustus* has seven sacral vertebræ, like the *Bradypus didactylus.* The posterior expansion of the sacrum and its junction with the ischia, so as to circumscribe the great ischiatic notch, is a character common to the Order *Bruta.* The sacrum early anchyloses with the iliac bones in the Sloths; and the broad and expanded ilia of these animals, with the long and slender anterior parts of the ischia and pubes, circumscribing a large obturator foramen, and bounding a capacious pelvis in front by a narrow symphysis, are characters by which the Sloths, of all existing *Edentata*, offer most resemblance in their pelvis to the Megatherium.

The part of the skeleton in which the Sloths differ most from the Megatherium is the tail, which is so short as to be hardly visible in the entire animal; whilst the Anteaters and Armadillos present the extensive and complex development of caudal vertebræ which characterizes the Megatherioid skeletons. In all the existing species of the ground-dwelling Edentate families the tail is relatively longer and more slender than in the Megatherium; it is even prehensile in the *Manis longicaudatus* and in the *Myrmecophaga didactyla*; but of the modifications of the terminal vertebræ on which

* Description of the Skeleton of the *Mylodon robustus,* 4to. 1842.

that power depends there is no trace in the Megatherium any more than in the Mylodon.

The hæmapophyses are distinct, but short and stumpy, in the first caudal vertebra of the *Dasypus longicaudatus*, Pr. Max.: in the *Myrmecophaga jubata* they present proportions much more nearly resembling those in the first caudal vertebra of the Megatherium, and they are equally disjoined at their distal ends *.

The Mylodon in the number, as well as the proportions and structure of the caudal vertebræ, makes the nearest approach to the Megatherium; the hæmapophyses are equally distinct from each other in the first caudal †.

Upon the whole, our deductions from the characters of the parts of the skeleton described in the present section of this memoir, would lead to an inference that *Megatherium* was nearer akin to *Myrmecophaga* than to *Bradypus*; nevertheless the cervical vertebræ, the condition of the anterior ribs, the form of the manubrium sterni, and the pelvis, illustrate the intermediate nature of the giant's affinities, and afford an additional instance to many others which I have had occasion to point out, of a closer adherence to a common type in extinct animals than in the existing species to which they may be most nearly allied.

Description of the Plates.

PLATE XVII.

Skeleton of the Megatherium, on the scale of one inch to a foot. (The vertebræ concealed by the scapula are added in outline.)

C 1–7. Cervical vertebræ.
D 1–16. Dorsal vertebræ.
L 1–3. Lumbar vertebræ.
S. Sacrum.
Cd 1–18. Caudal vertebræ.
38. Stylohyal.
51. Scapula.
53. Humerus.
54. Ulna.
55. Radius.
s. Scaphoid part } of Scaphotrapezium.
t. Trapezial part }
l. Lunare.
c. Cuneiforme.
p. Pisiforme.
d. Trapezoides.
m. Magnum.
u. Unciforme.
I. Rudimentary metacarpal of first digit or pollex.
II. Second digit, or index.
III. Third digit, or medius.
IV. Fourth digit, or annularis.
V. Fifth digit, or minimus.
62. Ilium.
63. Ischium.

* See Philosophical Transactions, 1851, Plate LIII. fig. 60.

† Memoir on the Mylodon, 4to. p. 69, pl. 8, fig. 5, *b*, *b*.

64. Pubis.
65. Femur.
66. Tibia.
66′. Patella.
67. Fibula.
67′. Fabella.
a. Astragalus.
cl. Calcaneum.
s. Scaphoides.
ce. Ectocuneiform.
b. Cuboides.
II. Rudimentary metatarsal of second digit.
III. Third digit.
IV. Fourth digit.
V. Fifth digit. (There is no rudiment of the first or innermost toe of the hind foot in the Megatherium.)

PLATE XVIII.

Typical vertebræ in the Megatherium. One-fourth natural size.

Fig. 1. Fifth dorsal vertebra, or natural segment of the skeleton. *c.* Centrum, *c′.* articular tubercle for head rib; *n,* neurapophysis, *n′,* neurapophysial surface for neck of rib; *d,* diapophysis, *d′,* diapophysial surface for tubercle of rib; *z,* anterior zygapophysis; *m,* metapophysis; *ns,* neural spine; *pl,* pleurapophysis or 'vertebral rib'; *c″,* head; *n″,* articular surface on upper part of the neck; *d″,* articular surface on tubercle; *h,* hæmapophysis or 'sternal rib,' *s′, s′,* its condyles; *hs,* hæmal spine, or 'sternal bone;' *s″, s″,* articular surfaces for hæmapophysial condyles.

Fig. 2. First caudal vertebra. *n.* Neural arch; *p,* parapophysial part, *d,* diapophysial part, *pl,* pleurapophysial part, of compound transverse process; *hy,* hypapophyses; *h,* hæmapophysis, *hy′,* articular surface for its own vertebra, which is in advance; *hy″,* articular surface for succeeding vertebra.

PLATE XIX.

Fig. 1. Seventh dorsal vertebra, front view.
Fig. 2. Seventh dorsal vertebra, back view.
Fig. 3. Seventh dorsal vertebra, side view.
Fig. 4. Sixteenth dorsal vertebra, front view.
Fig. 5. Sixteenth dorsal vertebra, back view.

All the figures are drawn, minus the hæmal arch, one-fourth natural size.

c′. Articular tubercle for the head of the rib; *n′,* neurapophysial articular surface for the neck of the rib; *d,* diapophysis, *d′,* diapophysial articular surface for tubercle of rib; *m,* metapophysis, *mz′,* metapophysial articular surface; *a,* anapophysis, *a′,* anapophysial articular surface; *p,* parapophysis, *pa,* parapophysial articular surface; *z,* anterior, *z′,* posterior, *mz,* mid-anterior, *mz′,* mid-posterior, zygapophyses; *ns,* neural spine.

PLATE XX.

Fig. 1. The atlas, upper view.
Fig. 2. The atlas, under view.
Fig. 3. The atlas, back view.
Fig. 4. The atlas, side view.
Fig. 5. The seven cervical and first dorsal vertebræ, upper view.
Fig. 6. The seventh cervical vertebra, back view.
Fig. 7. The seventh cervical vertebra, side view.

One-fourth natural size; *c*, centrum; *n*, neurapophysis; *ns*, neural spine; *d*, diapophysis, d_3–d_7, of the third to the seventh cervicals inclusive; *pl*, pleurapophysis; *pl'*, (fig. 7) articular surface for head of first dorsal rib; *m*, metapophysis; *nz* (fig. 2), anterior articular surface; *z'*, posterior zygapophysis; *hy*, hypapophysis; *o*, (figs. 1 & 3) its articular surface for the odontoid or true body of the atlas; *r*, division of the foramen alare posterius, for the passage of the posterior branch of the occipital artery; *s*, division of the same foramen for the passage of a branch of the second spinal nerve; *v*, foramen alare anterius, communicating with *q* the canal for the vertebral artery.

PLATE XXI.

Fig. 1. The axis, or vertebra dentata, front view.
Fig. 2. The axis, or vertebra dentata, side view.
Fig. 3. The third cervical vertebra, front view.
Fig. 4. The third cervical vertebra, side view.
Fig. 5. The sixth cervical vertebra, front view.
Fig. 6. The sixth cervical vertebra, side view.

One-fourth natural size; *o*, odontoid process (centrum of atlas); *zn*, analogues of anterior zygapophyses, in the dentata; *z*, anterior zygapophysis; *z'*, posterior zygapophysis; *d*, diapophysis; *p*, parapophysis; *pl*, pleurapophysis, *pl'*, its anterior production; *m*, metapophysis; *ns*, neural spine.

PLATE XXII.

First sacral vertebra, with part of its hæmal arch (ilium and pubis).

One-fourth natural size; *c*, centrum; ** exogenous growths; *ma*, metapophysial process and articular surface.

PLATE XXIII.

Fig. 1. The five sacral vertebræ, upper view.

Fig. 2. The five sacral vertebræ, back view.

One-fourth natural size; *c*, centrum; n_1, n_4, coalesced neurapophyses; ns_1, ns_4, coalesced neural spines, of the four anterior vertebræ; ns_5, neural spine of fifth vertebra; $d_{1, 2, 3, 4, 5}$, diapophyses; m_2–m_5, metapophyses; o_1–o_5, posterior outlets of nerve-canals; *z*, anterior, z', posterior, zygapophyses.

PLATE XXIV.

Fig. 1. The second caudal vertebra, back view.

Fig. 2. The second caudal vertebra, under view, minus the hæmal arch.

Fig. 3. The eleventh caudal vertebra, back view.

Fig. 4. The eleventh caudal vertebra, side view.

Fig. 5. The eleventh caudal vertebra, upper view.

Fig. 6. The eleventh caudal vertebra, under view, minus the hæmal arch.

Fig. 7. The thirteenth caudal vertebra, upper view.

Fig. 8. The sixteenth caudal vertebra, upper view.

Fig. 9. The sixteenth caudal vertebra, under view.

Fig. 10. The seventeenth caudal vertebra, upper view.

Fig. 11. The seventeenth caudal vertebra, under view.

Fig. 12. The eighteenth caudal vertebra, back view.

Fig. 13. The eighteenth caudal vertebra, front view.

One-fourth natural size; *c*, centrum; *hy*, anterior, hy', posterior hypapophyses; h', their hæmal articular surface; *n*, neural arch and neurapophyses; *ns*, neural spine; *d*, diapophysis; *pl*, pleurapophysis; *m*, metapophysis; *z*, anterior, z', posterior zygapophysis; *h*, hæmal arch, hy', (fig. 1) its anterior hypapophysial articular surface, hy'', its posterior hypapophysial surface; *hs*, hæmal spine.

PLATE XXV.

Fig. 1. First dorsal rib.

Fig. 2. Second dorsal rib.

Fig. 3. Third dorsal rib; one-fourth natural size.

Figs. 1*a*, 2*a*, 3*a*, their upper articulations.

Figs. 1*b*, 2*b*, 3*b*, their lower articulations; one-half natural size.

c'', articular surface on the head for the centrum; n'', articular surface on the neck for the neurapophysis; d'', articular surface on the tubercle for the diapophysis; s', anterior condyle for sternum, s'', posterior condyle for sternum; *pl*, pleurapophysial part, *h*, hæmapophysial part, of rib.

which have a dark yellowish-brown tinge. On adding to this mass a small quantity of cold water, part of it dissolves with a deep yellow or reddish-yellow colour, while a yellow powder remains undissolved. The latter consists of a peculiar acid, to which I shall give the name of *Rubianic Acid*. After being collected on a filter and washed with cold water, it is purified by solution in boiling water, to which a little animal charcoal may be added. On filtering the solution boiling hot and allowing to cool, it crystallizes in beautiful lemon-yellow needles, which if the solution was at all concentrated entirely fill the liquid. The substance dissolved by the cold water is left on evaporation in the shape of a reddish-yellow or brownish-yellow substance, resembling rubian itself in appearance and all its properties. As analysis showed it to be formed from rubian by the elimination of several equivalents of water, I shall call it *Rubidehydran*.

The liquid filtered from the red flocks, consisting of the baryta compounds of the two substances just named, has still a dark brownish-yellow colour, and contains a third organic substance in solution. In order to ascertain whether the whole quantity of the two former substances has been separated, and whether the solution still contains any unchanged rubian, it is well to add caustic baryta to the liquid, to supersaturate the latter with carbonic acid and to evaporate again. If no more red flocks separate on evaporation, but only a deposit of carbonate of baryta is formed, then the process is completed. On now adding to the filtered solution basic acetate of lead a red precipitate falls, while the liquid becomes colourless. The former being separated by filtration and washed with water, is to be decomposed with sulphuric acid in the cold, and the excess of acid having been removed with carbonate of lead, sulphuretted hydrogen is passed through the filtered liquid, and the latter, after being filtered again from the sulphuret of lead, is evaporated to dryness, when it leaves a dark brownish-yellow substance, resembling rubian in appearance, but differing in being somewhat deliquescent. To this substance I will give the name of *Rubihydran*. The liquid filtered from the lead compound of this substance sometimes contains sugar, but this is entirely a secondary product of decomposition, formed at the cost of one or more of the primary products, and indicates the formation at the same time either of rubiadine or alizarine.

The same products of decomposition may be obtained by adding to the solution of rubian a solution of bicarbonate of baryta made from chloride of barium and bicarbonate of soda, and evaporating in contact with the air until red flocks begin to appear, or by adding caustic baryta to the solution, allowing the mixture to stand exposed to the air for some time and filtering, when the insoluble baryta compound left on the filter will yield rubianic acid and rubidehydran, and the liquid rubihydran. The same process of decomposition takes place, if caustic soda, ammonia, or lime water be added to a watery solution of rubian, and the mixtures be allowed to stand exposed to the air for some time. On now adding chloride of barium to any one of them a dark red precipitate falls, which being collected on a filter, washed with water and then treated in the same manner as the red flocks obtained by means of bicarbonate of baryta, affords in each case rubianic acid and rubidehydran, while the liquid contains rubihydran. If a small quantity of any acid be added to a watery solution of rubian, and if the acid be then neutralized with carbonate of baryta, the small quantity of bicarbonate of baryta formed is sufficient to induce a decomposition of the rubian, for the filtered solution on exposure to the air very soon begins to deposit red flocks, which consist of rubianic acid and rubidehydran in combination with baryta. It is for this reason that I have recommended the employment of carbonate of lead instead of carbonate of baryta for the purpose of neutralizing the sulphuric acid used in purifying rubian*. Even oxide of lead is a sufficiently strong base to cause rubian to undergo this process of decomposition, when oxygen is present at the same time. If rubian be precipitated from its watery solution by means of basic acetate of lead, and the lead compound be left exposed to the air for a short time, it will be found no longer to contain unchanged rubian. If the compound be decomposed with sulphuric acid in the cold, and the excess of acid be neutralized with carbonate of lead, the filtered solution deposits during evaporation crystals of rubianic acid, and leaves at last a brown deliquescent mass, which unless it be strongly dried is with difficulty removed from the vessel containing it†. In short, whenever rubian is brought into contact at the same time with oxygen and an alkaline or other strong base, it undergoes decomposition. Hence it follows, that in preparing the so-called xanthine according to the methods proposed by Kuhlmann, Berzelius, Runge and Higgin, as well as the ruberythric acid of Rochleder, the use either of alkaline earths or basic acetate of lead being prescribed by all these chemists, products of the decomposition of rubian must in every case be formed.

For the purpose of preparing the three bodies which result from this process of decomposition, it is not necessary to employ pure rubian. If madder be extracted with boiling water, and sugar of lead be added to the extract, a purple precipitate is produced, and ammonia being added to the filtered liquid, the whole of the rubian is precipitated together with some chlorogenine in combination with oxide of lead. The precipitate is decomposed with sulphuric acid in the cold, and the excess of the acid is removed by means of carbonate of lead. To the filtered solution a quantity of baryta-water is to be added, and the baryta is then converted into bicarbonate of baryta by a stream of carbonic acid gas. The solution is then left exposed to the air for some time and then slowly evaporated. The red flocks which are deposited are collected on a filter as they form, and after being treated in the same way as the flocks from pure rubian, yield rubianic acid and rubidehydran. The liquid filtered

* See Philosophical Transactions for 1851, p. 440.

† In purifying rubian by precipitation with basic acetate of lead, as described in the first part of this paper, care must be taken to wash the precipitate with alcohol and not with water. If the former be employed, decomposition is almost entirely prevented, whereas in using water the rubian in the precipitate undergoes complete decomposition during the short time necessary for edulcoration.

from these flocks contains chlorogenine as well as rubihydran, and the separation of the latter becomes therefore rather more difficult. This may however be effected by precipitating it with basic acetate of lead, filtering, washing the precipitate with water, then redissolving it in warm acetic acid, and again precipitating by means of a little ammonia. The last precipitate contains hardly any chlorogenine, and after being put on a filter and washed, is decomposed with sulphuric acid in the cold. The excess of acid is removed by carbonate of lead, the liquid is filtered, sulphuretted hydrogen is passed through it, it is again filtered from the sulphuret of lead, and on being evaporated leaves the rubihydran free from chlorogenine. Should it still contain a little of the latter substance, in which case its watery solution turns green on being boiled with muriatic or sulphuric acid, the precipitation with basic acetate of lead must be repeated. It frequently happens, that besides the usual products of decomposition, a quantity of rubiadine is also formed. In this case the red flocks deposited on evaporating the barytic solution yield, on being decomposed with sulphuric acid, besides the substances soluble in water, a yellow powder, which is insoluble both in cold and boiling water, and which consists of impure rubiadine; but this, like the sugar, which is also sometimes formed during the process, is without doubt a secondary product of decomposition. Indeed the formation of the one is most probably dependent on that of the other.

The process of decomposition just described is always accompanied by an absorption of oxygen. In order to ascertain the quantity of the latter which an alkaline solution of rubian was capable of absorbing, I took 4·2280 grms. of rubian, which after deducting the inorganic matter with which it was contaminated, was equivalent to 4·1049 grms. of the pure substance, dissolved it in hot water, poured the solution into a graduated tube, added to it a solution of about 6 grms. hydrate of baryta, filled the rest of the tube with mercury, and then inverted it over mercury. Oxygen gas was then introduced, and the liquid was from time to time agitated with the gas, in order to bring every portion, and especially the red flocculent precipitate produced by the baryta, and which sank to the bottom of the solution, into frequent contact with it. After 143 days I found that 147 cubic centimetres of gas had been absorbed. The contents of the tube were then removed and filtered. There remained on the filter a red baryta compound, which, on being treated as usual, yielded rubianic acid and rubidehydran, as well as a small quantity of alizarine. To the red liquid sulphuric acid was added; the excess of acid was removed with carbonate of lead, and sulphuretted hydrogen having been passed through the filtered liquid, the latter, after being filtered again from the sulphuret of lead, was submitted to distillation. The distillate was acid, and after being neutralized with carbonate of soda and evaporated, left a saline residue having all the characters of acetate of soda. Acetic acid is therefore another product formed in this process, but whether this acid is an essential product of decomposition or not, still remains doubtful. The residue of the distillation contained rubihydran, which was separated by precipitation with basic acetate of lead, as before described, and a little sugar, which was obtained from the liquid filtered from the lead precipitate. The sugar and the alizarine were probably secondary products of decomposition formed from the rubianic acid by the action of the alkali.

Rubianic Acid.—This substance really merits the name of an acid, for though its acid properties are not well marked, the fact of its giving crystallized compounds with the alkalies is a sufficient indication of the class in which it should be placed. It crystallizes from its watery solution in silky needles of a pure lemon-yellow colour, which when dry form a light, bulky interwoven mass. Sometimes it is slowly deposited from its watery solution in grains and masses of an indistinctly crystalline form, which leave an orange tinge. This difference in appearance is due to some impurity, which may be removed by redissolving the acid in boiling water and adding a little animal charcoal, when the acid crystallizes rapidly from the filtered solution in needles, as just mentioned. The watery solution has a light yellow colour, reddens litmus paper slightly, and has a distinctly bitter taste, though not so intensely bitter as that of a solution of rubian. It is soluble in alcohol, but not in ether. When heated on platinum foil it melts and then burns with a smoky flame, leaving a slight carbonaceous residue. When heated in a tube it melts to a brownish-red liquid, which on being allowed to cool becomes solid and crystalline, but on being heated again more strongly gives fumes, which condense on the colder parts of the tube to a liquid, which soon solidifies, forming a mass of shining needles. When slowly heated between two watch-glasses, there is formed on the upper glass a sublimate of shining orange-coloured crystals, having the appearance and all the properties of alizarine, while a considerable carbonaceous residue is left on the lower glass. Concentrated sulphuric acid dissolves rubianic acid easily even in the cold, forming a dark red solution, which on being boiled becomes of a dark reddish-brown colour, without evolving much sulphurous acid. If sulphuric acid be added to a watery solution of rubianic acid, and the solution be boiled, dark yellow or orange-coloured flocks begin to be deposited, which increase as the boiling continues. On allowing to cool they often separate in such quantities as to render the liquid quite thick. These flocks are quite uncrystalline, though sometimes small yellow crystalline grains are found among them, which consist of undecomposed acid, and disappear on continuing the action for some time longer. These flocks consist of alizarine almost in a state of purity. On dissolving them in alcohol and evaporating spontaneously, a quantity of beautifully crystallized alizarine is obtained. The liquid filtered from the flocks contains sugar, which is obtained with its usual appearance and properties on neutralizing the acid with carbonate of lead, filtering, passing sulphuretted hydrogen through the liquid, filtering again, and evaporating. Muriatic acid acts in precisely the same manner on rubianic acid as sulphuric acid. Nitric acid dissolves rubianic acid even in the cold, forming a yellow solution, which on being boiled disengages nitrous fumes and becomes colourless. The solution on evaporation leaves a brown syrup, which contains oxalic acid. Rubianic acid is not decomposed on being treated with boiling solutions of phosphoric, acetic, oxalic or tartaric acids; it merely dissolves in them, and crystallizes out again unchanged on the solutions cooling. If chlorine gas be passed through a watery solution of rubianic acid, the yellow colour of the solution slowly disappears. If there are any crystals of undissolved acid floating in the liquid, these disappear very slowly, and the gas must be passed through for a long time in order to effect their decomposition. The solution now appears colourless, but slightly milky. No yellow flocks are formed, as in the case of rubian, but the solution, on standing for some time, usually deposits a quantity of brownish-yellow crystals. If a solution of chloride of lime be added to a watery solution of rubianic acid, the latter turns blood-red, but in a few moments the colour disappears, and the solution becomes quite colourless.

Rubianic acid dissolves in caustic potash and soda in the cold with a lively cherry-red colour. If caustic potash be added to a strong watery solution of the acid, nothing separates on standing; but if carbonate of potash be employed instead of caustic potash, there is deposited almost immediately a quantity of dark-coloured crystalline needles. These needles are the potash salt. When collected on a filter, slightly washed with water and dried, they form a beautiful puce-coloured silky mass. When dried either in the water-bath or *in vacuo*, the colour changes to a bright red, but after a few minutes' exposure to the atmosphere the original colour is restored. Rubianic acid is so weak an acid, that water alone is sufficient to separate it from its combination with potash. When the potash salt is treated with boiling water it dissolves with a red colour, but the solution on cooling and standing deposits yellow crystals of the acid itself. Mere treatment with cold water produces to some extent the same effect; for if the salt be placed on a filter and washed with cold water, numerous small yellow specks, caused by the separation of the acid, become visible. When caustic or carbonate of soda are added to a boiling watery solution of the acid, the solution deposits on cooling a quantity of small bright red spherical grains, which generally form a mass so bulky as to render the liquid thick and gelatinous. When this mass, which consists of the soda salt, is placed on a filter, an almost colourless liquid runs through, leaving the salt on the filter in a soft spongy state, from its retaining mechanically a large quantity of water. The round grains of which it consists, though they have a somewhat crystalline appearance, are not in reality crystalline. In consequence of its sparing solubility in water, which is less than that of the potash salt, it may be washed with cold water without dissolving. When dry it is dark red, and gives a red powder. It dissolves again in boiling water with a red colour, and the solution on cooling forms a thick jelly, which however is converted, on standing, into a mass of yellow crystalline grains consisting of acid. If a solution of rubianic acid in caustic potash or soda be supersaturated with any stronger acid, the solution becomes yellow without depositing anything, but after some time the acid separates in yellow crystals. But if the alkaline solution, containing a considerable excess of alkali, be boiled, its colour changes gradually from red to purple, and after prolonged boiling to violet, similar to that of a solution of alizarine in caustic alkali, the colour being generally so intense as to deprive the solution of its transparency. Acids now produce an immediate precipitate of dark yellow flocks, while the liquid becomes almost colourless. If the action of caustic alkali has not been continued long enough to produce a complete decomposition of the acid, then after some time crystals of the latter appear in the midst of the flocculent precipitate. The flocks consist of alizarine, but less pure than when the decomposition is effected by means of sulphuric or muriatic acid. On decomposing a considerable quantity of the acid by means of caustic soda, and treating the precipitated flocks in the same manner as those formed by the decomposition of rubian with acids or alkalies, I obtained, besides alizarine, a trace of verantine and a small quantity of a substance resembling rubianine or rubiadine *. The liquid filtered from the flocks contains sugar. Rubianic acid dissolves in caustic ammonia with the same colour as in caustic potash or soda, but with much greater difficulty than in the latter. The solution does not change its colour, however long it may be boiled, and on adding an excess of a stronger acid, the rubianic acid crystallizes out unchanged. On evaporating the ammoniacal solution to dryness it loses part of its ammonia, leaving some yellow crystals of acid surrounded by a red uncrystalline coating like gum. The latter dissolves in cold water, but on evaporation the solution again gives yellow crystals. If to a boiling watery solution of the acid carbonate of ammonia be added, the solution becomes red, and deposits on cooling crystals of the ammonia salt, which have the same colour and appearance as the potash salt. These crystals may be dried without losing their ammonia, but, like the other alkaline salts, are decomposed on redissolving them in boiling water, the solution depositing on cooling crystals of the acid.

Baryta water gives in a watery solution of rubianic acid a beautiful crimson flocculent precipitate, while the liquid becomes colourless. If carbonic acid be passed through the liquid the precipitate gradually dissolves, forming a clear yellow solution; but if this solution be left exposed to the air, its surface becomes covered with a red film, and on evaporation it deposits a quantity of red flocks, until at length the whole of the acid seems to be again precipitated in combination with baryta. Lime water produces in a watery solution of the acid a light red precipitate, while the liquid becomes colourless. If carbonic acid be now passed through the liquid the precipitate dissolves, forming a yellow solution, which on exposure to the air becomes red, but gives no red film, and on evaporation deposits no flocks, but leaves at last a red mass, which dissolves again in boiling water. Acetate of alumina and peracetate of iron produce no change in a watery solution of the acid. Nevertheless the latter

* This substance is light yellow and crystalline. When heated it is volatilized, giving a sublimate of yellow shining needles and scales. It is insoluble in boiling water, but dissolves in boiling nitric acid, and crystallizes out of the solution again on cooling in yellow needles. It is precipitated from its alcoholic solution by acetate of copper, but not by acetate of lead.

is entirely removed from its solution by hydrate of alumina, to which it communicates a light red colour, and also by hydrated peroxide of iron, though not so easily as by alumina. Rubianic acid dissolves easily in a boiling solution of perchloride of iron, forming a dark greenish-brown liquid, which contains protochloride of iron. The solution deposits nothing on standing, and on evaporation only a small quantity of black powder separates, which is probably a compound of alizarine and oxide of iron. Acetate of copper gives in watery and alcoholic solutions of the acid a brownish-red precipitate, which is soluble in boiling acetic acid. Neutral acetate of lead turns the watery solution of the acid red, but produces no precipitate; basic acetate of lead, however, gives a copious red flocculent precipitate, just as in a solution of rubian. An alcoholic solution of the acid gives with an alcoholic solution of acetate of lead a light red precipitate, which after filtration of the alcoholic liquid dissolves in pure water. On adding nitrate of silver to an aqueous solution of the acid and boiling no change takes place, but on the addition of a few drops of ammonia a dark reddish-brown precipitate slowly subsides as a fine powder, which is soluble in an excess of ammonia, forming a red solution, which does not change, and shows no signs of any reduction taking place when it is boiled. On adding chloride of gold to a watery solution of the acid no change takes place even on boiling, but on the addition of a little caustic alkali metallic gold is deposited, partly as a brown powder, partly in shining scales. If rubianic acid be mixed with water to which a quantity of erythrozym has been added, the yellow crystals of the acid gradually disappear, and are converted into alizarine. The liquid is found to contain sugar.

It is impossible to dye with rubianic acid. Mordants acquire in boiling solutions of the acid no more colour than in solutions of rubian or rubianine.

On submitting the acid to analysis the following results were obtained:—

I. 0·4605 grm., formed by the action of bicarbonate of baryta on rubian, dried in the water-bath and burnt with chromate of lead, gave 0·9445 carbonic acid and 0·2250 water.

II. 0·4030 grm. of another preparation, obtained by the action of bicarbonate of baryta, gave 0·8240 carbonic acid and 0·2020 water.

III. 0·2750 grm., formed by the action of caustic soda on rubian, gave 0·5605 carbonic acid and 0·1390 water.

IV. 0·4750 grm., obtained by means of ammonia, gave 0·9670 carbonic acid and 0·2295 water.

V. 0·2920 grm., obtained by means of caustic lime, gave 0·6000 carbonic acid and 0·1490 water.

These analyses give in 100 parts—

	I.	II.	III.	IV.	V.
Carbon	55·93	55·76	55·58	55·52	56·04
Hydrogen . . .	5·42	5·56	5·61	5·36	5·66
Oxygen	38·65	38·68	38·81	39·12	38·30

corresponding to the formula $C_{52} H_{29} O_{27}$, as the following calculation shows:—

	Eqs.		Calculated.
Carbon	52	312	56·01
Hydrogen . . .	29	29	5·20
Oxygen	27	216	38·79
		557	100·00

Rubianate of potash was prepared by dissolving the acid in as small a quantity of water as possible, adding an excess of carbonate of potash, allowing the salt to crystallize, collecting it on a filter, washing it with just sufficient water to remove the excess of alkali, and drying. On analysing it I obtained the following results:—

I. 0·3580 grm., dried in the water-bath and burnt with chromate of lead, gave 0·6830 carbonic acid and 0·1590 water.

0·5590 grm. gave 0·0780 sulphate of potash.

II. 0·4850 grm. of another preparation, dried *in vacuo*, gave 0·9250 carbonic acid and 0·2070 water.

0·8140 grm. gave 0·1150 sulphate of potash.

These numbers lead to the following composition:—

	Eqs.		Calculated.	I.	II.
Carbon . . .	52	312	52·42	52·03	52·01
Hydrogen . . .	28	28	4·70	4·93	4·74
Oxygen . . .	26	208	34·97	35·50	35·62
Potash	1	47·2	7·91	7·54	7·63
		595·2	100·00	100·00	100·00

The soda salt, prepared in the same way as the potash salt, was also analysed, but the analysis led to no satisfactory results.

A compound with baryta was obtained by dissolving the acid in water and adding ammonia and chloride of barium. The precipitate, which was bulky and of a fine crimson colour, was placed on a filter and washed with water. During the washing its bulk diminished and the colour changed to red, probably in consequence of its losing part of its baryta. Analysis showed it to be a basic compound, in which the acid is to the base as 2 : 3. The neutral baryta salt was obtained by adding chloride of barium to a watery solution of rubianate of potash, when a red precipitate fell, which was collected on a filter, washed with water, and dried *in vacuo*.

I. 0·6020 grm. of the latter compound, burnt with chromate of lead, gave 1·0800 carbonic acid and 0·2580 water.

0·5180 grm. gave 0·0925 sulphate of baryta.

II. 0·4440 grm. gave 0·7870 carbonic acid and 0·1900 water.

0·4890 grm. gave 0·0870 sulphate of baryta.

These numbers lead to the formula $C_{52} H_{28} O_{26}+BaO+HO$, as the following comparison of the experimental results with the theoretical composition shows:—

	Eqs.		Calculated.	I.	II.
Carbon	52	312	49·24	48·92	48·34
Hydrogen . .	29	29	4·57	4·76	4·75
Oxygen . . .	27	216	34·11	34·60	35·24
Baryta	1	76·6	12·08	11·72	11·67
		633·6	100·00	100·00	100·00

A lead compound was prepared by adding to an alcoholic solution of the acid acetic acid and acetate of lead and then a little ammonia, taking care to leave an excess of rubianic acid in solution. The red precipitate was collected on a filter, washed with alcohol and then dried, at first *in vacuo*, and then for several hours in the water-bath. Its analysis gave the following results:—

0·5490 grm. gave 0·6760 carbonic acid and 0·1580 water.

0·6600 grm. gave 0·3760 sulphate of lead.

These numbers correspond in 100 parts to—

Carbon	33·58
Hydrogen	3·19
Oxygen	21·32
Oxide of lead	41·91

The most probable formula for this compound is $2C_{52}H_{28}O_{26}+7PbO$, though the numbers found by experiment agree better with the formula $2C_{52}H_{27}O_{25}+7PbO$, as will be seen from the following calculation:—

	Eqs.		Calculated.	Eqs.		Calculated.
Carbon	104	624	33·22	104	624	33·55
Hydrogen . .	56	56	2·98	54	54	2·90
Oxygen	52	416	22·17	50	400	21·51
Oxide of lead .	7	781·9	41·63	7	781·9	42·04
		1877·9	100·00		1859·9	100·00

Another specimen of the lead salt, prepared by precipitating an alcoholic solution of the acid with an alcoholic solution of basic acetate of lead, was found to have a composition agreeing tolerably well with the formula $C_{52}H_{28}O_{26}+9PbO$. When this precipitate was redissolved in a mixture of alcohol and acetic acid, and a fresh precipitate was produced by means of a little ammonia, the latter was found to contain acid and oxide of lead in the proportion of 1 equiv. of the former to 6 equivs. of the latter. It appears, therefore, that these lead precipitates are by no means uniform in composition.

The silver salt was obtained by dissolving the acid in boiling water, adding a small quantity of ammonia and then nitrate of silver, when it fell in the form of a dark chocolate-coloured flocculent precipitate, which, when viewed under a lens, was sometimes seen to consist of small crystalline needles. The precipitate was collected on a filter and slightly washed with water. As it was found to dissolve in water with a reddish-brown colour, as soon as the excess of nitrate of silver was removed, the edulcoration was completed with alcohol, in which the salt is insoluble. After being dried *in vacuo*—

0·6300 grm. gave 0·1370 grm. chloride of silver, equivalent to 0·11076 or 17·58 per cent. of oxide of silver. The formula $C_{52}H_{28}O_{26}+AgO$ requires 17·47 per cent.

The alizarine and sugar obtained by the decomposition of rubianic acid with acids and alkalies, I found to have the usual composition, as the following analyses will show:—

I. 0·2190 grm. alizarine, obtained by the action of sulphuric acid on rubianic acid, dried at 100° C., yielded 0·5550 carbonic acid and 0·0850 water.

II. 0·3320 grm. alizarine, formed by the decomposition of rubianic acid with caustic soda, dried at 100° C., gave 0·8480 carbonic acid and 0·1195 water.

These numbers give in 100 parts—

	I.	II.
Carbon	69·11	69·66
Hydrogen	4·31	4·00
Oxygen	26·58	26·34

The sugar derived from the decomposition of rubianic acid with sulphuric acid was prepared for analysis by heating it, after the solution had been evaporated to a syrup, for some time at 100° C. until it became sufficiently brittle to be pulverized, when

0·3600 grm. yielded 0·5840 carbonic acid and 0·2060 water, corresponding with the following composition:—

	Eqs.		Calculated.	Found.
Carbon	12	72	44·44	44·24
Hydrogen	10	10	6·17	6·35
Oxygen	10	80	49·39	49·41
		162	100·00	100·00

This specimen of sugar had therefore the same composition as that obtained by the action of erythrozym on rubian*.

The analysis of the acid, its compounds and products of decomposition just detailed, lead to the conclusion that its composition must be expressed by the formula $C_{52}H_{29}O_{27}$, and that it belongs to the class of the so-called glucosides, the copula contained in it being alizarine. Its decomposition, by means of strong acids and alkalies into alizarine and sugar, is symbolized by the following equation:—

$$C_{52}H_{29}O_{27}+5HO=2C_{14}H_5O_4+2C_{12}H_{12}O_{12}.$$

In order to remove all doubt concerning the true formula of this substance, I deter-

* Philosophical Transactions for 1853, p. 79.

mined the quantities of alizarine which were obtained by decomposition of weighed quantities of the acid with sulphuric acid and with caustic soda.

1·7210 grm. rubianic acid, dried at 100° C., gave, when decomposed with sulphuric acid, 0·7310 grm. dry alizarine=42·47 per cent.

1·2020 grm. rubianic acid yielded, when decomposed with pure caustic soda, on precipitation with sulphuric acid, 0·5430 grm. alizarine=45·17 per cent.

Assuming the formula $C_{52} H_{29} O_{27}$ to be correct, 100 parts of acid should, according to theory, afford 43·44 of dry alizarine. If STRECKER's formula for alizarine, $C_{20} H_6 O_6$, were the correct one, then the only possible formula for rubianic acid, though the calculated composition would then not agree very well with that found in my experiments, would be $C_{52} H_{27} O_{27}$, in which case the quantity of alizarine derived from 100 parts of acid would be 62·70. It will be seen, therefore, that the view which I have adopted of the constitution of the acid derives considerable support from these determinations.

A comparison of the composition of rubian and rubianic acid shows that the latter can only be derived from the former by means of oxidation. In order however to remove all doubt on this point, I made the following experiment. A solution of rubian was divided into two equal parts. One half was boiled to expel all the air it might contain, then mixed with a certain quantity of caustic soda, and immediately put into a bottle, which the liquid just sufficed to fill, and which was then closed air-tight. The other half was mixed with the same quantity of caustic soda and left exposed to the air. After twenty-one days, both liquids were examined in the same manner. The soda was supersaturated with acetic acid, the liquid was again rendered alkaline with ammonia, and then chloride of barium was added, which gave in each case a red precipitate. This precipitate was treated as usual with sulphuric acid and carbonate of lead, and the filtered liquid was evaporated to dryness. From that part of the solution which had been enclosed in the bottle I obtained in this manner a quantity of a substance resembling rubian, which was probably rubidehydran, but not a trace of rubianic acid; whereas, by adding baryta water to the liquid filtered from the precipitate with chloride of barium, leaving the mixture to stand exposed to the air for some time, filtering and treating the substance left on the filter with sulphuric acid as usual, I obtained a small quantity of the acid. On the other hand, the precipitate with chloride of barium from the second half of the solution, which had been left in an open vessel, yielded at once 0·60 grm. of rubianic acid as well as a quantity of rubidehydran, but the liquid filtered from this precipitate, on being mixed with baryta water and treated as before, gave no more acid. Hence it follows that the presence of oxygen is as essential as that of alkalies to the formation of this acid.

As regards the manner in which the formation of rubianic acid takes place, we may suppose it to be effected in two ways. Assuming 1 equivalent of rubian to absorb 10 equivalents of oxygen, it may then yield 1 equivalent of rubianic acid, 4 equivalents of carbonic acid, and 5 of water, as the following equation will show:—

$$C_{56}H_{34}O_{30}+10O=C_{52}H_{29}O_{27}+4CO_2+5HO.$$

But since acetic acid is found among the products of decomposition, it is possible that this acid takes the place of carbonic acid, and the equation will then appear as follows:—

$$C_{56}H_{34}O_{30}+2O=C_{52}H_{29}O_{27}+C_4H_3O_3+2HO.$$

It is a very remarkable circumstance, that a body like rubianic acid, which belongs to a class of a highly complex nature, and having in general a high atomic weight, should owe its formation to a process of oxidation. It is probably the first known instance in which the formation of a body of this class by means of oxidation has actually been observed.

Some years since Rochleder described a body obtained from madder to which he gave the name of *ruberythric acid**. This body, like rubianic acid, is decomposed by strong acids into alizarine and sugar, and the other properties mentioned by Rochleder are similar to those of rubianic acid. It is therefore very probable that the two acids are identical. But the description given by Rochleder of his acid is not sufficiently minute to enable me to come to a decision as to their identity, and the composition ascribed to it by him (C 54·48, H 5·16, O 40·36 per cent.) differs so much from that of rubianic acid as almost to lead one to believe that they are not the same. Until therefore the properties and composition of ruberythric acid have been more accurately studied, it will be impossible to arrive at any positive conclusion on this point, and for the present the two acids must be considered as distinct†. Rochleder has ventured to express his conviction that rubian is nothing but impure ruberythric acid‡. The experiments which I have here described must lead to the conclusion that such an opinion cannot for an instant be entertained. If rubianic acid and ruberythric acid are identical, then Rochleder has only committed the common error of mistaking a product for an educt.

Rubidehydran.—As obtained by the method above described, this body is not perfectly pure. It may be purified by redissolving it in water, evaporating the solution to a syrup, and then adding a quantity of alcohol, which precipitates a reddish-yellow glutinous substance mixed with sulphates of lime, magnesia, and soda. The alcoholic liquid having been allowed to clear, is decanted and evaporated to dryness, when it leaves a dark yellow or reddish-yellow, brittle, transparent residue like gum or varnish, which cannot be distinguished from rubian in appearance. It still gives, when burnt, a quantity of ash, consisting of sulphates of lime and magnesia, but I did not attempt

* Berichte der Wiener Academie, April 1851.

† The easiest way of arriving at a conclusion would be by determining the amount of alizarine which ruberythric acid by its decomposition is capable of yielding. If Rochleder's formula $C_{72}H_{40}O_{40}$ be accepted, it should give 67·04 per cent. of alizarine. Variations of several per cent. in the amount of alizarine obtained would still allow a positive decision to be arrived at.

‡ Ann. der Chem. und Pharm. B. lxxxii. S. 215.

to purify it any further, for fear of inducing a change in its composition. Rubidehydran resembles rubian exactly in most of its properties. It is not in the least deliquescent. If it shows any tendency to deliquesce on exposure to the air, this must be attributed to its containing some impurity. It has a strongly bitter taste. The watery solution is yellow, and when boiled, with the addition of sulphuric or muriatic acid, deposits yellow flocks, and after cooling appears almost colourless. Like rubian, is also decomposed on boiling the watery solution with the addition of caustic potash or soda, the solution becoming purple and depositing yellow flocks on supersaturating the alkali with acid. It is not precipitated from its aqueous solution by any earthy or metallic salt, with the exception of basic acetate of lead, which produces a red precipitate, while the liquid becomes colourless. The alcoholic solution, however, gives, with an alcoholic solution of neutral acetate of lead, a red precipitate, similar to the last, which is soluble in water. When exposed in a dry state to the action of heat, and to that of chlorine in its watery solution, rubidehydran behaves exactly like rubian. It may, however, be distinguished from rubian by its not yielding a trace of rubianic acid, when its watery solution is mixed with caustic soda or baryta and left exposed to the atmosphere for some time. Its products of decomposition with acids differ also from those of rubian. If a solution of the substance in water be boiled with the addition of sulphuric acid, yellow flocks are deposited, which consist principally of alizarine, with some rubiadine and a little verantine and rubiretine, but not a trace of rubianine, while the filtered liquid contains sugar. Rubidehydran yields therefore the same products with acids as rubian does with alkalies, which makes it very probable that rubian, when acted on by alkalies, is first changed, in part at least, into rubidehydran.

In calculating the composition of this substance from the analysis, I took it for granted that the lime and magnesia found in the ash are contained originally in it as sulphates, and therefore always treated the ash obtained in the first instance with sulphuric acid, in order to replace that portion of the sulphuric acid which might have been reduced during the process of ignition, and then made the corrections accordingly.

I. 0·4360 grm. rubidehydran, formed by the action of bicarbonate of baryta on rubian, dried at 100° C. and burnt with chromate of lead, gave 0·8005 carbonic acid and 0·2130 water.

0·2790 grm., on being incinerated, left 0·0150 grm. ash, which after being treated with sulphuric acid and heated again, weighed 0·0325 grm. =11·64 per cent.

II. 0·4950 grm., formed by the action of lime-water on rubian, gave 0·9750 carbonic acid and 0·2320 water.

0·6220 grm. left 0·0200 grm. ash, which after treatment with sulphuric acid weighed 0·0340 grm. =5·46 per cent.

III. 0·4795 grm., obtained by means of caustic baryta, gave 0·9540 carbonic acid and 0·2320 water.

1·0785 grm. left 0·0245 grm. ash, which after treatment with sulphuric acid, weighed 0·0455 grm. =4·21 per cent.

IV. 0·4990 grm., obtained by means of ammonia, gave 0·9780 carbonic acid and 0·2320 water.

0·7385 grm. left 0·0155 grm. ash, which after treatment with sulphuric acid, weighed 0·0395 grm. =5·34 per cent.

After making the due corrections for the ash in the way just described, these numbers correspond in 100 parts to—

	I.	II.	III.	IV.
Carbon	56·66	56·81	56·06	56·46
Hydrogen	6·13	5·50	5·54	5·45
Oxygen	37·21	37·69	38·40	38·09

leading to the formula $C_{56} H_{32} O_{28}$, as the following calculation shows:—

	Eqs.		Calculated.
Carbon	56	336	56·75
Hydrogen . . .	32	32	5·40
Oxygen	28	224	37·85
		592	100·00

The lead compound was prepared by dissolving the substance in a small quantity of water, adding an alcoholic solution of acetate of lead, separating the red precipitate which was formed, by filtration, adding a little ammonia to the filtered liquid, taking care to leave an excess of rubidehydran in solution, filtering the precipitate and washing it with alcohol.

0·5705 grm. of this precipitate, dried in the water-bath, gave 0·6110 carbonic acid and 0·1465 water.

0·3180 grm., treated with sulphuric acid and heated, left 0·2200 grm. residue, which on being treated with hot water, yielded to the latter 0·0140 grm. sulphate of magnesia, leaving 0·2060 sulphate of lead.

These numbers lead to the following composition:—

	Eqs.		Calculated.	Found.
Carbon	56	336	29·20	29·20
Hydrogen . . .	32	32	2·78	2·85
Oxygen	28	224	19·48	,,
Oxide of lead . .	5	558·5	48·54	47·66
		1150·5	100·00	

The deficiency in the oxide of lead in this analysis arose without doubt from the oxide being in part replaced by magnesia.

It appears therefore that rubidehydran differs from rubian merely by containing the elements of two equivalents of water less, and the origin of the different products of decomposition to which it gives rise may therefore be explained in the same manner as in the case of rubian.

Of its products of decomposition with acids I only obtained alizarine in quantities and of the degree of purity requisite for analysis. It had the usual composition of that substance, as the following analysis will show:—

0·2500 grm. of the crystallized substance lost, on being heated in the water-bath, 0·0460 water =18·40 per cent.

0·2800 grm. of the dry substance gave 0·7110 carbonic acid and 0·1030 water, corresponding in 100 parts to—

Carbon	69·25
Hydrogen	4·08
Oxygen	26·67

Rubihydran.—This substance, like rubidehydran, bears a great resemblance to rubian, the body from which it is derived. It is obtained on evaporation of its solutions as an uncrystalline, transparent mass like gum, having a dark brownish-yellow colour and a bitter taste. When quite dry it is brittle and may be easily pulverized, but on exposure to the atmosphere at the ordinary temperature it rapidly attracts moisture and becomes soft, a property by which it may be distinguished from rubian and rubidehydran. On being heated in a tube it gives less crystalline sublimate than rubian does. Its watery solution, on being boiled with the addition of sulphuric or muriatic acid, becomes muddy, and slowly deposits a quantity of yellow flocks mixed with some brown resinous drops. The liquid must be boiled for a considerable time in order to effect the entire decomposition of the rubihydran contained in it and make it appear colourless. The flocks on being collected on a filter are found to consist chiefly of rubiretine, verantine, and rubiadine, with only a very small quantity of alizarine, while the filtered liquid contains sugar. The products of decomposition with acids are therefore the same as those of rubidehydran, the only difference being in the relative proportions of the products formed. It is not decomposed when treated with boiling phosphoric, oxalic, tartaric or acetic acids. If a watery solution of rubihydran be boiled with caustic potash or soda, the colour of the solution, which was red on the first addition of alkali, changes to reddish- or yellowish-brown, but only a few purple flocks are deposited. This circumstance also serves to distinguish this substance from rubian and rubidehydran, the watery solutions of which, when mixed with caustic alkali and boiled, deposit an abundance of the purple compound of alizarine and alkali. Nevertheless the rubihydran is completely decomposed by the caustic alkali, for on adding an excess of acid to the liquid a quantity of yellow or light brown flocks are precipitated like those produced by the action of acids, while the liquid becomes almost colourless. When chlorine gas is passed through the watery solution, it produces exactly the same effects as in watery solutions of rubian or rubidehydran. It is the behaviour to chlorine, which more than any other reaction proves that these substances have a similar composition, and forms perhaps the most characteristic mark of distinction between them and rubianic acid. Rubihydran is not precipitated from its aqueous solution by any reagent except basic acetate of lead, which produces a light brownish-red precipitate, which is paler and less bright than the precipitates produced by the same reagent in solutions of rubian or rubidehydran. The liquid filtered from this precipitate has still a light yellow colour, and ammonia produces in it a fresh precipitate of a pale pink colour, after which it appears colourless. Rubihydran is soluble in alcohol, but not very easily. From the manner in which it is prepared it may be inferred, that it is incapable of yielding rubianic acid by the influence of the same agents which lead to the formation of the latter from rubian. In order to obtain this substance in a perfectly dry state, it is necessary to heat it in the water-bath continuously for a day or two. In two analyses I obtained numbers corresponding respectively with the formulæ $C_{56} H_{45} O_{41}$ and $C_{56} H_{42} O_{38}$. I only succeeded once in obtaining it of the same composition, or as free from water as when in combination with oxide of lead. On this occasion it gave the following results:—

0·3940 grm. gave 0·7440 carbonic acid and 0·2130 water, numbers which lead to the following composition:—

	Eqs.		Calculated.	Found.
Carbon	56	336	51·29	51·50
Hydrogen . . .	39	39	5·95	6·00
Oxygen	35	280	42·76	42·50
		655	100·00	100·00

On being burnt this specimen left no ash.

The lead compound was prepared by dissolving the substance in alcohol and then adding acetate of lead and ammonia, taking care to leave an excess of rubihydran in solution, filtering, washing the precipitate with alcohol and drying, at first *in vacuo*, and then for several hours in the water-bath.

I. 0·8740 grm. of this compound gave 0·6720 carbonic acid and 0·1970 water.
0·7270 grm. gave 0·5850 sulphate of lead.

II. 0·8860 grm. gave 0·6820 carbonic acid and 0·1930 water.
0·8465 grm. gave 0·6805 sulphate of lead.

In 100 parts:—

	I.	II.
Carbon	20·96	20·99
Hydrogen	2·50	2·42
Oxygen	17·34	17·44
Oxide of lead	59·20	59·15
	100·00	100·00

The amount of oxide of lead here stands in no simple relation to that of the other constituents. If the oxide of lead be deducted, then the rubihydran combined with it will have the following composition in 100 parts :—

	I.	II.
Carbon	51·37	51·38
Hydrogen	6·12	5·92
Oxygen	42·51	42·70
	100·00	100·00

It will be seen that this is exactly the composition of the substance itself according to the analysis given above, and it follows that the composition of rubihydran, both in a perfectly dry state and when in combination with oxide of lead, is expressed by the formula $C_{56}H_{39}O_{35}$. It differs therefore from rubian by containing the elements of 5 equivalents more of water. That it should yield the same products of decomposition as rubian and rubidehydran is therefore not at all extraordinary.

The rubiadine which is formed by the decomposition of rubihydran with acids may be obtained in a state of greater purity and with much greater facility from this, than from any other source. It is separated from the other products of decomposition in the manner I have described when treating of the action of alkalies on rubian. It is purified by dissolving it in boiling alcohol and adding to the boiling solution hydrated oxide of lead, when it crystallizes from the filtered solution in beautiful golden-yellow, glittering scales, some of which assume the form of regular four-sided tables, possessing all the properties of rubiadine*, but evidently freer from impurities than when obtained by the action of alkalies on rubian.

I. 0·3060 grm. of this substance, dried at 100° C., gave 0·7795 carbonic acid and 0·1380 water.

II. 0·2580 grm. of the same, recrystallized from boiling alcohol, gave 0·6600 carbonic acid and 0·1190 water.

From these numbers it may be inferred that its composition is as follows :—

			Calculated.	I.	II.
Carbon	32	192	69·31	69·47	69·76
Hydrogen	13	13	4·69	5·01	5·12
Oxygen	9	72	26·00	25·52	25·12
		277	100·00	100·00	100·00

The formula $C_{32}H_{13}O_9$, to which these analyses lead, differs from the one formerly given, $C_{32}H_{12}O_8$, by 1 equivalent of water. I prefer the former, since the substance employed in the last analyses was evidently purer than that used in any previous one. It will be seen also, that the formula just given is confirmed by an examination of the products of decomposition of rubian with chlorine.

* In examining these crystals I discovered a property of rubiadine which I had not previously observed. If the ammoniacal solution, which is red, be mixed with chloride of barium, it deposits a quantity of dark brownish-red needle-shaped crystals, which are evidently the baryta compound of rubiadine, while the liquid loses almost all its colour. The rubiadine derived directly from rubian also yields these crystals.

Action of Chlorine on Rubian.—If a current of chlorine gas be passed through a watery solution of rubian, the latter begins immediately to deposit flocks of a lemon-yellow or orange colour. These flocks continue to be formed as long as the solution retains any portion of its yellow colour. When the action is completed the liquid appears colourless. The flocks, the quantity of which is considerable compared with that of the rubian employed, consist almost entirely of one substance, which I shall call *Chlororubian*, though this name is not perfectly appropriate, since it is not formed from rubian simply by the substitution of hydrogen by chlorine. If these flocks, after being collected on a filter and washed until all the acid and chlorine are removed, be treated with a little cold alcohol, the latter dissolves a small quantity of a substance, which after the evaporation of the alcohol is left as a yellow or yellowish-brown resinous residue. This substance melts at the temperature of boiling water; it contains chlorine, and dissolves in caustic alkalies with a dirty purple colour. The chlororubian may be purified by simply dissolving it in boiling alcohol. It crystallizes on the solution cooling in small orange-coloured needles, which increase very much in quantity after standing for several hours. The acid liquid, filtered from the yellow flocks formed by the action of chlorine, contains sugar, which may be obtained by neutralizing the acid with carbonate of lead, filtering, evaporating the liquid to a small volume, decolorizing with animal charcoal, filtering, evaporating to dryness, and treating the residue with alcohol. The alcohol after filtration and evaporation leaves a yellow syrup having all the properties of sugar, as usually obtained by the decomposition of rubian. Chlororubian may be prepared as well from impure as from pure rubian. It is only necessary to extract madder with boiling water, add sugar of lead to the extract, add ammonia to the liquid filtered from the precipitate, decompose the red precipitate produced by ammonia with sulphuric acid, and pass chlorine gas through the filtered liquid. The first portions of chlorine generally produce a dirty yellow flocculent precipitate, which, being separated by filtration, is found to consist of the resinous easily fusible substance just mentioned. On passing chlorine through the filtered liquid, pure yellow flocks of chlororubian are precipitated, which are purified as before by crystallization from boiling alcohol.

Chlororubian has the following properties. After crystallization from alcohol and drying, it forms a mass of a light orange colour, consisting of small crystalline needles. It has no perceptible taste at first, but on chewing it for some time it produces a slightly bitter taste. When heated on platinum it melts and burns with a smoky flame slightly tinged with green, and leaves a considerable quantity of charcoal. On being heated in a tube it melts to a brown liquid, and gives fumes which condense on the colder parts of the tube to a white crystalline sublimate, consisting of star-shaped masses, while much carbonaceous residue is left. On being treated with boiling water chlororubian dissolves in considerable quantity, forming a yellow solution, which on cooling deposits a great part of the substance, not in crystals, but in amorphous masses consisting of spherical grains. The boiling alcoholic solution, if very concentrated, also deposits part of the substance on cooling in amorphous, spherical, translucent grains, which have the appearance of drops of oil, but by re-dissolving these in fresh alcohol, crystals of the usual appearance are obtained. The alcoholic solution does not redden litmus paper in the least. The watery solution gives no precipitate with nitrate of silver, but if chlororubian be treated with boiling nitric acid it is decomposed with an evolution of nitrous acid, forming a colourless solution, in which nitrate of silver produces a precipitate of chloride of silver. The action of sulphuric and muriatic acids, caustic alkalies and chlorine on chlororubian, I shall treat of presently. Chlororubian dissolves in boiling solutions of the carbonates of potash, soda and ammonia, forming blood-red solutions, which deposit nothing on cooling. Baryta water imparts to the watery solution a deep red colour, and on boiling dark red flocks are deposited, while the liquid becomes almost colourless. Lime water turns the watery solution red without producing any precipitate, but the ammoniacal solution gives with chloride of calcium a light red flocculent precipitate, while the supernatant liquid becomes colourless. The watery solution gives no precipitate with the acetates of alumina and peroxide of iron. On being treated with a boiling solution of perchloride of iron, chlororubian dissolves with a brownish-yellow colour, which after some time becomes dark brown, while a black powder is deposited. The alcoholic solution of chlororubian does not change on being mixed with an alcoholic solution of acetate of lead, but the watery solution gives with basic acetate of lead a light red precipitate, the liquid retaining a reddish colour. The alcoholic solution gives no precipitate with acetate of copper. An alkaline solution of chlororubian reduces chloride of gold to the metallic state, even in the cold. Chlororubian produces no effect on mordants, on trying to dye with it in the usual manner.

On submitting chlororubian to analysis the following results were obtained:—

I. 0·4100 grm., dried in the water-bath and burnt with chromate of lead, gave 0·7690 carbonic acid and 0·1860 water.

0·4760 grm., burnt with lime, gave 0·1260 chloride of silver.

II. 0·4780 grm. of the same, recrystallized from boiling alcohol, gave 0·8950 carbonic acid and 0·2100 water.

III. 0·5470 grm. of another preparation gave 1·0270 carbonic acid and 0·2400 water.

0·7720 grm., burnt with lime, gave 0·2015 chloride of silver.

IV. 0·4425 grm. of the same preparation as the last gave 0·8320 carbonic acid and 0·1960 water.

0·5450 grm., on decomposition with fuming nitric acid to which a little nitrate of silver was added, gave 0·1380 chloride of silver.

V. 0·4975 grm. of a new preparation gave 0·9330 carbonic acid and 0·2240 water.

0·8580 grm., burnt with lime, gave 0·2180 chloride of silver.

VI. 0·5320 grm. gave 1·0000 carbonic acid and 0·2380 water.

0·9370 grm., burnt with lime, gave 0·2450 chloride of silver.

These numbers correspond in 100 parts to—

	I.	II.	III.	IV.	V.	VI.
Carbon . . .	51·15	51·06	51·20	51·27	51·14	51·26
Hydrogen . .	5·04	4·88	4·87	4·92	5·00	4·97
Chlorine . .	6·54	„	6·45	6·26	6·28	6·46
Oxygen . . .	37·27	„	37·48	37·55	37·58	37·31

There are several formulæ which give a composition in 100 parts agreeing tolerably well with these numbers, but only one which at the same time explains the manner in which chlororubian is formed. This formula is $C_{44}H_{27}ClO_{24}$, which gives the following composition:—

	Eqs.		
Carbon . . .	44	264	50·92
Hydrogen . .	27	27	5·20
Chlorine . . .	1	35·4	6·82
Oxygen . . .	24	192	37·06
		518·4	100·00

Assuming this formula to be correct, then rubian, when acted on by chlorine, loses 1 equivalent of hydrogen, which is replaced by chlorine, at the same time taking up 6 equivalents of water and splitting up into chlororubian and sugar, as will be seen from the following equation:—

$$C_{56}H_{34}O_{30}+6HO+2Cl=C_{44}H_{27}ClO_{24}+C_{12}H_{12}O_{12}+ClH.$$

Action of Acids on Chlororubian.—If chlororubian be treated with boiling dilute sulphuric or muriatic acid, it first dissolves to a yellow liquid, but on continuing to boil, the solution suddenly becomes milky and deposits a large quantity of yellow crystalline flocks. The filtered liquid is almost colourless, and contains sugar. The flocks consist entirely of a body, to which, as it has the composition of rubiadine in which 1 equivalent of hydrogen is substituted by chlorine, I shall give the name of *Chlororubiadine.* It is purified by collecting the flocks on a filter, washing them with water, and dissolving them in boiling alcohol, which on cooling and standing, deposits yellow shining crystals, which are larger than those of chlororubian.

Chlororubiadine has the following properties. When crystallized from alcohol and dried, it has the appearance of a yellow mass, consisting of small shining crystalline needles and scales. When heated on platinum it melts to a brown liquid, and then burns with a yellow flame bordered with green, leaving much carbonaceous residue. When heated in a tube it melts and gives penetrating fumes, smelling of muriatic acid, and forming on the colder parts of the tube a sublimate which is at first oily but soon becomes crystalline. It is insoluble in boiling water. Dilute nitric acid does not affect it, even on boiling. Nitric acid of sp. gr. 1·52, however, dissolves it, even in the cold, forming a dark orange-coloured solution. If nitrate of silver be added to this solution, no precipitate is produced; but if the solution be boiled, an evolution of nitrous acid takes place, the solution becomes turbid, and gives a copious deposit of chloride of silver. Concentrated sulphuric acid dissolves it in the cold, forming an orange-coloured solution, from which it is precipitated again by water in bright yellow flocks. If the solution in the acid be boiled its colour changes to a deep purple, without much sulphurous acid being evolved, a small quantity of a white crystalline sublimate making its appearance after some time on the sides of the vessel near the surface of the liquid. Chlororubiadine is easily dissolved by caustic soda with a purplish-red colour, and by ammonia and the carbonates of potash, soda and ammonia, with a blood-red colour. The ammoniacal solution loses its ammonia on evaporation, and leaves the substance behind as a bright yellow residue. On adding chloride of barium to the ammoniacal solution, the baryta compound crystallizes out on standing in long needles, arranged in large fan-shaped or star-shaped masses of a beautiful red colour, while the liquid becomes almost colourless. The baryta compound, when treated with boiling water, only dissolves in part, some chlororubiadine being left undissolved. If a current of carbonic acid be passed through the filtered solution, the whole of the chlororubiadine is precipitated in yellow flocks, the liquid becoming colourless. The ammoniacal solution gives with chloride of calcium, after some time, a dark red uncrystalline deposit, and also loses its colour. A boiling solution of perchloride of iron does not dissolve chlororubiadine, nor does the colour of the solution change during boiling. The alcoholic solution of chlororubiadine reddens litmus paper. The solution gives no precipitate with acetate of lead, not even on the addition of ammonia, but with acetate of copper it gives after some time a copious light brown precipitate. Acetate of alumina and peracetate of iron produce no change in the alcoholic solution. An alkaline solution of chlororubiadine reduces chloride of gold to the metallic state.

The composition of chlororubiadine was determined by the following analyses:—

I. 0·5650 grm., dried at 100° C. and burnt with chromate of lead, gave 1·2630 carbonic acid and 0·2130 water.

0·4850 grm., burnt with lime, gave 0·2200 chloride of silver.

II. 0·4620 grm. of a second preparation gave 1·0260 carbonic acid and 0·1760 water.

III. 0·5475 grm., made from the last by dissolving it in ammonia, adding chloride of barium, filtering the liquid from the dark flocks which were precipitated, allowing the baryta compound to crystallize, decomposing it with muriatic acid, and crystallizing what was left by the acid from boiling alcohol, gave 1·2280 carbonic acid and 0·2100 water.

0·6260 grm. gave 0·2775 chloride of silver.

IV. 0·4980 grm. of another preparation, obtained like the last from the baryta compound, gave 1·1130 carbonic acid and 0·1905 water.

0·7680 grm. gave 0·3455 chloride of silver.

Hence the following composition may be deduced:—

	Eqs.		Calculated.	I.	II.	III.	IV.
Carbon .	32	192	61·65	60·96	60·56	61·17	60·95
Hydrogen.	12	12	3·85	4·18	4·23	4·26	4·25
Chlorine . .	1	35·4	11·36	11·21	„	10·95	11·10
Oxygen .	9	72	23·14	23·65	„	23·62	23·70
		311·4	100·00	100·00		100·00	100·00

The baryta compound I found to have no very simple composition. It was prepared, as above described, by dissolving crystallized chlororubiadine in ammonia, adding chloride of barium, filtering from a few flocks that were precipitated, and allowing to crystallize in an air-tight flask, filtering, washing with water, and drying *in vacuo*.

0·8370 grm. of the crystals lost, on being heated for some hours in the water-bath, 0·0690 water=8·24 per cent.

0·5770 grm. of the substance thus dried gave, when burnt with chromate of lead, 1·0965 carbonic acid and 0·1790 water.

0·5660 grm. gave 0·1350 sulphate of baryta.

These numbers lead to the formula $4C_{32}H_{12}ClO_9+3BaO$, as the following comparison between the theoretical composition and the experimental results will show:—

	Eqs.		Calculated.	Found.
Carbon	128	768	52·05	51·82
Hydrogen . . .	48	48	3·25	3·44
Chlorine	4	141·6	9·59	„
Oxygen	36	288	19·54	„
Baryta	3	229·8	15·57	15·65
		1475·4	100·00	

The sugar which is formed from chlororubian together with chlororubiadine may be obtained in a crystallized state, which is not the case with the different specimens of sugar derived from the other processes of decomposition to which I have subjected rubian. If sulphuric acid be employed for the decomposition of chlororubian, and the acid after filtration of the flocks of chlororubiadine be neutralized with carbonate of lead, the filtered liquid yields on evaporation a sweet syrup. If this syrup be treated with alcohol, a part of it dissolves with a yellow colour. If the alcoholic solution, after separation from the insoluble part, be mixed with several times its volume of ether, it becomes milky and deposits again a yellow syrup, which after standing some time becomes filled with small yellowish crystals, so as almost to form a solid mass. This mass is pressed between blotting-paper, in order to remove the mother-liquor, and the crystals are dissolved in boiling alcohol, to which a little animal charcoal is added. The filtered solution on evaporation gives a syrup, which is soon converted into a mass of white crystals. These crystals have the properties and composition of crystallized grape sugar.

0·5015 grm. of the crystals, dried *in vacuo*, gave 0·6830 carbonic acid and 0·3380 water.

These numbers lead to the following composition:—

	Eqs.		Calculated.	Found.
Carbon . . .	12	72	36·36	37·14
Hydrogen . .	14	14	7·07	7·48
Oxygen . . .	14	112	56·57	55·38
		198	100·00	100·00

It will now admit, I think, of little doubt, that the uncrystallizable sugar obtained from rubian in other processes of decomposition is merely modified grape sugar.

The formation of chlororubiadine and sugar from chlororubian is a very simple process. The latter loses three equivalents of water and splits up into chlororubiadine and sugar, as will be evident from the following equation:—

$$C_{44}H_{27}ClO_{24}=C_{32}H_{12}ClO_{9}+C_{12}H_{12}O_{12}+3HO.$$

I have adopted the name of chlororubiadine, under the assumption that the true formula for rubiadine is $C_{32}H_{13}O_{9}$. Nevertheless I have not succeeded in converting the latter into chlororubiadine by means of chlorine, nor in substituting the chlorine in chlororubiadine by hydrogen and thus forming rubiadine. If chlororubiadine be suspended in water to which an amalgam of potassium (1 part of potassium to 100 mercury) is added, it dissolves with a dirty red colour without much hydrogen being evolved. The liquid gives a greenish-yellow flocculent precipitate on the addition of nitric acid. This precipitate contains no chlorine, but it does not contain any rubiadine, since it gives, after being dried and heated between two watch-glasses, none of the crystalline sublimate characteristic of rubiadine. The liquid filtered from this precipitate gives an abundant precipitate with nitrate of silver.

Hydrosulphate of ammonia dissolves chlororubiadine, forming a red solution, which on standing becomes of a fine purple, and after some hours brownish-red. If nitric acid be added to the solution as soon as it has acquired a purple colour, an orange-coloured flocculent precipitate falls. This precipitate is free both from sulphur and chlorine, the chlorine of the chlororubiadine being found in the filtered liquid, but it contains no rubiadine. It is only partly soluble in boiling alcohol, but dissolves easily in boiling nitric acid, the solution depositing on standing some long dark yellow sword-shaped crystals.

If crystallized rubiadine, obtained by the decomposition of rubihydran with acid, be dissolved in caustic alkali and reprecipitated with acid, and if the precipitated flocks after filtering and washing be suspended in water, and a current of chlorine gas be passed through the liquid, the flocks become somewhat paler in colour. If they be now collected on a filter and washed with water and then treated with cold alcohol, the greatest part dissolves, leaving undissolved a small quantity of a white powder, which has all the properties of a body which I shall describe presently, and which is formed by the action of chlorine on chlororubian. If the alcoholic solution be evaporated to dryness and a little cold alcohol be again added to the residue, the alcohol again dissolves a great part, leaving undissolved a yellowish-green granular powder, which resembles but is not identical with chlororubiadine. The alcoholic solution leaves on evaporation a brown, transparent, resinous substance. This substance contains a large quantity of chlorine. It dissolves in caustic soda with a brown colour, and is reprecipitated by acids in yellow flocks, which melt in the boiling liquid to brown oily drops. I found it to contain the following quantities of carbon and hydrogen:—

0·3090 grm., dried at 100° C. and burnt with chromate of lead, gave 0·6090 carbonic acid and 0·1060 water, corresponding in 100 parts to—

Carbon	53·75
Hydrogen	3·81

The formula $C_{32}H_{13}Cl_2O_{10}=C_{32}H_{12}Cl_2O_9+HO$ requires in 100 parts—

Carbon	53·96
Hydrogen	3·65
Chlorine	19·89
Oxygen	22·50

I may mention by the way, that the brown resinous substance, which, as I stated above, is formed in small quantities along with chlororubian by the action of chlorine on rubian, is very similar both in properties and composition to this substance. It was prepared simply by passing chlorine through a solution of rubian, collecting the yellow flocks which were formed in the first instance separately, washing them with water, and treating with cold alcohol. The filtered solution was evaporated to dryness, when it left a resinous substance resembling rubiretine.

0·4130 grm. of this substance, dried in the water-bath, gave 0·8200 carbonic acid and 0·1500 water.

0·3010 grm., burnt with lime, gave 0·2160 chloride of silver.

These numbers correspond in 100 parts to—

Carbon	54·14
Hydrogen	4·03
Chlorine	17·74
Oxygen	24·09

By the action of chlorine on chlororubiadine, a body very similar to these but differing in composition is formed. If finely pounded chlororubiadine be suspended in water and a stream of chlorine gas be passed through the liquid for some time, the powder becomes lighter in colour, but not white. If it now be collected on a filter, washed with water, and treated with boiling alcohol, it dissolves in the latter, forming a yellow solution, which deposits nothing on cooling, but on evaporation leaves a transparent dark yellow amorphous substance like resin, which remains soft for a long time, and only becomes hard and brittle after being heated in the water-bath for some time. This substance, when heated on platinum, burns with a yellow flame, leaving much charcoal. When heated in a tube, it melts and gives acid fumes and a yellow oily sublimate, in which on cooling some white crystals make their appearance. The alcoholic solution gives no precipitate with nitrate of silver, but the substance on being treated with boiling nitric acid is dissolved and decomposed, and nitrate of silver now gives an abundant precipitate. It is soluble in concentrated sulphuric acid with a brown colour, but on boiling the solution no sulphurous acid is evolved. Caustic soda dissolves it easily with a brown colour. On being analysed it yielded the following results:—

0·3910 grm., dried in the water-bath, gave 0 6675 carbonic acid and 0·1100 water.

0·4055 grm. gave 0·4990 chloride of silver.

In 100 parts it contained therefore—

Carbon	46·55
Hydrogen	3·12
Chlorine	30·42
Oxygen	19·91

Since the carbon here is to the chlorine as $32\ C : 3\frac{1}{2}\ Cl$, this substance must either have been a mixture, or it must have lost chlorine during the process of drying.

Action of Caustic Alkalies on Chlororubian.—The action of alkalies on chlororubian differs essentially from that of acids. The chlorine in chlororubian is so loosely combined that the affinity of the alkaline metal is sufficient to remove it, and hence all the organic products of decomposition formed by the alkali are free from chlorine.

If chlororubian be treated with a solution of caustic soda it dissolves easily, forming a red solution. If this solution be heated for some time, it deposits a quantity of dark reddish-brown flocks. When these flocks cease to be formed, the liquid, which is still red, is filtered, the flocks are washed with water until the excess of soda is removed, and they are then treated with boiling muriatic acid, by which their colour is changed to yellowish brown. After being collected on a filter and washed with water, they are then treated with boiling alcohol, in which they are but little soluble, placed on a filter, washed with alcohol and dried. After drying, there is obtained a yellowish-brown powder which has the following properties. It is almost insoluble both in alcohol and in caustic alkalies, though the latter impart to it a dark reddish-brown colour. It is not dissolved by hydrosulphate of ammonia. It contains no chlorine. When heated in a tube it gives a yellow crystalline sublimate, which dissolves easily in caustic alkalies. Its composition was determined by the following analyses:—

I. 0·3980 grm., dried at 100° C. and burnt with chromate of lead, gave 1·0330 carbonic acid and 0·1390 water.

II. 0·3910 grm. gave 1·0150 carbonic acid and 0·1370 water.

III. 0·4000 grm. of a new preparation gave 1·0350 carbonic acid and 0·1435 water.

In 100 parts it contained therefore—

	I.	II.	III.
Carbon	70·78	70·79	70·56
Hydrogen	3·88	3·89	3·98
Oxygen	25·34	25·32	25·46

It is doubtful whether this body contains 44 or 32 equivalents of carbon. The formulæ $C_{44}H_{14}O_{12}$, $C_{32}H_{10}O_8$ and $C_{32}H_{11}O_9$ require in 100 parts respectively the following amounts of the three constituents:—

	$C_{44}H_{14}O_{12}$.	$C_{32}H_{10}O_8$.	$C_{32}H_{11}O_9$.
Carbon	70·58	72·18	69·81
Hydrogen	3·74	3·75	4·00
Oxygen	25·68	24·07	26·19

If the first formula be adopted, then this substance is formed from chlororubian by the latter losing its chlorine, which combines with sodium and is replaced by oxygen, while 13 equivalents of water are eliminated, in accordance with the following equation:—

$$C_{44}H_{27}ClO_{24}+NaO=C_{44}H_{14}O_{12}+NaCl+13HO.$$

If one of the two latter formulæ be adopted, then chlororubian first splits up into chlororubiadine and sugar, and the former then loses its chlorine which is replaced by oxygen, while either five or four equivalents of water separate, as will be seen from the following equations:—

$$C_{44}H_{27}ClO_{24}+NaO=C_{32}H_{10}O_8+C_{12}H_{12}O_{12}+NaCl+5HO,$$

$$C_{44}H_{27}ClO_{24}+NaO=C_{32}H_{11}O_9+C_{12}H_{12}O_{12}+NaCl+4HO.$$

The first formula agrees best with the results of analysis. Nevertheless, it seems improbable that chlororubian, when subjected to the action of so potent an agent as caustic soda, should not, in the first instance at least, be decomposed in the same manner as with strong acids. On the other hand, if this were the case, chlororubiadine should, by the action of caustic alkalies, be converted into the same body as chlororubian, which is not the case. If chlororubiadine be treated with caustic soda in the same way as chlororubian, it first dissolves with a purple colour, but the solution on boiling slowly deposits reddish-brown flocks and loses the greatest part of its purple colour. The flocks, on being treated with boiling muriatic acid, acquire an orange colour, and after being collected on a filter and washed, are found to be almost insoluble in boiling alcohol; but notwithstanding their resemblance to the body formed from chlororubian, they differ from the latter in containing a large quantity of chlorine, only a small quantity of the latter having been abstracted by the alkali.

0·3610 grm. of this substance, dried at 100° C., gave 0·8620 carbonic acid and 0·1060 water.

0·2920 grm. gave 0·1110 chloride of silver.

In 100 parts it contained therefore—

Carbon	65·12
Hydrogen	3·26
Chlorine	9·39
Oxygen	22·23

Should the body formed from chlororubian by caustic alkalies be found to contain 44 equivalents of carbon, the most appropriate name for it would be *Oxyrubian*.

The liquid filtered from this body is still red. On adding to it sulphuric acid, a yellowish-brown flocculent precipitate falls. This precipitate consists of several bodies. If after being filtered and washed it be treated with boiling alcohol, a part dissolves, leaving undissolved a dark brown substance, which after drying becomes black. This substance is doubtless a product of the decomposition of sugar, as it has the same properties and very nearly the same composition as the body, insoluble in alcohol, which I obtained by the decomposition of rubian with caustic soda*.

0·1760 grm. of this substance gave 0·4360 carbonic acid and 0·0650 water, corresponding in 100 parts to—

Carbon	67·56
Hydrogen	4·10
Oxygen	28·34

On adding acetate of lead to the liquid filtered from this precipitate, a brown precipitate falls, which after being filtered off, washed with alcohol and decomposed with boiling muriatic acid, yields brown flocks. These, on being dried and treated with cold alcohol, yield to the latter a body resembling and probably identical with rubiretine, while a brown powder is left undissolved, having the properties and composition of verantine.

0·1755 grm. of the latter, after being purified by redissolving in a boiling mixture of alcohol and ammonia, then adding an excess of acetic acid, collecting the pulverulent deposit formed on cooling, and washing with alcohol, gave 0·4170 carbonic acid and 0·0680 water.

In 100 parts it contained therefore—

Carbon	64·80
Hydrogen	4·30
Oxygen	30·90

The liquid filtered from the lead precipitate is yellow. It gives with water a yellow precipitate, which, after being filtered off and washed with water, dissolves again in boiling alcohol, with the exception of a little brown flocculent matter. The alcoholic solution, on evaporation, leaves a yellow uncrystalline substance resembling impure rubiadine, which contains no chlorine, gives when heated a sublimate like that from

* Philosophical Transactions, 1853, p. 71.

rubiadine, and forms with baryta a compound which crystallizes in dark reddish-brown needles. I obtained so small a quantity of this substance that I was unable to arrive at any positive conclusion as to whether it is identical with rubiadine or not. The residue left on evaporation of the alcoholic solution was analysed without any attempt being made to purify it, when it was found to have a composition nearly approaching that of rubiadine.

0·2980 grm. gave 0·7525 carbonic acid and 0·1450 water, corresponding in 100 parts to—

Carbon	68·86
Hydrogen	5·40
Oxygen	25·74

The formation of this body, as well as that of verantine and rubiretine, from chlororubian, I am unable to explain in a satisfactory manner.

If the excess of sulphuric acid, with which the four last bodies have been precipitated, be neutralized with carbonate of lead, the filtered liquid leaves on evaporation a saline mass, which, on being pulverized and treated with warm alcohol, communicates to the latter a reddish-brown colour. The alcoholic liquid, after being filtered from the insoluble matter consisting of sulphate of soda, leaves on evaporation crystals of chloride of sodium surrounded by a brown syrup of sugar.

Action of Chlorine on Chlororubian.—If finely pounded chlororubian be mixed with water, and if, after the mixture has been placed in a large bottle or other suitable vessel, a current of chlorine gas be passed through it, no change is perceptible for some time; but if the bottle be closed, after the space above the liquid has been filled with chlorine, the latter is gradually absorbed, the colour of the chlororubian becomes paler, and after several days it appears perfectly white. The process is not accelerated by the action of sunlight, but frequent agitation of the liquid and powder with the gas assists it. The white powder into which the chlororubian is changed consists of a body to which I will give the name of *Perchlororubian.* After collecting it on a filter and washing out the chlorine and acid with water, it is to be dissolved in boiling alcohol, from which it crystallizes on the solution cooling in colourless, transparent, flat, four-sided tables, exhibiting a beautiful iridescence. If it should not be quite colourless, it must be redissolved in boiling alcohol, to which a little animal charcoal may be added. On filtering boiling hot and allowing to cool, the solution then yields perfectly colourless crystals. It may also be obtained directly from rubian, by continuing to pass chlorine through a watery solution of the latter, until the yellow precipitate produced at first has become white, but by this means it is not obtained as pure as from crystallized chlororubian.

Perchlororubian has the following properties. When heated on platinum it melts to a brown liquid, and then burns with a smoky yellow flame edged with green, leaving little carbonaceous residue. If slowly and carefully heated it may be entirely volatilized, yielding a sublimate of bright micaceous scales. But if it be suddenly heated, if, for instance, it be thrown into a red-hot tube, it is decomposed with a kind of explosion, giving off an acid smell, and forming a large quantity of soot with little or no crystalline sublimate. It is insoluble in water, but soluble in alcohol and ether. The alcoholic solution does not redden litmus paper. Concentrated sulphuric acid dissolves it on heating, the solution, on being heated to the boiling-point, becoming slightly brown, but giving off very little sulphurous acid. The colder parts of the tube become covered with a crystalline sublimate, consisting probably of the substance itself. Nitric acid of sp. gr. 1·37 has no effect on it, even on boiling. Nitric acid of sp. gr. 1·52 dissolves it on boiling without decomposing it, for on adding water, the substance is precipitated unchanged in the shape of a crystalline deposit, and nitrate of silver produces in the liquid no precipitate of chloride of silver. Perchlororubian is quite insoluble in strong caustic soda lye, even on boiling, as well as in ammonia. It dissolves easily, however, in hydrosulphate of ammonia on boiling, and on now adding nitric acid and boiling, nitrate of silver produces an abundant precipitate. The alcoholic solution gives no precipitate with an alcoholic solution of acetate of lead. Its analysis led to the following results:—

I. 0·4945 grm., dried at 100° C. and burnt with chromate of lead, gave 0·6730 carbonic acid and 0·0610 water.

0·4350 grm., burnt with lime, gave 0·7770 chloride of silver.

II. 0·3930 grm. of another preparation gave 0·5330 carbonic acid and 0·0585 water.

0·2730 grm. gave 0·4930 chloride of silver.

These numbers lead to the following composition:—

	Eqs.		Calculated.	I.	II.
Carbon . . .	44	264	37·09	37·11	36·98
Hydrogen . .	9	9	1·26	1·37	1·65
Chlorine . . .	9	318·6	44·77	44·16	44·64
Oxygen . . .	15	120	16·88	17·36	16·73
		711·6	100·00	100·00	100·00

It appears, therefore, that in passing into perchlororubian, chlororubian loses 9 equivalents of water and 9 of hydrogen, 8 of the latter being substituted by chlorine, since

$$C_{44}H_{27}ClO_{24}+17Cl=C_{44}H_{9}Cl_{9}O_{15}+9HO+9ClH.$$

It is a singular circumstance, that the 9 equivalents of chlorine in this compound are much more firmly combined with the other constituents than the 1 equivalent contained in chlororubian, which the mere action of alkali is sufficient to separate.

From the experiments just described it may be inferred, that chlororubian is a conjugate compound containing sugar. It resembles Piria's chlorosalicine, which, by the action of acids, yields chlorosaligenine and sugar, just as chlororubian gives chlororubiadine and sugar. Though chlororubian is not, strictly speaking, a product of substitution of rubian, still it retains some of the properties of the latter; for instance, that of giving, with alkalies, products of decomposition differing from those formed by acids. In all the processes of decomposition previously described, rubian is decomposed in no less than three different modes, just as if it were a compound or mixture of three different bodies, whereas, when acted on by chlorine, it yields only one series of products. It behaves in the latter case as if it were simply a conjugate compound containing sugar. It splits up into sugar and a chlorinated body, and the latter, by the action of acids, again splits up into sugar and a second chlorinated compound. This series of products corresponds exactly with one of the three series in the other processes of decomposition, the bodies belonging to the two other series not making their appearance even in the form of products of substitution.

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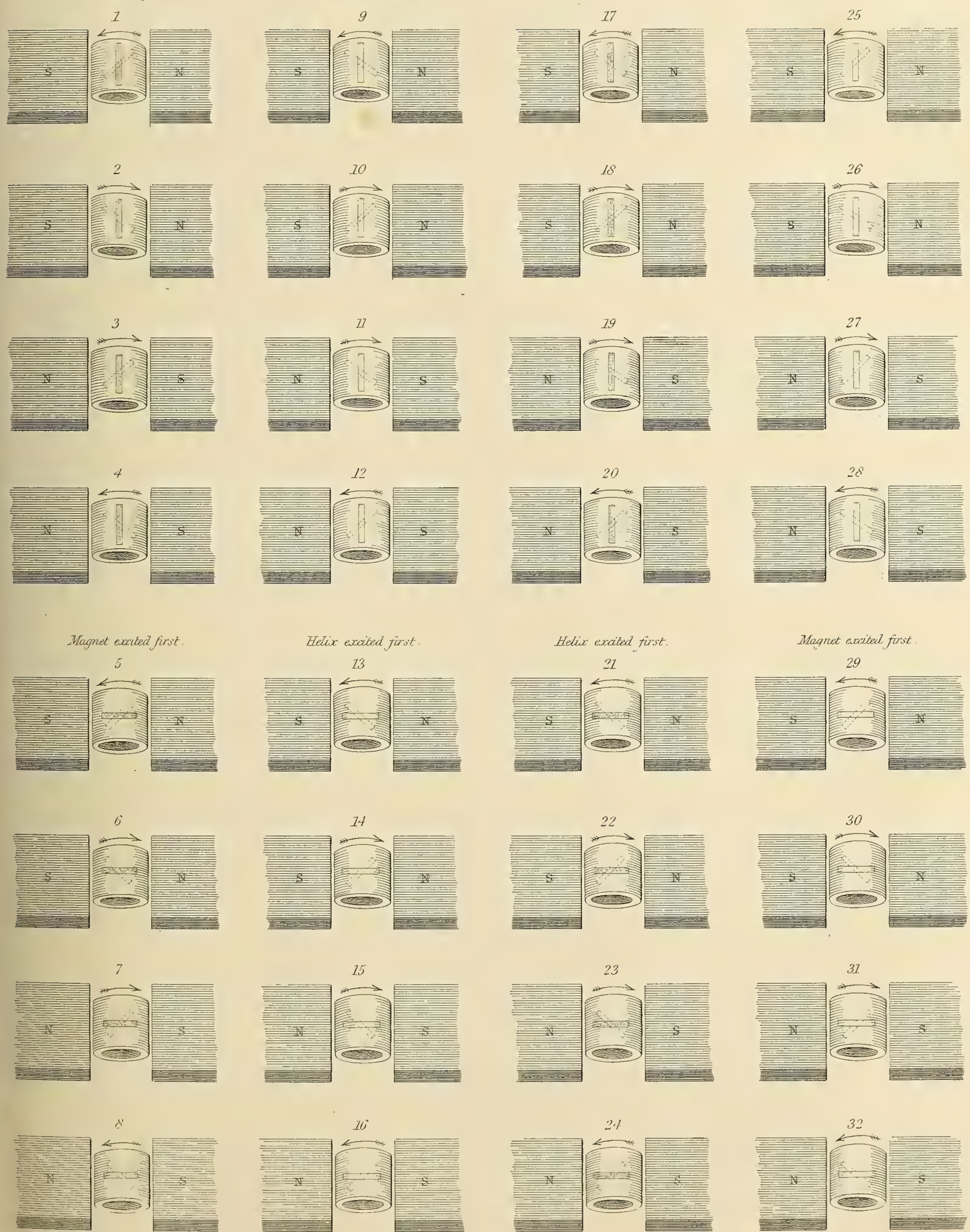
Normal Paramagnetic Bar.	Normal Diamagnetic Bar.	Abnormal Paramagnetic Bar.	Abnormal Diamagnetic Bar.
Helix excited first.	*Magnet excited first.*	*Magnet excited first.*	*Helix excited first.*



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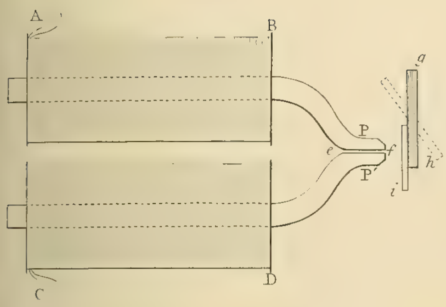
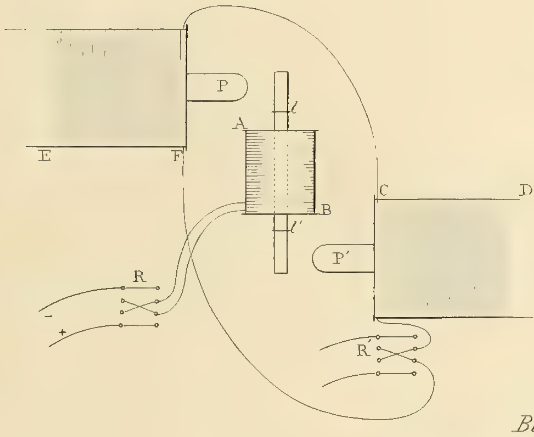


Fig. 39.



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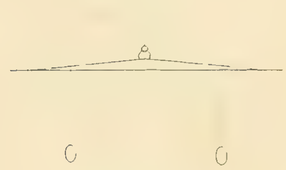


Bar of Iron.

Bar of Bismuth.

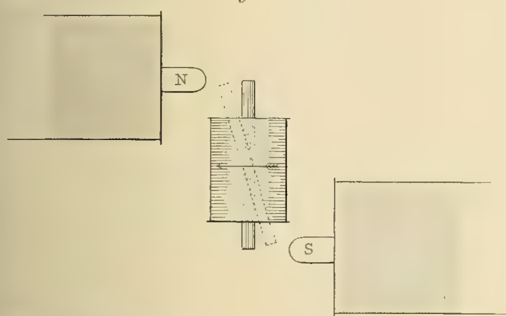
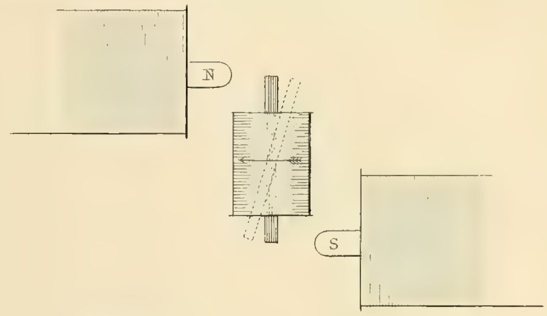


Fig. 40.



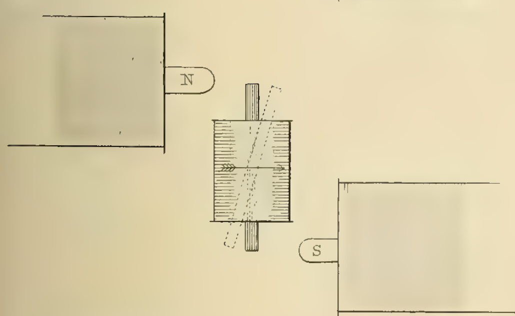
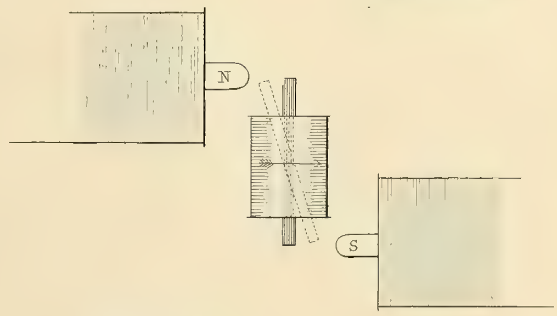


Fig. 41.



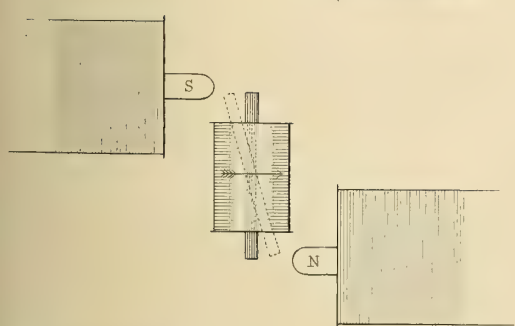
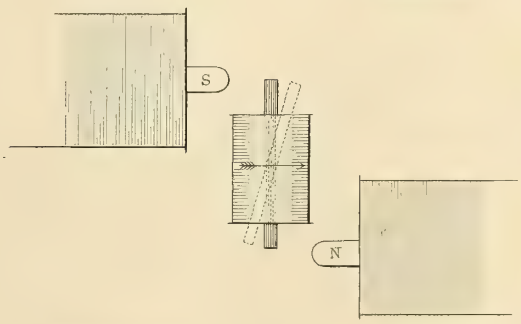


Fig. 42.



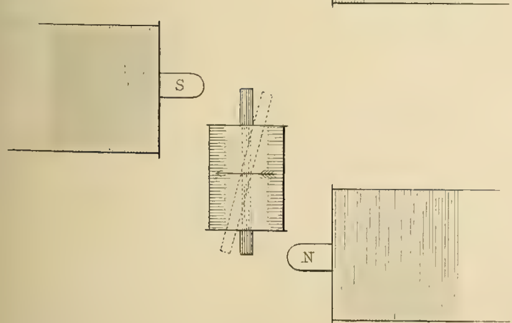
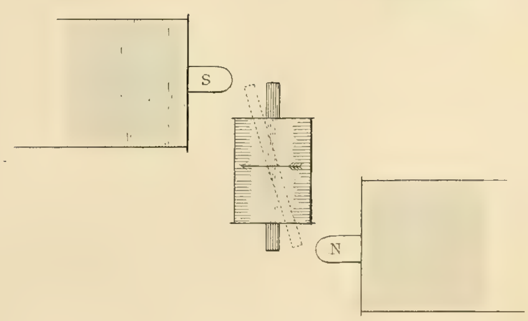
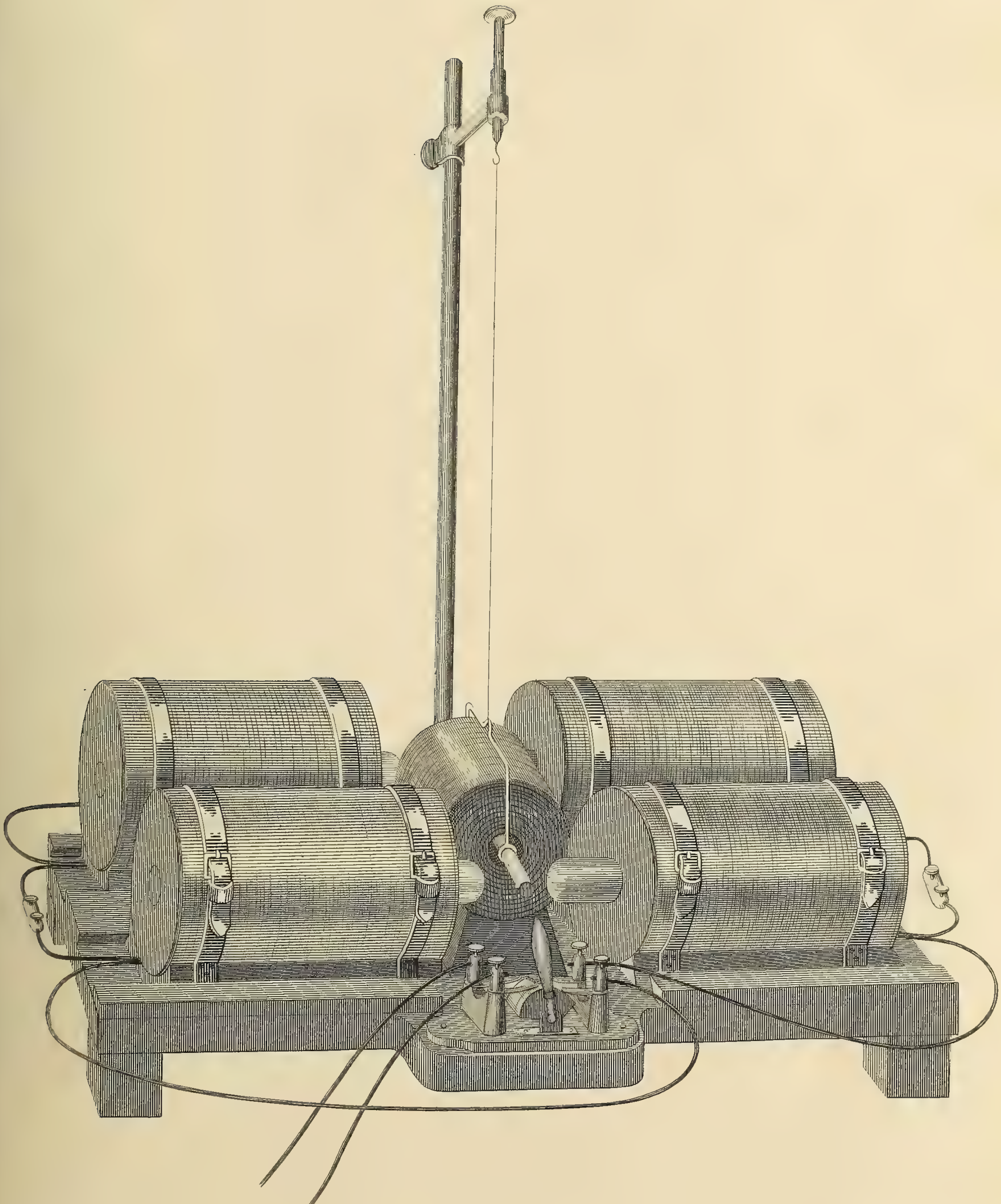


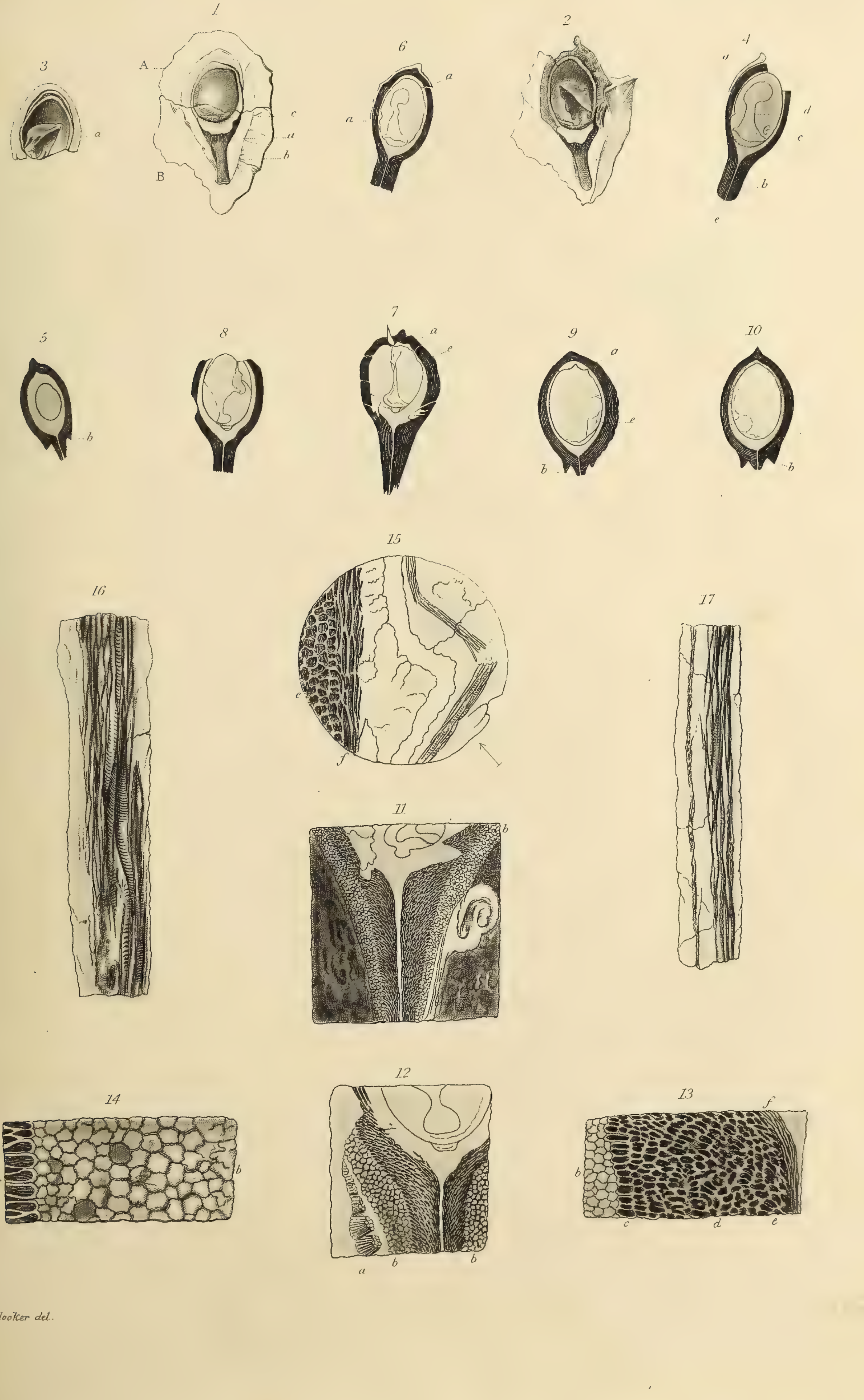
Fig. 43.



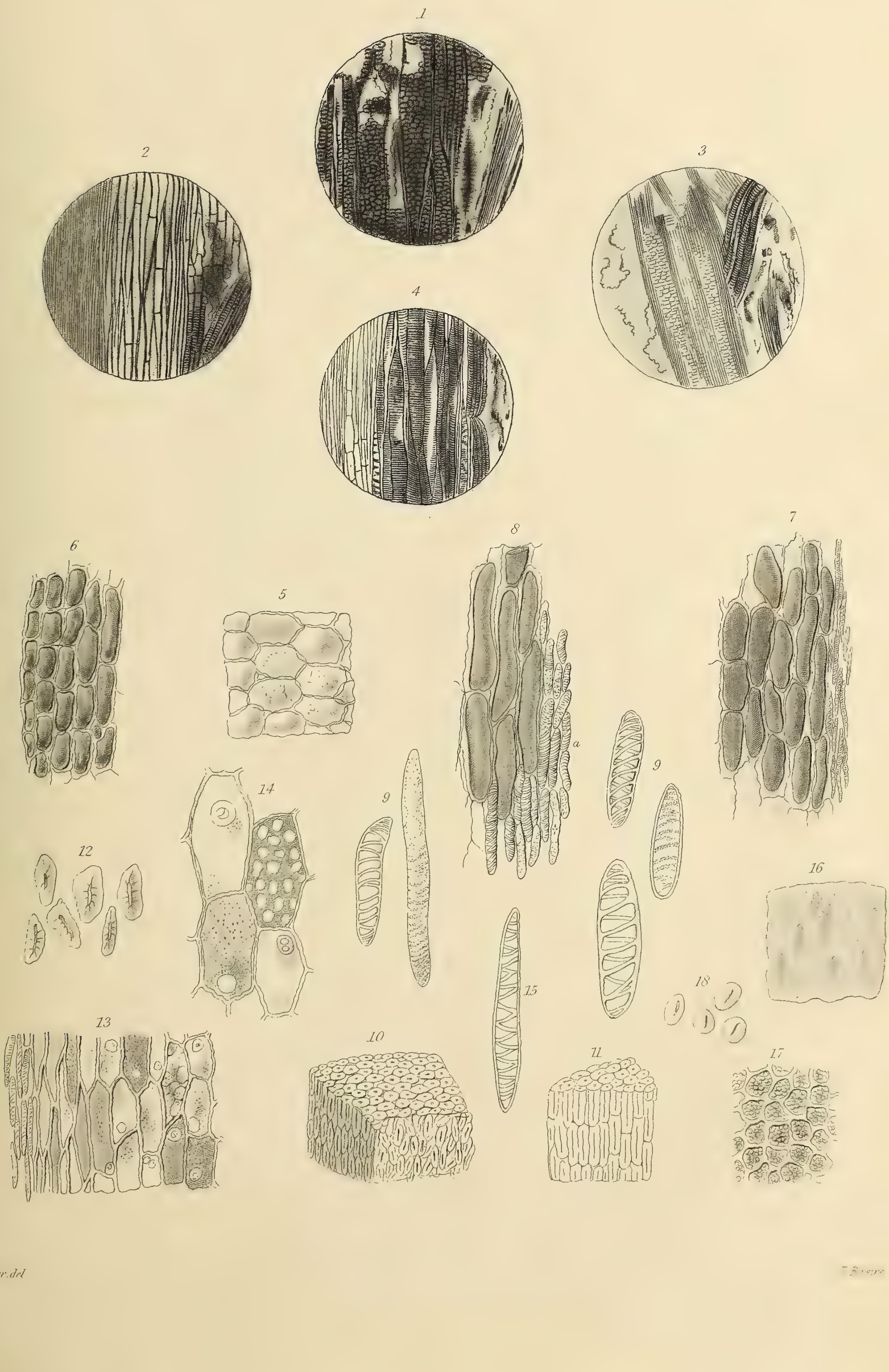


C. Berker, del.

J. Basire, sc.



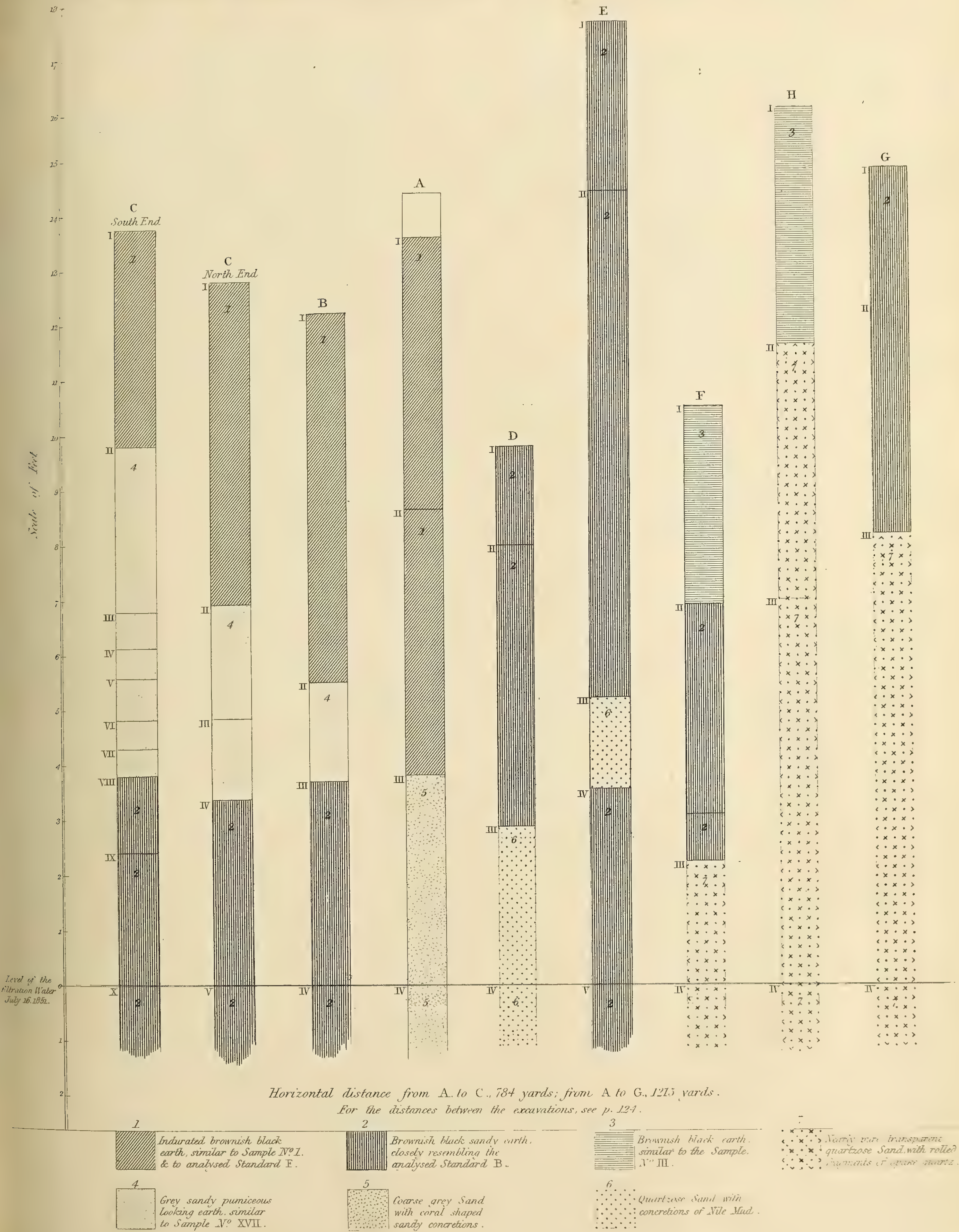
J. D. Hooker del.



J. D. Hooker, del.

[illegible] sc.

Vertical Sections of the Excavations at Heliopolis from the surface of the ground at the mouth of each Pit, to the level of the filtration water, shewing the variety in the soils passed through at the same levels.



Ferric Sulphocyanide.

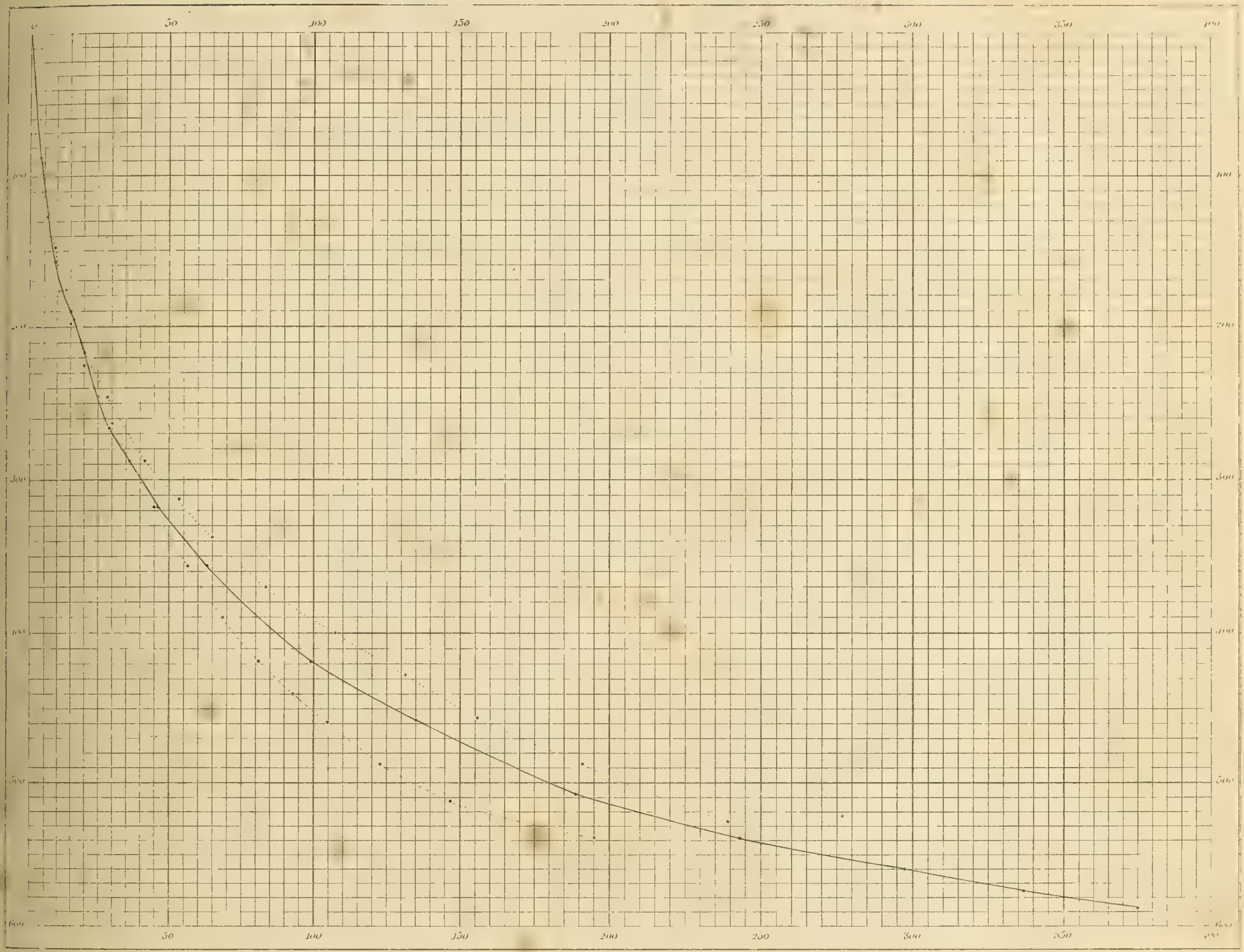
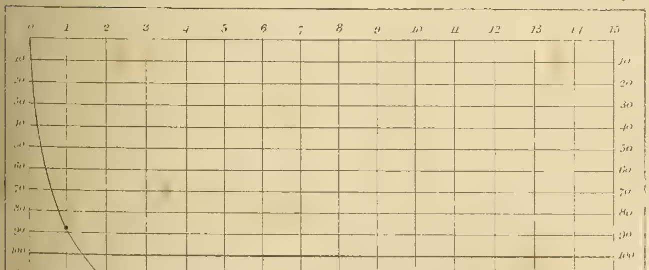
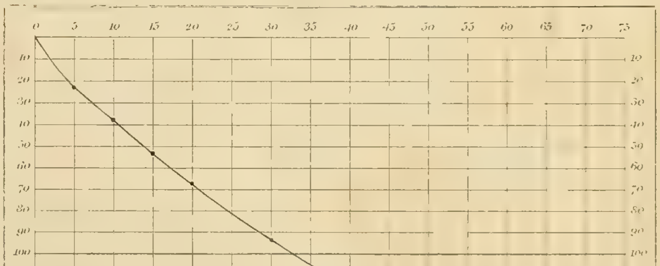


Fig. II. Ferric Sulphocyanide.

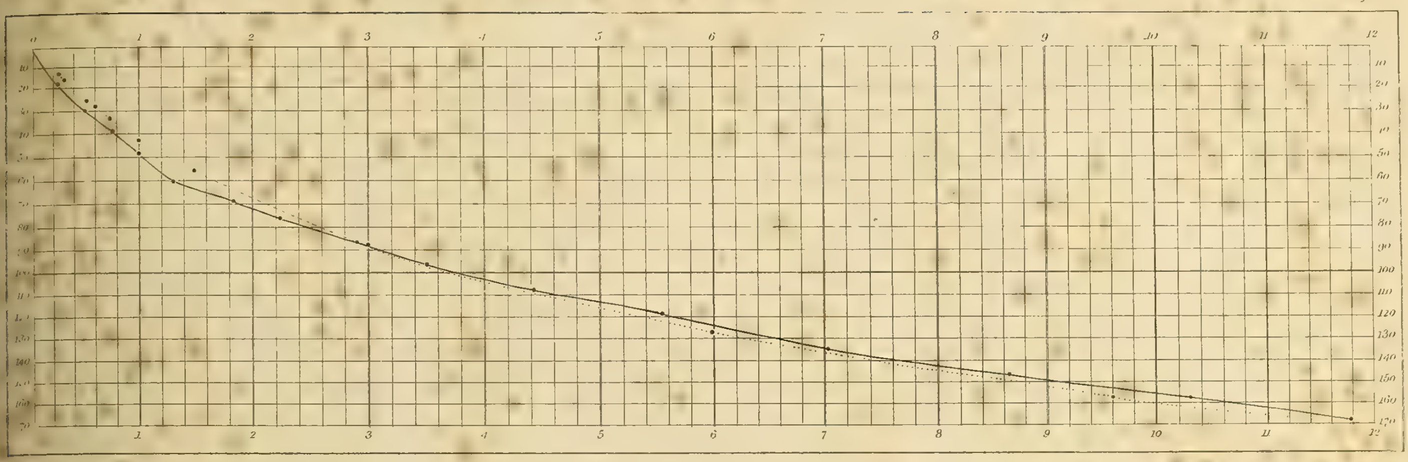


Ferric Sulphocyanide. Fig. III.



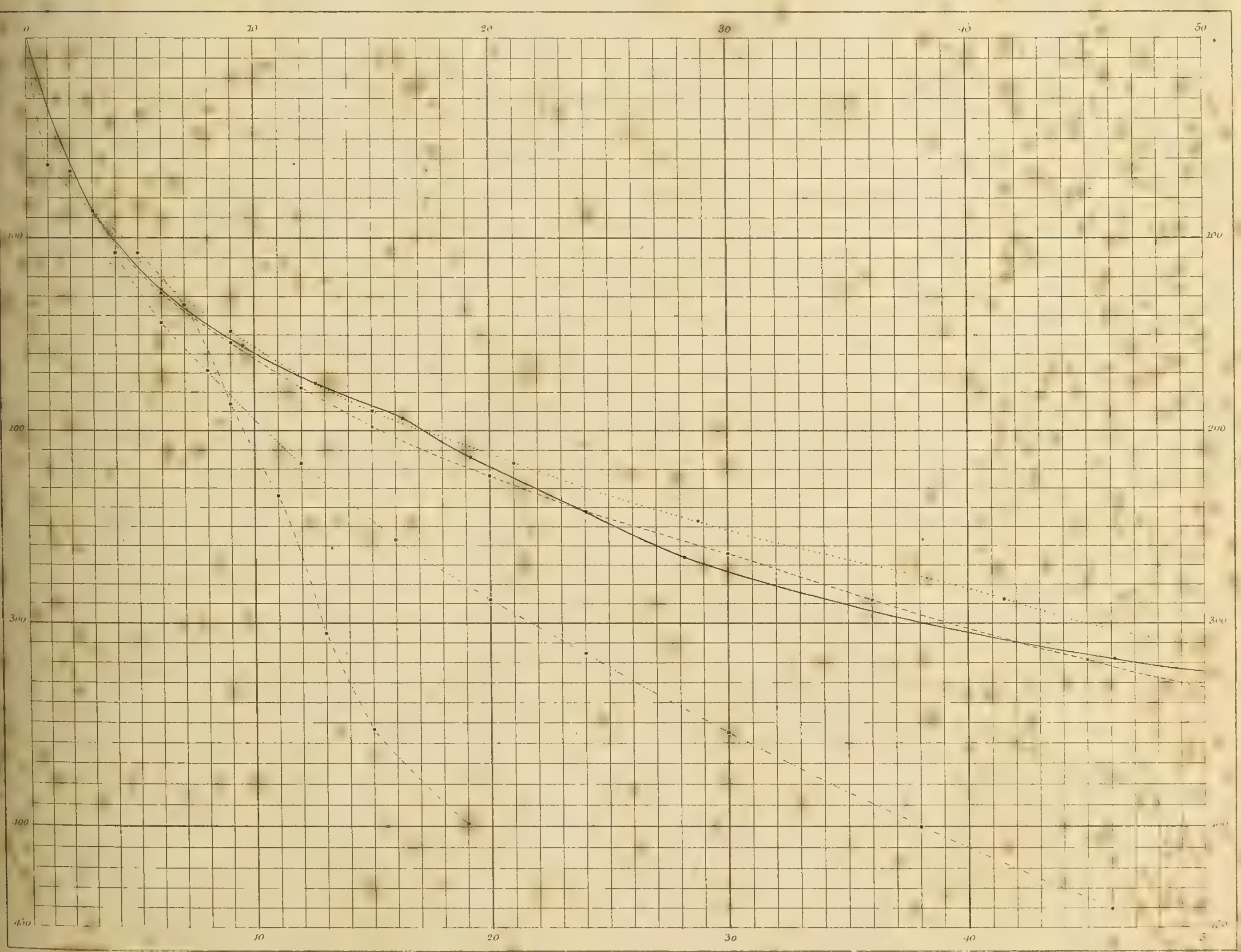
Ferric Sulphocyanide.

Fig. I.

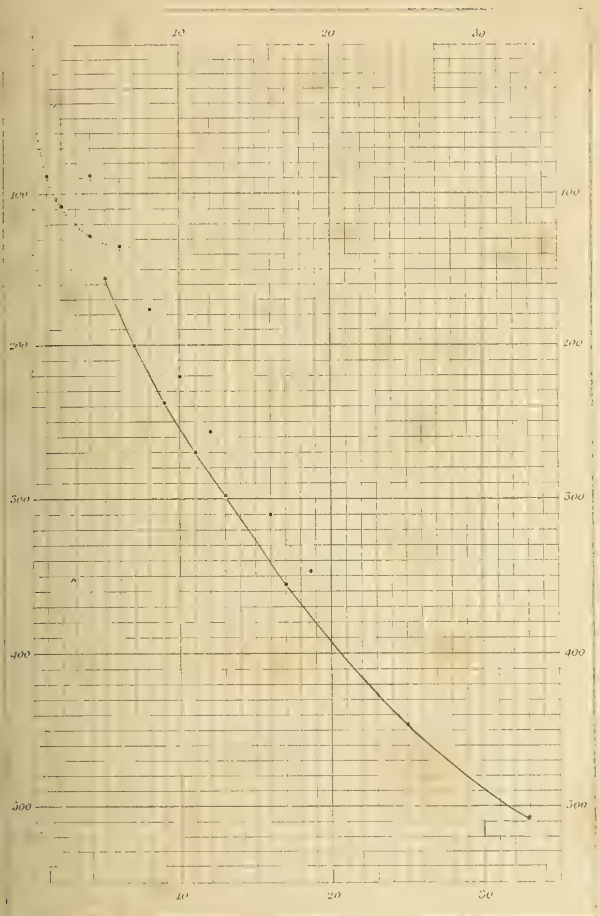


Ferric Sulphocyanide.

Fig. II.

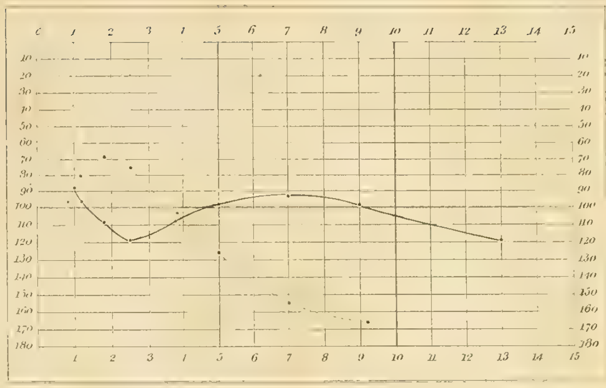


Ferric Gallate.

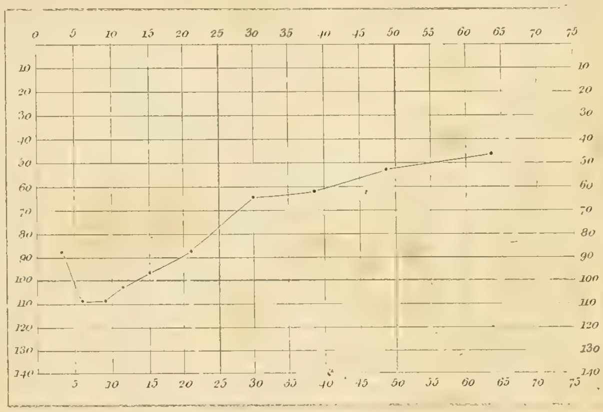


Ferric Meconate.

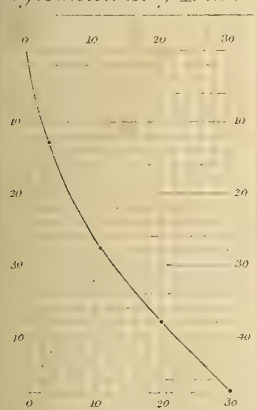
Fig. II.



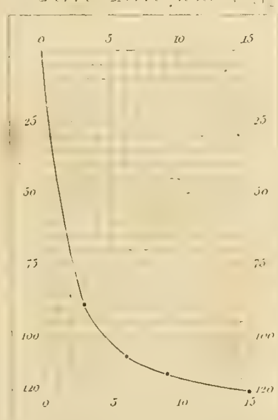
Ferric Acetate.



Pyromeconate of Iron.



Ferric Ferrocyanide.



Bromide of Gold.

Fig. VI.

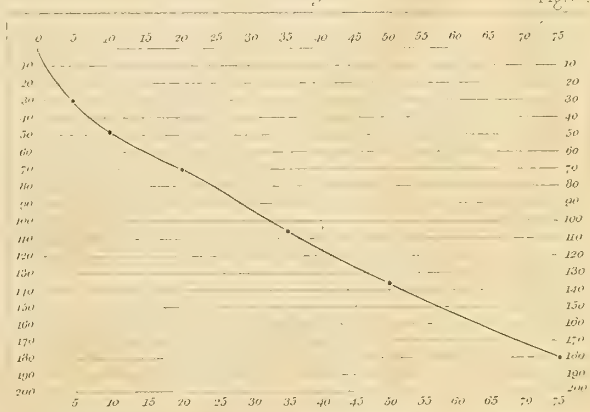
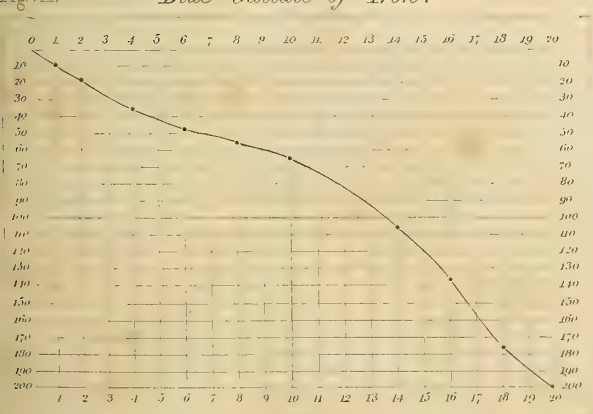


Fig. VII. Blue Gallate of Iron.



Deposited Oxalate of Lime.

Fig. VIII.

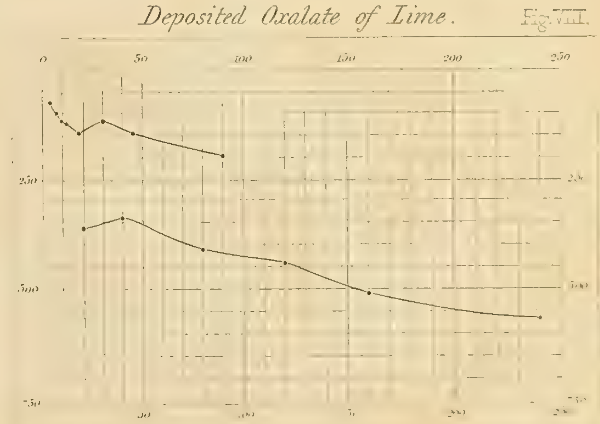


FIG. 1.

FIG. 2.

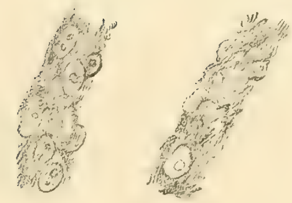


FIG. 3.

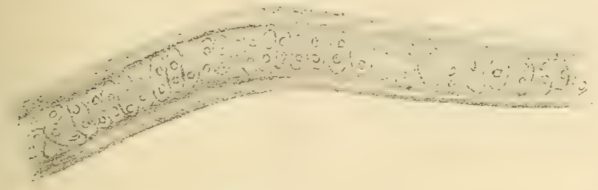


FIG. 5.

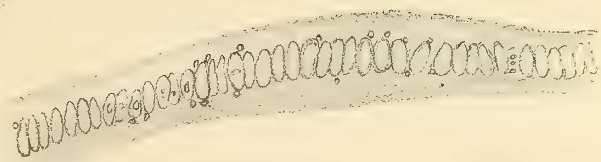


FIG. 4.

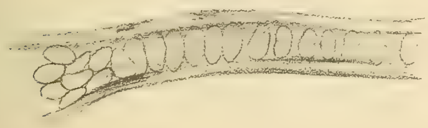


FIG. 7.

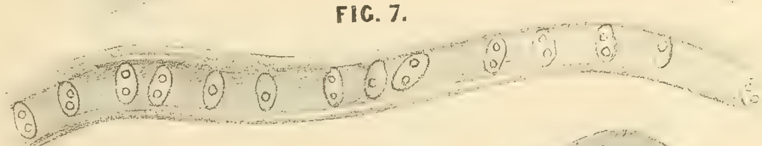


FIG. 8.

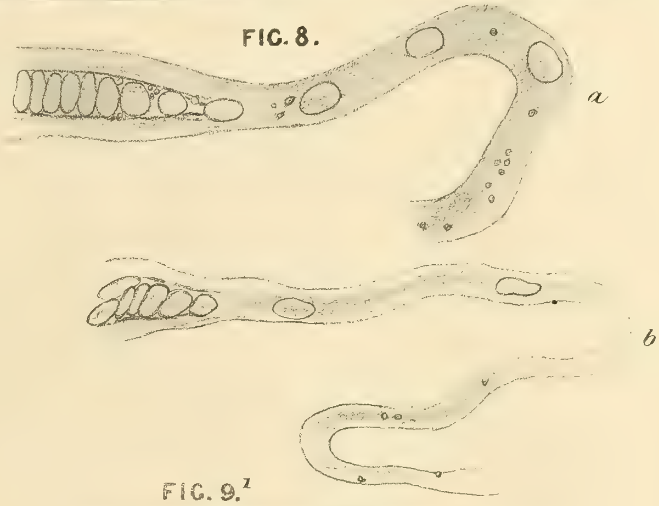


FIG. 6.

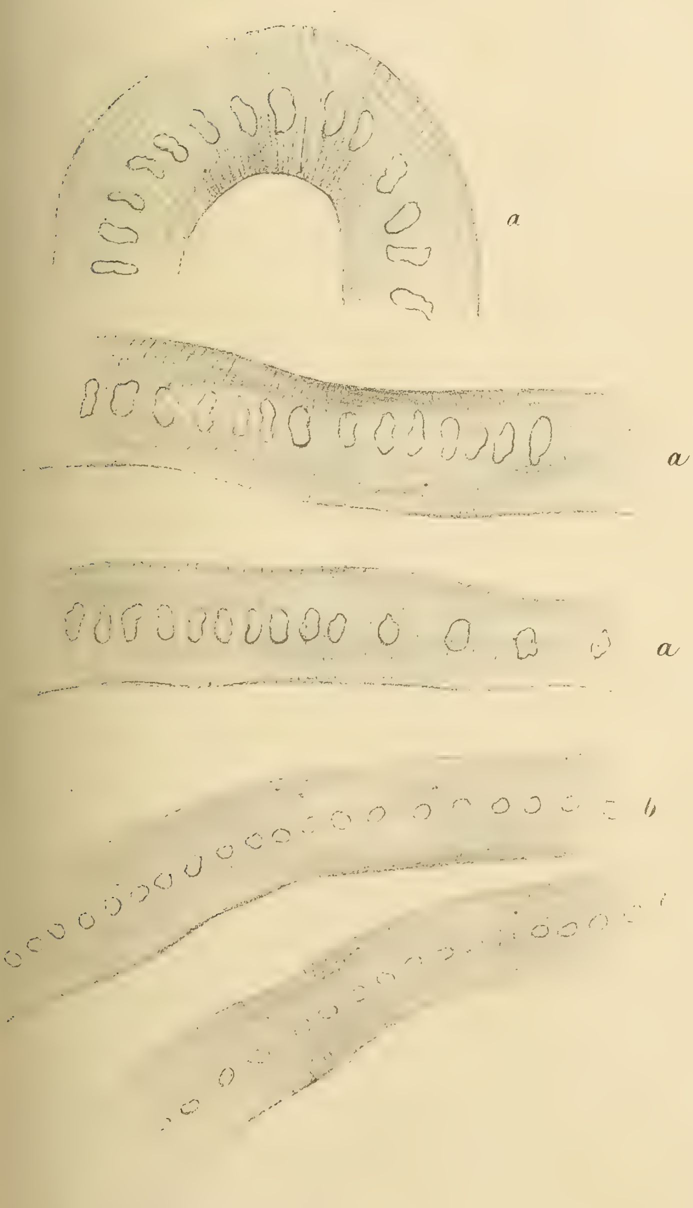
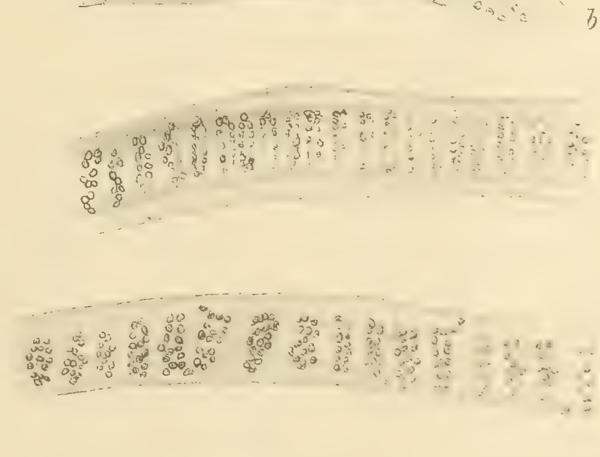


FIG. 9.



S. Savery del.t J. Basire, lith.

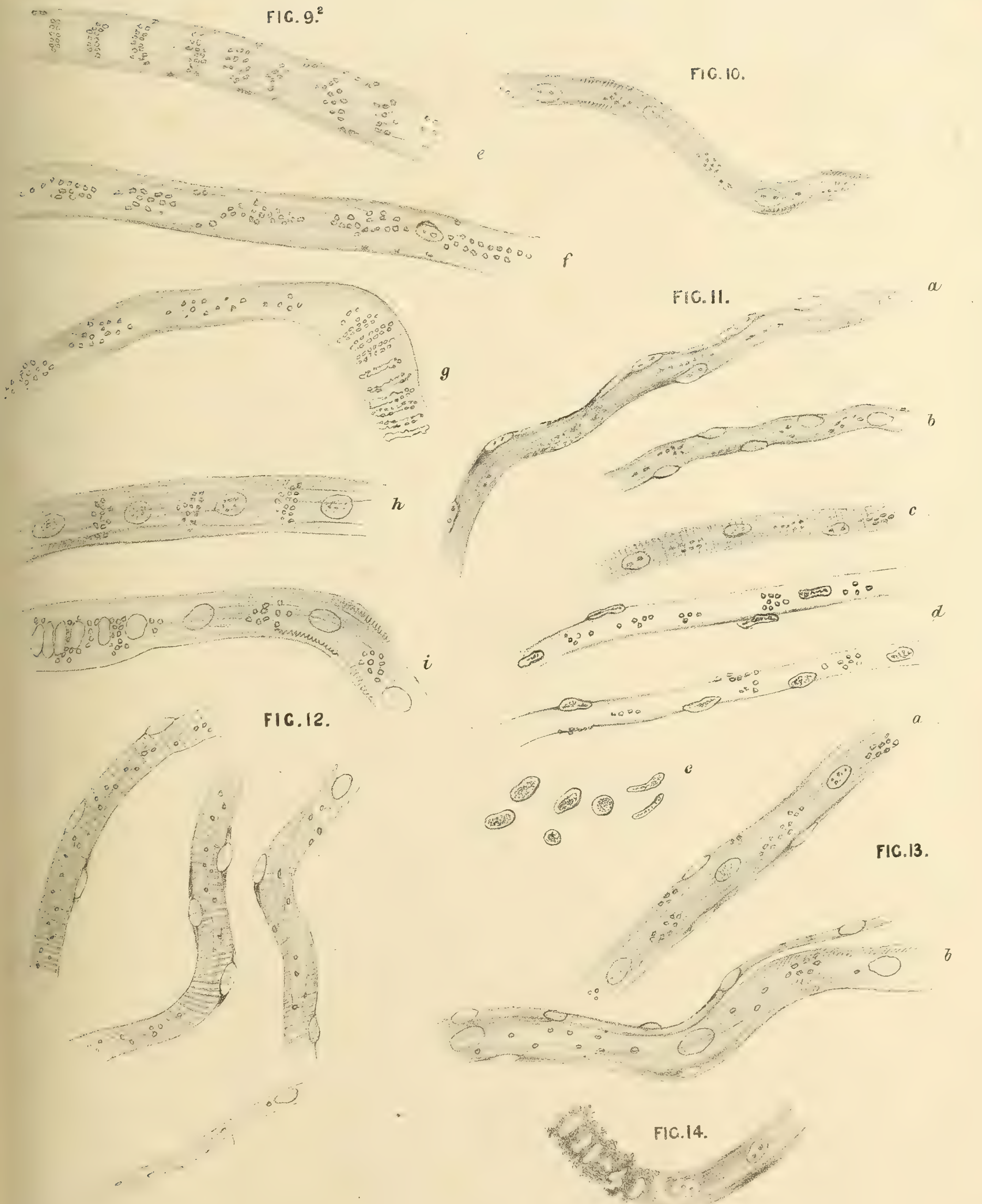
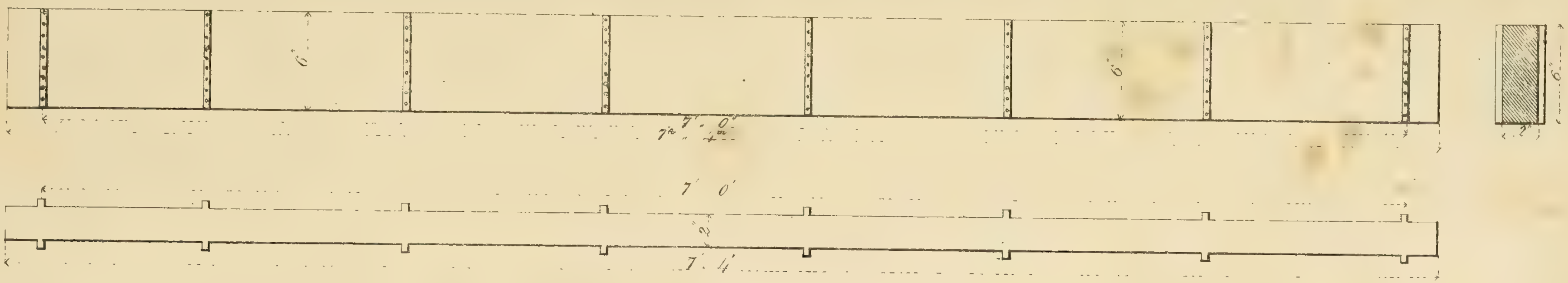


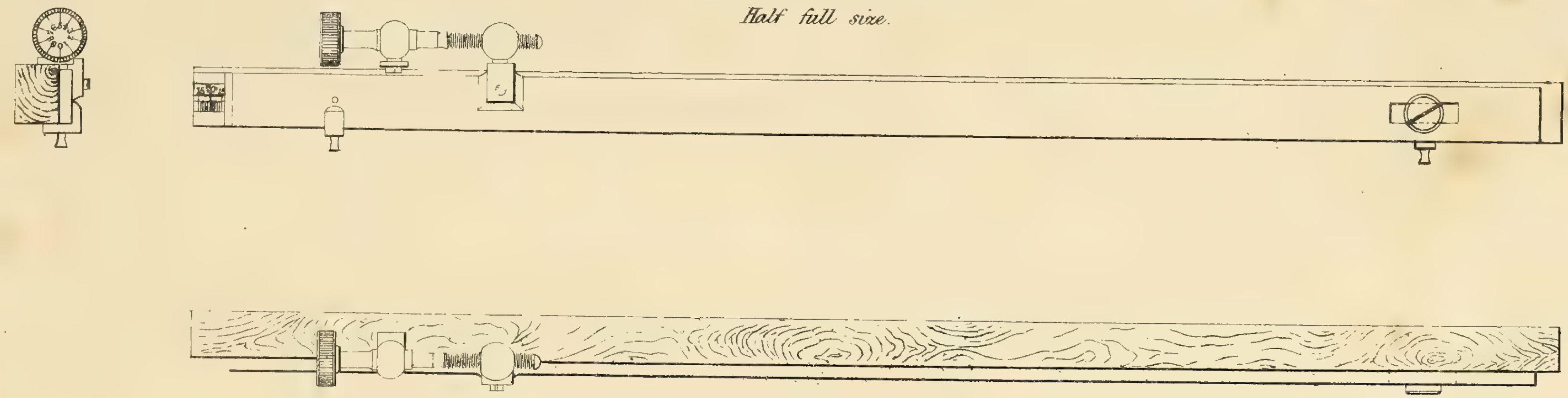
FIG. 9.²
e
f
g
h
i
FIG. 10.
FIG. 11.
a
b
c
d
a
e
FIG. 12.
FIG. 13.
b
FIG. 14.

Beam employed in determining the position of the neutral axis.



Measuring Instrument.

Half full size.



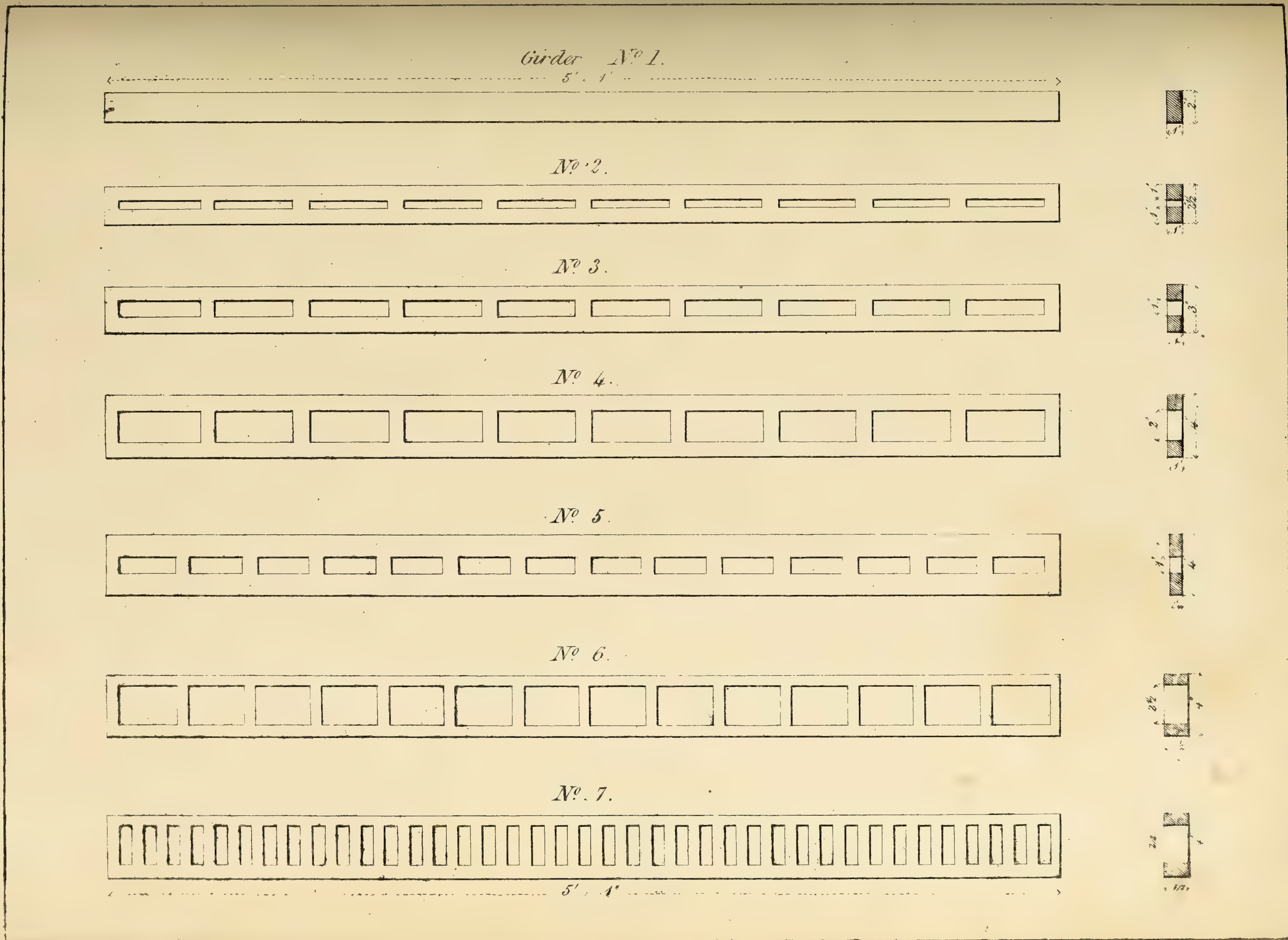

Girder No. 1.
5' , 1'
No. 2.
No. 3.
No. 4.
No. 5.
No. 6.
No. 7.
5' , 1"

Fig. 1

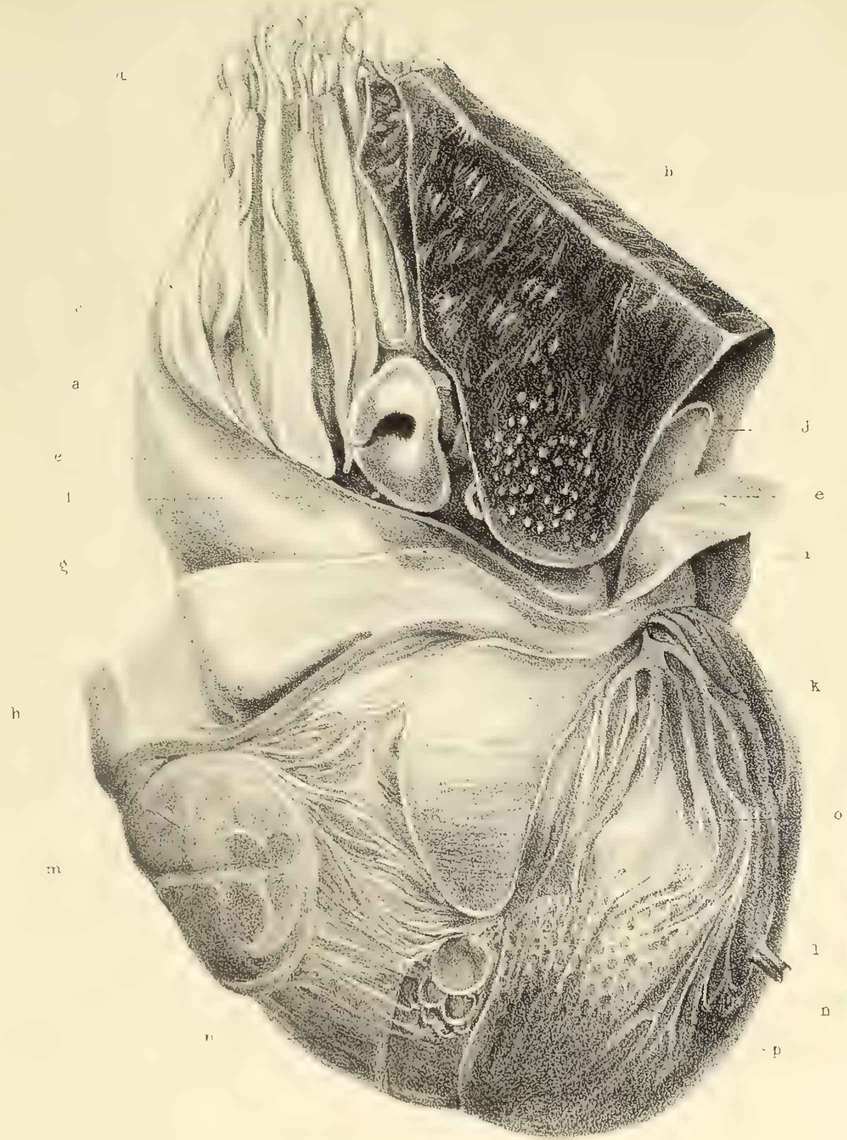


Fig. 2.

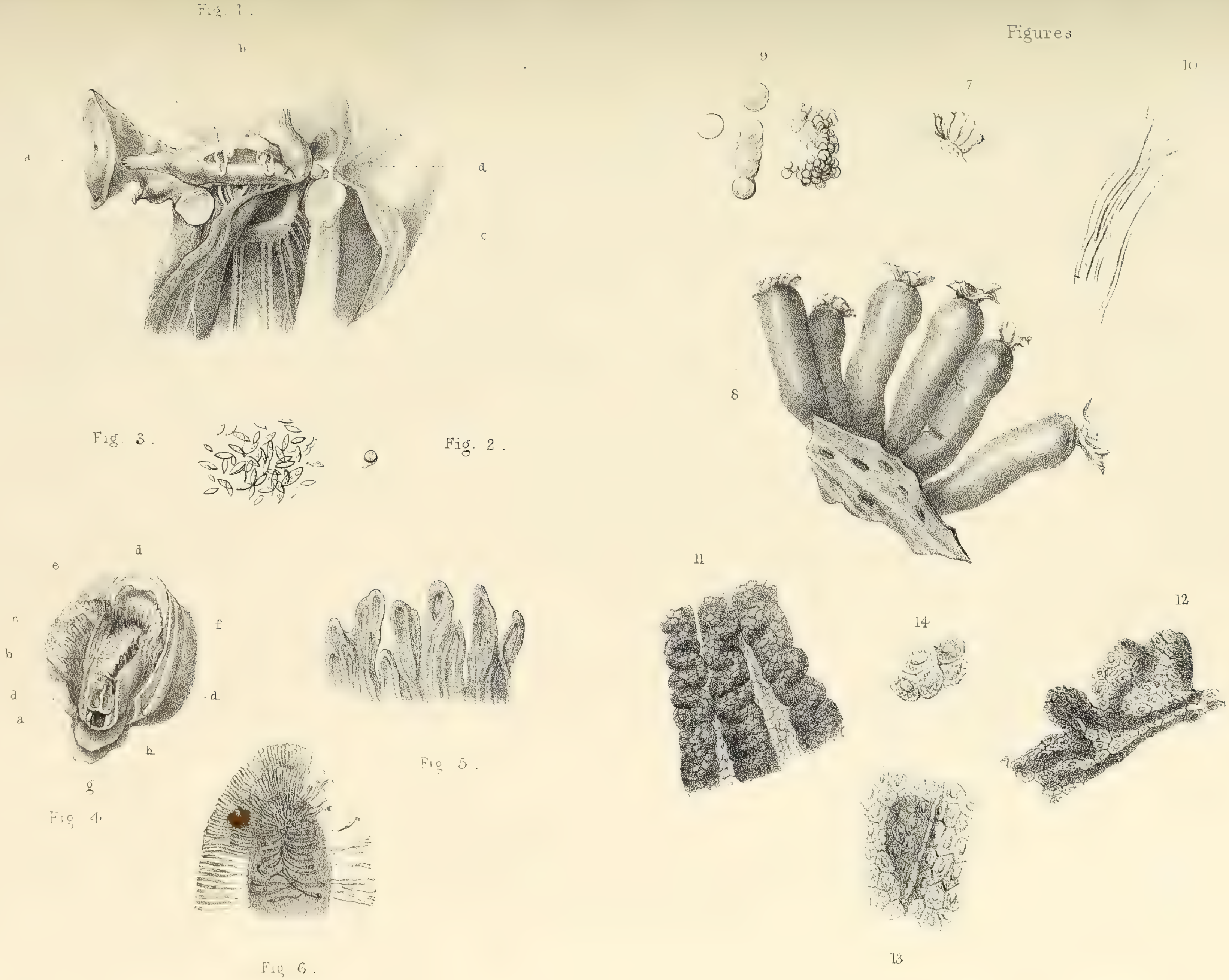


Fig. 1.
b
a
d
c
Fig. 3.
Fig. 2.
d
e
c
f
b
d
d
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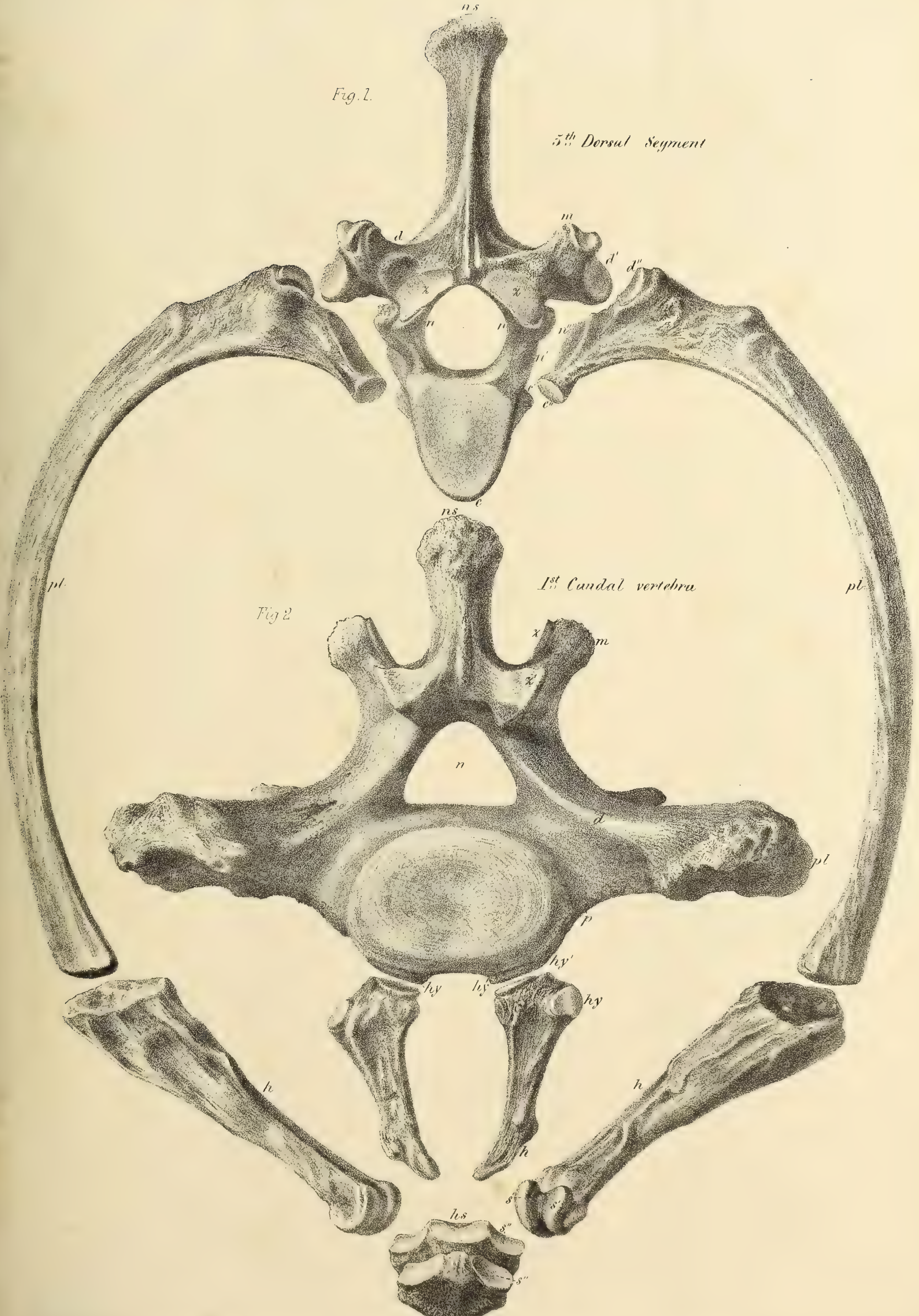
h
g
Fig. 4.
Fig. 5.
Fig. 6.
Figures
9
7
10
8
11
14
12
13

Figures.
Figure.

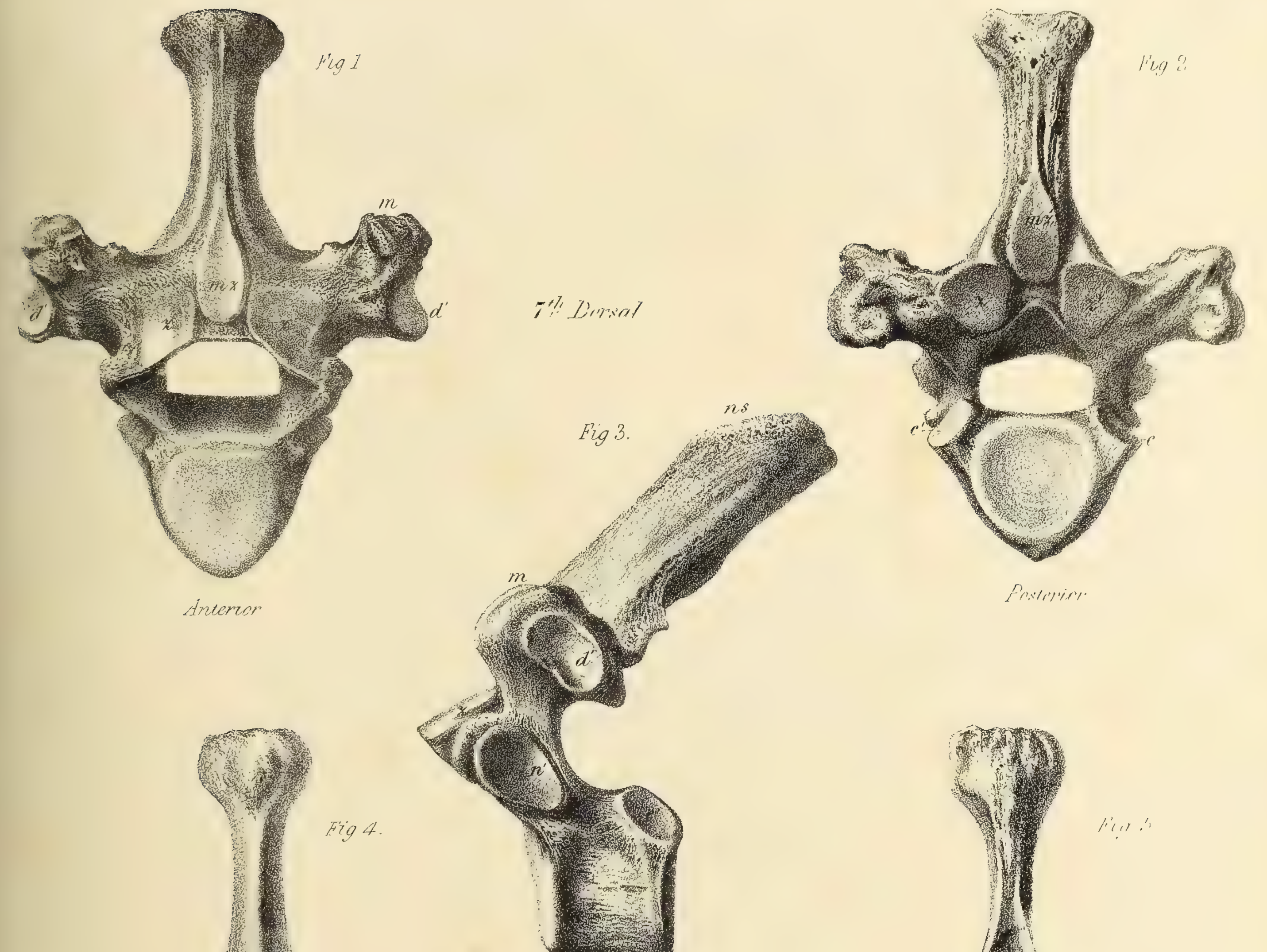
Plate XVI

MEGATHERIUM AMERICANUM



J. Dinkel, del. et lith.

Printed by Hanhart



J. Dinkel, del. et lith.

Fig. 1.

Fig. 2.

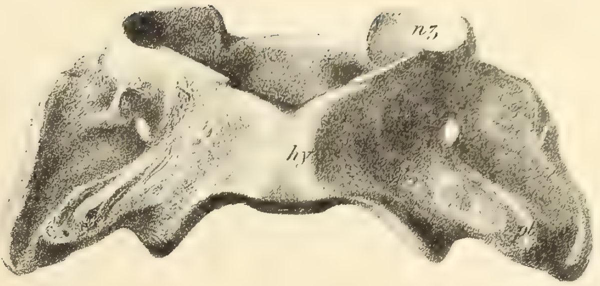


Fig. 3.

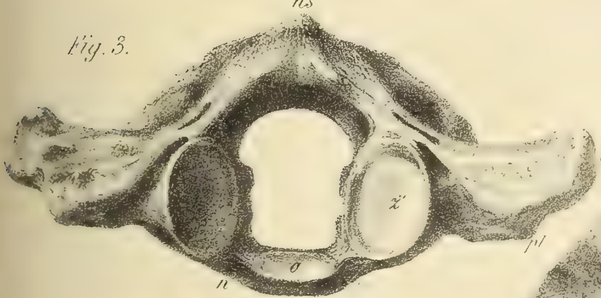


Fig. 4.

Fig. 5.

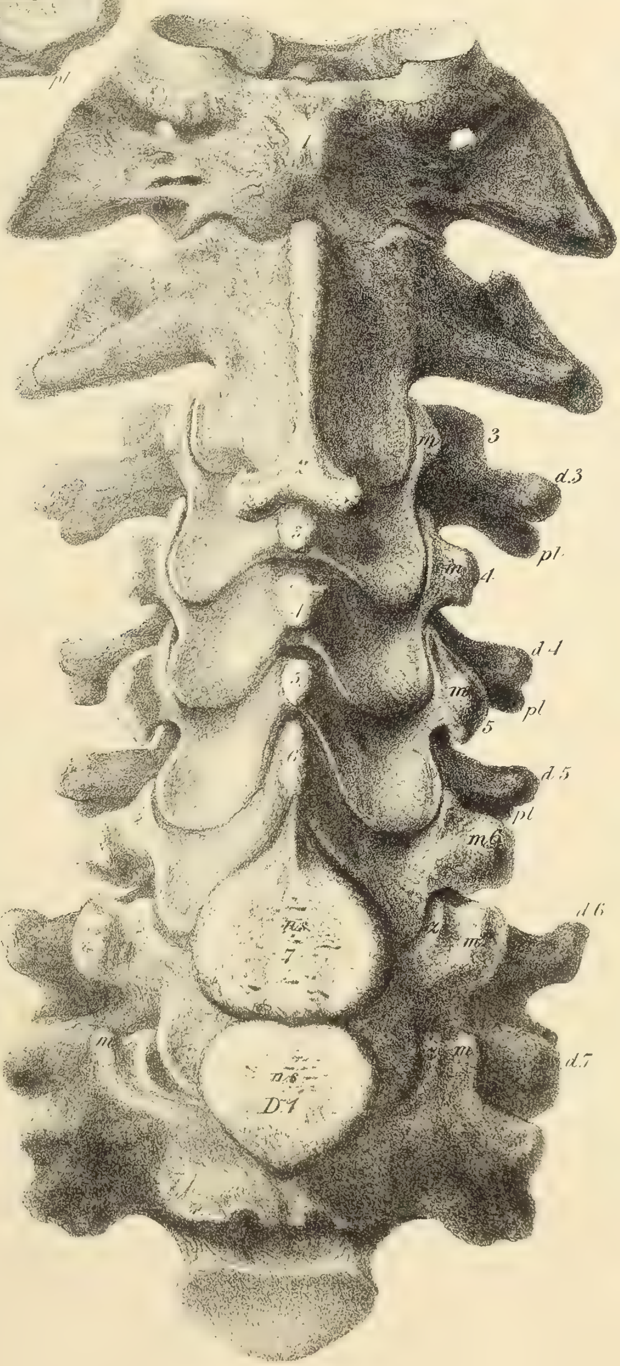


Fig. 6.

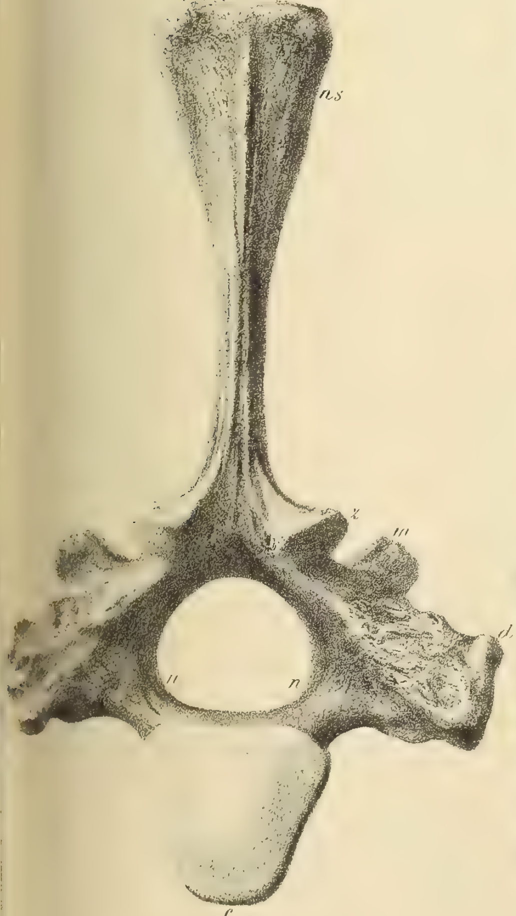


Fig. 7.

Jos. Dinkel, del. et lith.

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Fig. 1.

Fig. 2.

Fig. 3.

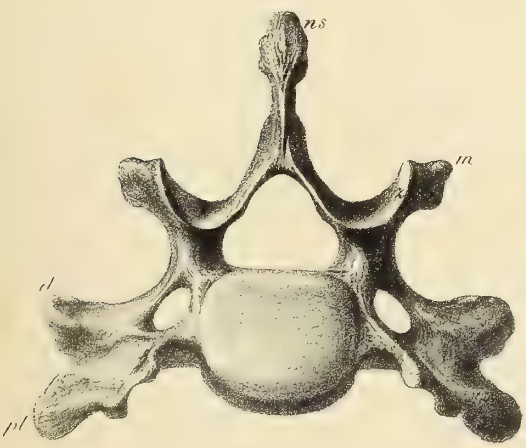


Fig. 4.

Fig. 5.

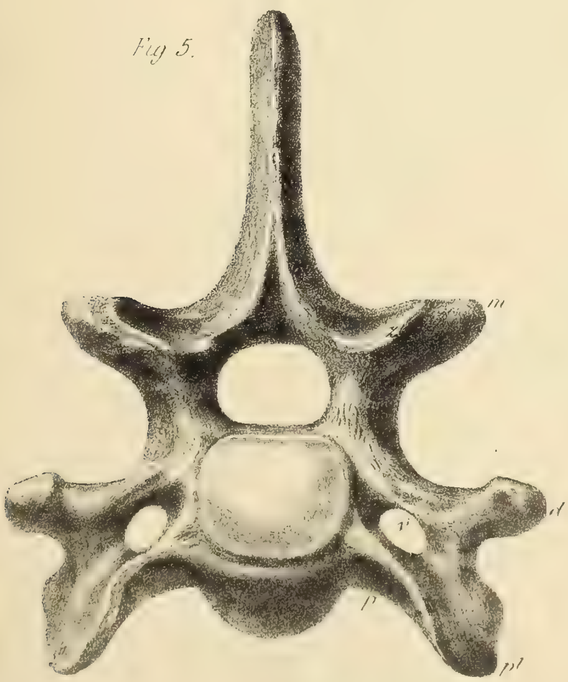
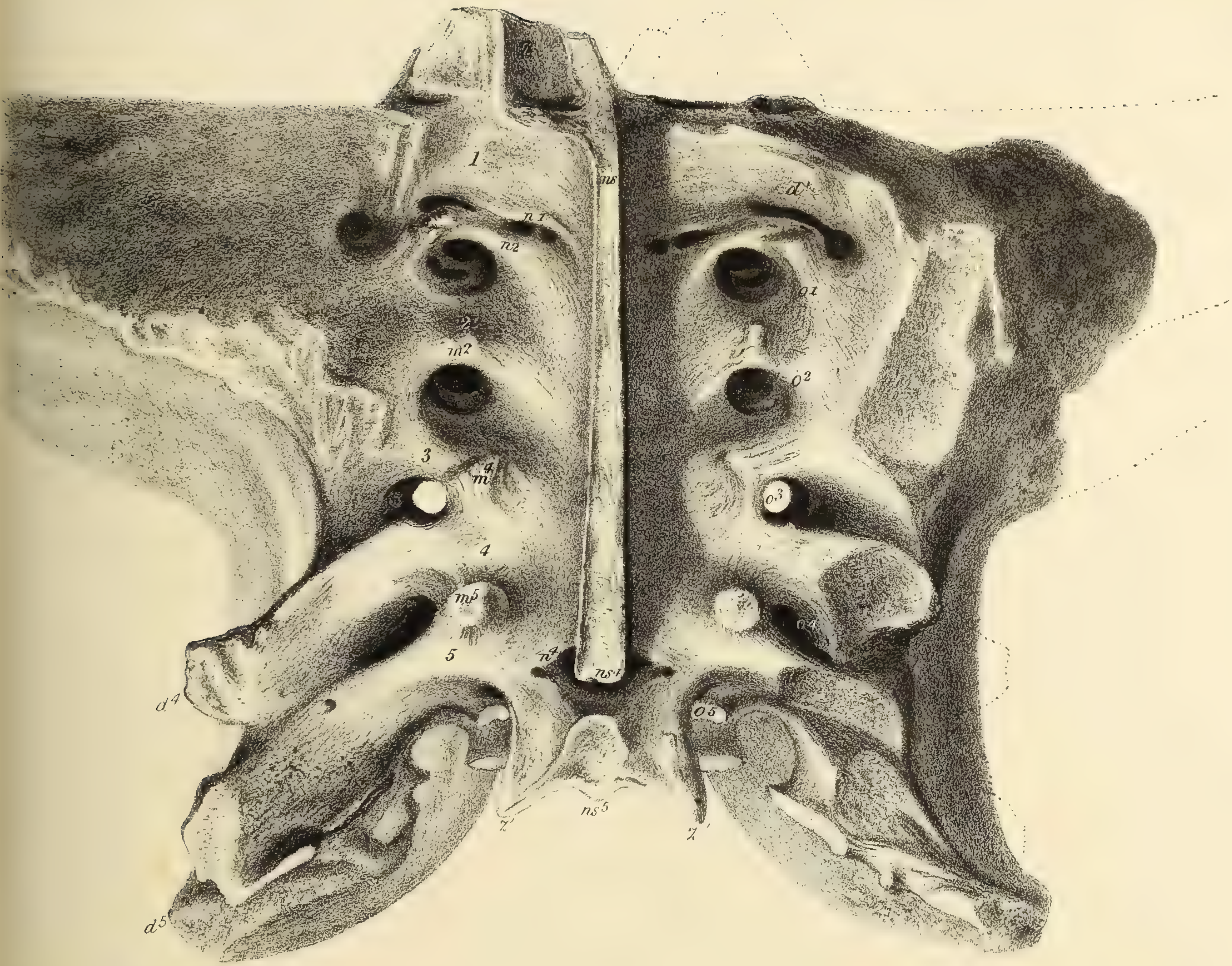


Fig. 6.


ma
c



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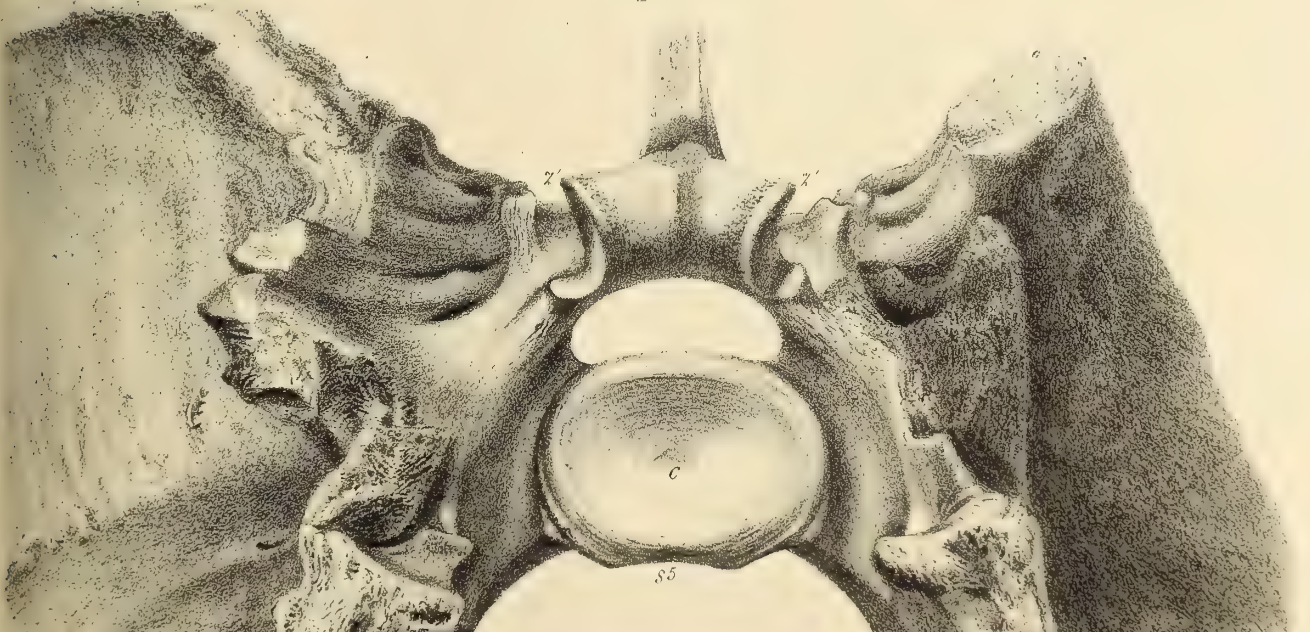
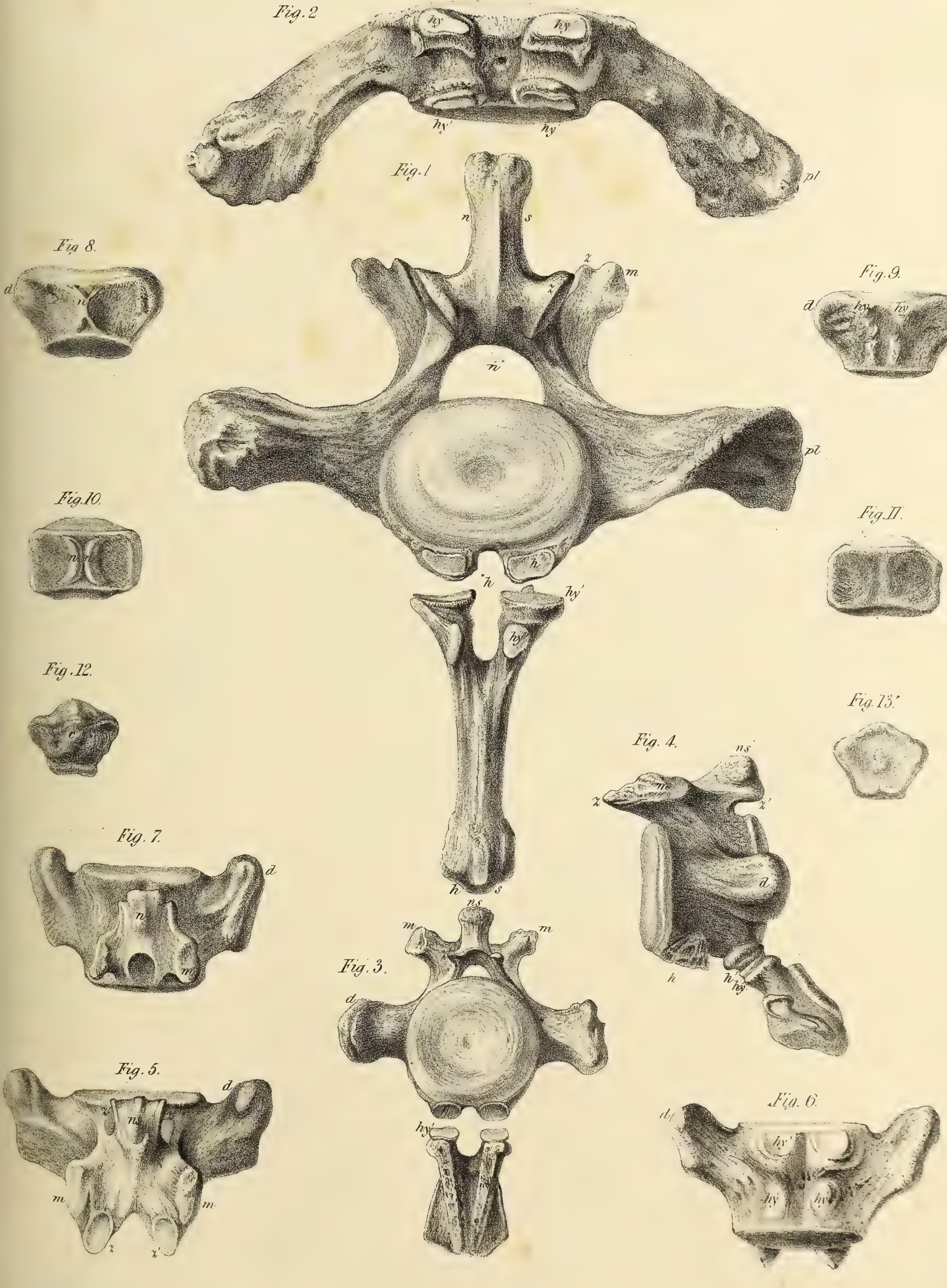


Jos.h Dinkel del.t



J. Dinkel del. et lith.

Printed by J. Basire.

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Fig. 1.

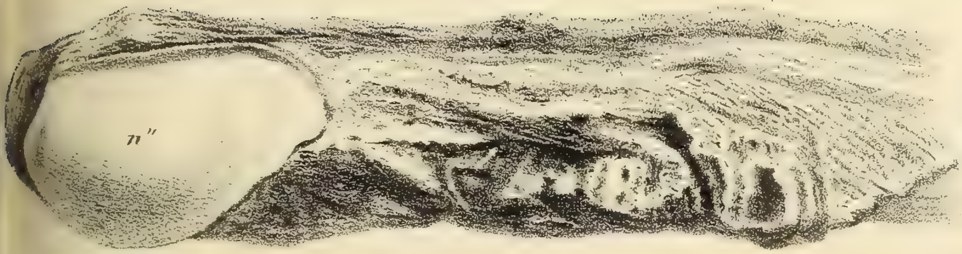


Fig. 2.

Fig. 3.

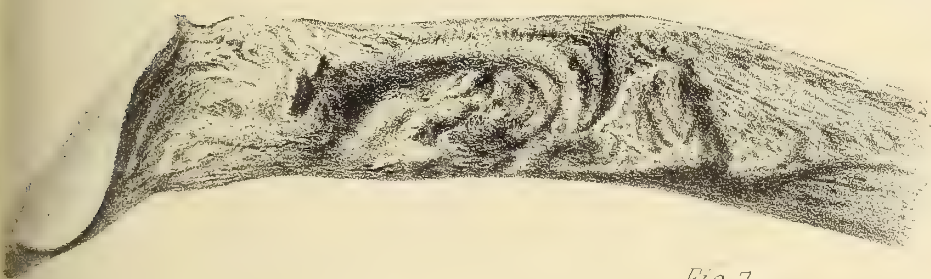
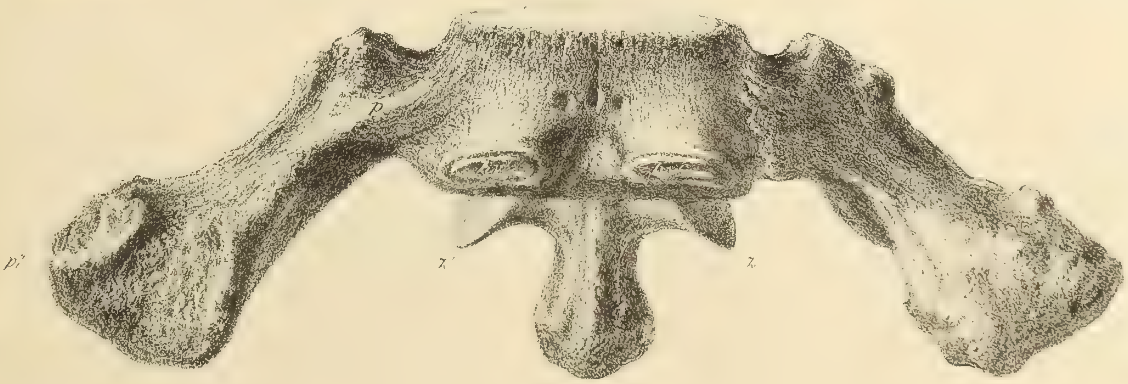


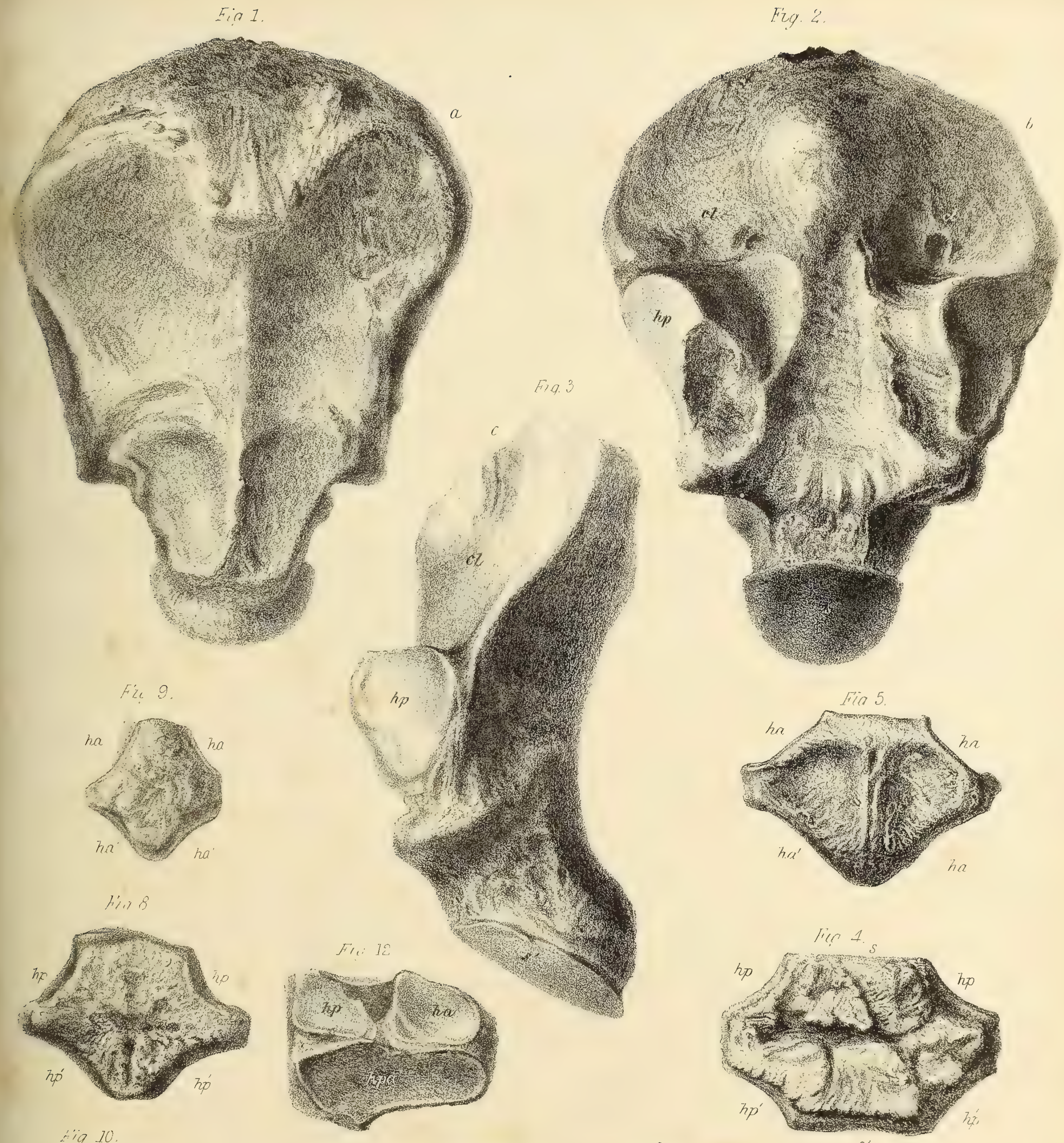
Fig. 5.

Fig. 7.

Fig. 4.

Fig. 6.





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